Computation of the vertex Folkman numbers $F(2,2,2,4;6)$ and $F(2,3,4;6)$

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Abstract

In this note we show that the exact value of the vertex Folkman numbers $F(2,2,2,4;6)$ and $F(2,3,4;6)$ is 14.

1 Notations

We consider only finite, non-oriented graphs, without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We call *p-clique* of G any set of p vertices, each two of which are adjacent. The largest natural number p, such that the graph G contains a p-clique, is denoted by $\mathrm{cl}(G)$ (the *clique number* of G). A set of vertices of a graph G is said to be *independent* if every two of them are not adjacent. The cardinality of any largest independent set of vertices in G is denoted by $\alpha(G)$ (the *independence number* of G).

If $W \subseteq V(G)$ then $G[W]$ is the subgraph of G induced by W and $G - W$ is the subgraph induced by $V(G) \setminus W$. We shall use also the following notation:

$\overline{G}$ - the complement of graph G;

K_n - complete graph of n vertices;

C_n - simple cycle of n vertices;

$N(v)$ - the set of all vertices adjacent to v;

$\chi(G)$ - the chromatic number of G;

$K_n - C_m, m \leq n$ - the graph obtained from K_n by deleting all edges of some cycle C_m.

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$, the graph G, for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x, y] : x \in V(G_1), y \in V(G_2)\}$.

2 Vertex Folkman numbers.

Definition 1. Let G be a graph, and let $a_1, \dots, a_r$ be positive integers, $r \geq 2$. An *r-coloring*

$$V(G) = V_1 \cup \ldots \cup V_r,\ V_i \cap V_j = \emptyset,\ i \neq j,$$

of the vertices of G is said to be $(a_1, \dots, a_r)$-free if for all $i \in \{1, \dots, r\}$ the graph G does not contain a monochromatic a_i-clique of color i. The symbol $G \to (a_1, \dots, a_r)$ means that every r-coloring of $V(G)$ is not $(a_1, \dots, a_r)$-free.

A graph G such that $G \to (a_1, \dots, a_r)$ is called a *vertex Folkman graph*. We define $F(a_1, \dots, a_r; q) = \min\{|V(G)| : G \to (a_1, \dots, a_r) \text{ and } \mathrm{cl}(G) < q\}$. Clearly $G \to (a_1, \dots, a_r)$ implies that $\mathrm{cl}(G) \geq \max\{a_1, \dots, a_r\}$. Folkman [2] proved that there exists a graph G, such that $G \to (a_1, \dots, a_r)$ and $\mathrm{cl}(G) = \max\{a_1, \dots, a_r\}$. Therefore, if $q > \max\{a_1, \dots, a_r\}$ then the numbers $F(a_1, \dots, a_r; q)$ exist and they are called *vertex Folkman numbers*.

Let $a_1, \dots, a_r$ be positive integers, $r \geq 2$. Define

$$m = \sum_{i=1}^{r} (a_i - 1) + 1 \text{ and } p = \max\{a_1, \dots, a_r\}. \qquad (1)$$

Obviously $K_m \to (a_1, \dots, a_r)$ and $K_{m-1} \not\to (a_1, \dots, a_r)$. Hence, if $q \geq m + 1$, $F(a_1, \dots, a_r; q) = m$. For the numbers $F(a_1, \dots, a_r; m)$, the following theorem is known:

Theorem A([4]). *Let $a_1, \dots, a_r$ be positive integers, $r \geq 2$ and let m and p satisfy* (1)*, where $m \geq p + 1$. Then $F(a_1, \dots, a_r; m) = m + p$. If $G \to (a_1, \dots, a_r)$,* $\mathrm{cl}(G) < m$ *and $|V(G)| = m + p$, then $G = K_{m+p} - C_{2p+1}$.*

Another proof of Theorem A is given in [13]. It is true that:

Theorem B([13]). *Let $a_1, \dots, a_r$ be positive integers, $r \geq 2$. Let p and m satisfy* (1) *and $m \geq p + 2$. Then*

$$F(a_1, \dots, a_r; m - 1) \geq m + p + 2.$$

Observe that for each permutation φ of the symmetric group S_r, $G \to (a_1, \dots, a_r) \iff G \to (a_{\varphi(1)}, \dots, a_{\varphi(r)})$. Therefore, we can assume that $a_1 \leq \ldots \leq a_r$. Note that if $a_1 = 1$, then $F(a_1, \dots, a_r; q) = F(a_2, \dots, a_r; q)$. So, we will consider only Folkman numbers for which $a_i \geq 2, i = 1, \dots, r$.

