

# Dynamic Cage Survey

Geoffrey Exoo

Department of Mathematics and Computer Science  
Indiana State University  
Terre Haute, IN 47809  
ge@cs.indstate.edu

Robert Jajcay

Department of Mathematics and Computer Science  
Indiana State University  
Terre Haute, IN 47809  
robert.jajcay@indstate.edu

Department of Algebra  
Comenius University  
Bratislava, Slovakia  
robert.jajcay@fmph.uniba.sk

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## Abstract

A  $(k, g)$ -cage is a  $k$ -regular graph of girth  $g$  of minimum order. In this survey, we present the results of over 50 years of searches for cages. We present the important theorems, list all the known cages, compile tables of current record holders, and describe in some detail most of the relevant constructions.

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# 1 Origins of the Problem

The cage problem asks for the construction of regular simple graphs with specified degree and girth and minimum order. This problem was first considered by Tutte [107]. A variation of the problem in which the graphs were required to be Hamiltonian was later studied by Kártész [71]. At about the same time, the study of Moore graphs, first proposed by E. F. Moore, was developed by A. J. Hoffman and R. R. Singleton [63].

Their study begins with the observation that a regular graph of degree  $k$  and diameter  $d$  has at most

$$1 + k + k(k-1) + \cdots + k(k-1)^{d-1} \quad (1)$$

vertices, and graphs that achieve this bound must have girth  $g = 2d + 1$ .

One can turn this around and make a similar observation regarding the order,  $n$ , of a regular graph with degree  $k$  and girth  $g$ . Such a graph is called a  $(k, g)$ -graph. The precise form of the bound on the order  $n$  of a  $(k, g)$ -graph depends on the parity of  $g$ :

$$n \geq \begin{cases} 1 + \sum_{i=0}^{(g-3)/2} k(k-1)^i = \frac{k(k-1)^{(g-1)/2-2}}{k-2}, & g \text{ odd} \\ 2 \sum_{i=0}^{(g-2)/2} (k-1)^i = \frac{2(k-1)^{g/2-2}}{k-2}, & g \text{ even} \end{cases} \quad (2)$$

which is obtained by considering the vertices whose distance from a given vertex (edge) is at most  $\lfloor (g-1)/2 \rfloor$ .

The bound implied by (2) is called the *Moore bound*, and is denoted by  $M(k, g)$ . Graphs for which equality holds are called *Moore graphs*. Moore graphs are relatively rare.

**Theorem 1** ([14, 41]) *There exists a Moore graph of degree  $k$  and girth  $g$  if and only if*

- (i)  $k = 2$  and  $g \geq 3$ , *cycles*;
- (ii)  $g = 3$  and  $k \geq 2$ , *complete graphs*;
- (iii)  $g = 4$  and  $k \geq 2$ , *complete bipartite graphs*;
- (iv)  $g = 5$  and:

$k = 2$ , *the 5-cycle*,  
 $k = 3$ , *the Petersen graph*,  
 $k = 7$ , *the Hoffman-Singleton graph*,  
*and possibly  $k = 57$ ;*

- (v)  $g = 6, 8$ , or  $12$ , *and there exists a symmetric generalized  $n$ -gon of order  $k - 1$  (see 2.2).*

Regarding  $(v)$ , it should be noted that the only known symmetric generalized  $n$ -gons have prime power order.

The problem of the existence of Moore graphs is closely related to the *degree/diameter problem* surveyed in [87] (which also contains a further discussion of the history of the above theorem).

As Moore graphs do not exist for all parameters, one is naturally led to consider the more general problem of determining the minimum order of  $(k, g)$ -graphs. We denote this minimum value by  $n(k, g)$  and refer to a graph that achieves this minimum as a  $(k, g)$ -cage.

In cases where the order of the  $(k, g)$ -cage is not known, we denote the order of the smallest known  $k$ -regular graph of girth  $g$  by  $rec(k, g)$  (the current record holder).

The existence of a  $(k, g)$ -cage for any pair of parameters  $(k, g)$  is not immediately obvious, and it was first shown by Sachs [101]. Almost immediately thereafter, Sachs' upper bound was improved by Erdős, who proved the following theorem in a joint paper with Sachs [44].

**Theorem 2 ([44])** *For every  $k \geq 2$ ,  $g \geq 3$ ,*

$$n(k, g) \leq 4 \sum_{t=1}^{g-2} (k-1)^t$$

The proof in [44] follows from the stronger assertion of the existence of a  $k$ -regular graph of girth at least  $g$  and order  $2m$  for every  $m \geq 2 \sum_{t=1}^{g-2} (k-1)^t$ . This claim is proved using an interesting induction argument that we present in full in Appendix C.

Curiously enough, the two papers [44, 101] contain two substantially different existence proofs, while cross-referencing each other. The proof in [44] attributed to Erdős is non-constructive. Sachs' proof in [101], on the other hand, is constructive and uses recursion on the degree. It so happened that the joint paper [44] somehow received more attention, and the natural recursive construction of Sachs was