The next theorem implies that, in the special situation where $a_1 = \ldots = a_r = 2$ and $r \geq 5$, the inequality from Theorem B is exact.

Theorem C.

$$F(\underbrace{2,\dots,2}_{r};r) = \begin{cases} 11, & r=3 \text{ or } r=4; \\ r+5, & r \geq 5. \end{cases}$$

Obviously $G \to (\underbrace{2,\dots,2}_{r}) \Leftrightarrow \chi(G) \geq r+1$.

Mycielski in [5] presented an 11-vertex graph G, such that $G \to (2,2,2)$ and $\mathrm{cl}(G)=2$, proving that $F(2,2,2;3) \leq 11$. Chvátal [1], proved that the Mycielski graph is the smallest such graph and hence $F(2,2,2;3)=11$. The inequality $F(2,2,2,2;4) \geq 11$ was proved in [8] and inequality $F(2,2,2,2;4) \leq 11$ was proved in [7] and [12] (see also [9]). The equality

$$F(\underbrace{2,\dots,2}_{r};r) = r+5,\ r \geq 5$$

was proved in [7], [12] and later in [4]. Only a few more numbers of the type $F(a_1,\dots,a_r;m-1)$ are known, namely: $F(3,3;4)=14$ (the inequality $F(3,3;4) \leq 14$ was proved in [6] and the opposite inequality $F(3,3;4) \geq 14$ was verified by means of computers in [15]); $F(3,4;5)=13$ [10]; $F(2,2,4;5)=13$ [11]; $F(4,4;6)=14$ [14].

In this note we determine two additional numbers of this type.

Theorem D. $F(2,2,2,4;6) = F(2,3,4;6) = 14$.

These two numbers are known to be less than 36 (see [4], Remark after Proposition 5).

We will need the following

Lemma. *Let $G \to (a_1,\dots,a_r)$ and let for some i, $a_i \geq 2$. Then*

$$G \to (a_1,\dots,a_{i-1},2,a_i-1,a_{i+1}\dots,a_r).$$

Proof. Consider an $(a_1,\dots,a_{i-1},2,a_i-1,a_{i+1}\dots,a_r)$-free $(r+1)$-coloring $V(G) = V_1 \cup \dots \cup V_{r+1}$. If we color the vertices of V_i with the same color as the vertices of V_{i+1}, we obtain an $(a_1,\dots,a_r)$-free coloring of $V(G)$, a contradiction.

3 Proof of Theorem D.

According to the lemma, it follows from $G \to (2,3,4)$ that $G \to (2,2,2,4)$. Therefore $F(2,2,2,4;6) \leq F(2,3,4;6)$ and hence it is sufficient to prove that $F(2,3,4;6) \leq 14$ and $F(2,2,2,4;6) \geq 14$.

1. Proof of the inequality $F(2,3,4;6) \leq 14$**.**

We consider the graph Q, whose complementary graph $\overline{Q}$ is given in Fig.1.

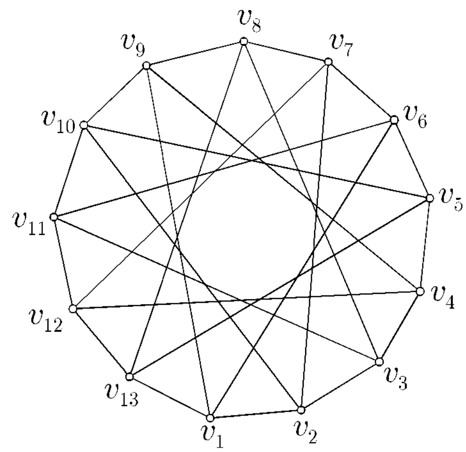

Fig. 1. Graph $\overline{Q}$

This is the well known construction of Greenwood and Gleason [3], which shows that the Ramsey number $R(3,5) \geq 14$. It is proved in [10] that $K_1 + Q \to (4,4)$. Together with the lemma, this implies that $K_1 + Q \to (2,3,4)$. Since $\mathrm{cl}(K_1+Q) = 5$ and $|V(K_1+Q)| = 14$, then $F(2,3,4;6) \leq 14$.

2. Proof of the inequality $F(2,2,2,4;6) \geq 14$.

Let $G \to (2,2,2,4)$ and $\mathrm{cl}(G) < 6$. We need to prove that $|V(G)| \geq 14$. It is clear from $G \to (2,2,2,4)$ that

$$G - A \to (2,2,4) \text{ for any independent set } A \subseteq V(G). \tag{2}$$

First we will consider some cases where the proof of the inequality $|V(G)| \geq 14$ is easy.

Suppose that $\mathrm{cl}(G-A) < 5$ for some nonempty independent set $A \subseteq V(G)$. According to (2) and the equality $F(2,2,4;5) = 13$ [11], $|V(G-A)| \geq 13$. Therefore, $|V(G)| \geq 14$. Hence in the sequel, without loss of generality, we will assume that

$$\mathrm{cl}(G-A) = \mathrm{cl}(G) = 5 \text{ for any independent set } A \subseteq V(G). \tag{3}$$

Next assume that there exist $u, v \in V(G)$, such that $N(u) \supseteq N(v)$. Observe that $[u,v] \notin E(G)$. Assume that $G - v \not\to (2,2,2,4)$ and let $V_1 \cup V_2 \cup V_3 \cup V_4$ be a $(2,2,2,4)$-free 4-coloring of $G - v$. If we color the vertex v with the same color as the vertex u, we obtain a $(2,2,2,4)$-free 4-coloring of the graph G, a contradiction. Therefore $G - v \to (2,2,2,4)$ and, according to Theorem B (with $m = 7$ and $p = 4$), $|V(G-v)| \geq 13$. Therefore, $|V(G)| \geq 14$. So:

$$N(v) \not\subseteq N(u), \forall u, v \in V(G). \tag{4}$$

From (3) it follows that $|N(v)| \neq |V(G)| - 1, \forall v \in V(G)$ and, according to (4), $|N(v)| \neq |V(G)| - 2, \forall v \in V(G)$. Hence

$$|N(v)| \leq |V(G)| - 3, \; \forall v \in V(G). \tag{5}$$

Since G cannot be complete we know that $\alpha(G) \geq 2$. Assume that $\alpha(G) \geq 3$ and let $\{a, b, c\} \subseteq V(G)$ be an independent set. We put $\widetilde{G} = G - \{a, b, c\}$. Assume that $|V(G)| \leq 13$. Then $|V(\widetilde{G})| \leq 10$. According to (2) and Theorem A (with $m = 6$ and $p = 4$), $\widetilde{G} = K_{10} - C_9 = K_1 + \overline{C_9}$. Let $V(K_1) = \{w\}$. From (5) it follows that w is not adjacent to at least one of the vertices a, b, c. Let, for example, a and w be not adjacent. Then $N(w) \supseteq N(a)$, which contradicts (4). Therefore, we obtain that if $\alpha(G) \geq 3$, then $|V(G)| \geq 14$. So, we can assume that

$$\alpha(G) = 2. \tag{6}$$

Hence, we need to consider only the case where the graph G satisfies conditions (3), (4), (5) and (6). According to Theorem B, $|V(G)| \geq 13$. Therefore, it is sufficient to prove, that $|V(G)| \neq 13$. Assume the contrary. Let a and b be two non-adjacent vertices of the graph G, and let $G_1 = G - \{a, b\}$.

Case 1. $G_1 \rightarrow (2, 5)$. According to (3), $\mathrm{cl}(G_1) = 5$. Since $|V(G_1)| = 11$, it follows from Theorem A that $G_1 = \overline{C_{11}}$. Let $V(C_{11}) = \{v_1, \ldots, v_{11}\}$ and $E(C_{11}) = \{[v_i, v_{i+1}] : i = 1, \ldots, 10\} \cup \{[v_1, v_{11}]\}$. From $\mathrm{cl}(G) = 5$ it follows that the vertex a is not adjacent to at least one of the vertices $v_1, \ldots, v_{11}$, say $[a, v_1] \notin E(G)$. Consider a 4-coloring $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{v_6, v_7\}$, $V_2 = \{v_8, v_9\}$, $V_3 = \{v_{10}, v_{11}\}$. Since V_1, V_2, V_3 are independent sets, then it follows from $G \rightarrow (2, 2, 2, 4)$ that V_4 contains a 4-clique. Since the set $\{v_1, v_2, v_3, v_4, v_5\}$ contains a unique 3-clique $\{v_1, v_3, v_5\}$ and the vertex a is not adjacent to v_1, the 4-clique containing in V_4 can be only $\{v_1, v_3, v_5, b\}$. Similarly, $\{v_1, v_8, v_{10}, b\}$ is a 4-clique too. Therfore $\{v_1, v_3, v_5, v_8, v_{10}, b\}$ is a 6-clique, a contradiction.

Case 2. $G_1 \not\rightarrow (2, 5)$. Let $V(G_1) = X \cup Y$ be a $(2, 5)$-free 2-coloring. According to (6), $|X| \leq 2$. From (5) and (6) it follows that we may assume that $|X| = 2$. Let $X = \{c, d\}$, $G_2 = G_1 - \{c, d\} = G[Y]$. According to (2), $G_1 \rightarrow (2, 2, 4)$ and therefore $G_2 \rightarrow (2, 4)$. Since Y contains no 5-cliques, then $\mathrm{cl}(G_2) < 5$. From Theorem A (with $m = 5$ and $p = 4$) it follows that $G_2 = \overline{C_9}$. Let $V(C_9) = \{v_1, \ldots, v_9\}$ and $E(C_9) = \{[v_i, v_{i+1}] : i = 1, \ldots, 8\} \cup \{[v_1, v_9]\}$. Denote $G_3 = G[a, b, c, d]$. From (6) it follows that $E(G_3)$ contains two independent edges. Without loss of generality we can assume that $[a, c]$, $[b, d] \in E(G_3)$. It is sufficient to consider next three subcases:

Subcase 2.a. $E(G_3) = \{[a, c], [b, d]\}$. From $\mathrm{cl}(G) = 5$ it follows that one of the vertices a, c is not adjacent to at least one of the vertices $v_1, \ldots, v_9$, say $[a, v_1] \notin E(G)$. Consider a 4-coloring $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{v_6, v_7\}$, $V_2 = \{v_8, v_9\}$ and $V_3 = \{c, d\}$. Since the sets V_1, V_2, V_3 are independent sets, it follows from $G \rightarrow (2, 2, 2, 4)$ that V_4 contains a 4-clique. Since $\{v_1, v_3, v_5\}$ is the unique 3-clique in $V_4 - \{a, b\}$ and $a \notin N(v_1)$, then this 4-clique can be only $\{v_1, v_3, v_5, b\}$. Similarly we obtain also that $\{v_1, v_6, v_8, b\}$ is a 4-clique. Hence, we may conclude that

$$v_1, v_3, v_5, v_6, v_8 \in N(b). \tag{7}$$

In the same way we can prove that $v_1, v_3, v_5, v_6, v_8 \in N(d)$ which, together with (7), implies that $\{v_1, v_3, v_5, v_8, b, d\}$ is a 6-clique, contradicting $\mathrm{cl}(G) < 6$.

Subcase 2.b. $E(G_3) = \{[a,c],[b,d],[a,d]\}$. From $\mathrm{cl}(G) = 5$ it follows that one of the vertices a,d is not adjacent to at least one of the vertices $v_1, \dots, v_9$. Without loss of generality we may assume that v_1 and a are not adjacent. In the same way as in the *Subcase 2.a.* we can prove (7). Consider a 4-coloring $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{v_4, v_5\}$, $V_2 = \{v_6, v_7\}$, $V_3 = \{v_8, v_9\}$. Since V_1, V_2, V_3 are independent sets, it follows from $G \to (2,2,2,4)$ that V_4 contains a 4-clique L. It is clear that v_1, $v_3 \in L$. From $a \notin N(v_1)$ it follows that $a \notin L$. Therefore $d \in L$ and $L = \{v_1, v_3, b, d\}$. Similarly $\{v_1, v_8, b, d\}$ is a 4-clique. Therefore, $\{v_1, v_3, v_8, b, d\}$ is a 5-clique. This, together with (7) and $\mathrm{cl}(G) < 6$, implies that the vertex d is not adjacent to vertices v_5 and v_6, contradicting equality (6).

Subcase 2.c. $E(G_3) = \{[a,c],[b,d],[a,d],[c,b]\}$. As in the previous two subcases, we may assume that a and v_1 are not adjacent and also that (7) holds. Consider a 4-coloring $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = \{v_4, v_5\}$, $V_2 = \{v_6, v_7\}$, $V_3 = \{a, b\}$. V_1, V_2, V_3 are independent sets, which implies that V_4 contains a 4-clique L. Since $\{v_1, v_3, v_8\}$ is the unique 3-clique containing in $V_4 - \{c, d\}$, either $L = \{v_1, v_3, v_8, c\}$ or $L = \{v_1, v_3, v_8, d\}$. If $L = \{v_1, v_3, v_8, c\}$, then from (7) and $\mathrm{cl}(G) = 5$ it follows that the vertex c is not adjacent to vertices v_5 and v_6, which contradicts (6). The case $L = \{v_1, v_3, v_8, d\}$ similarly leads to a contradiction. The Theorem D is proved.

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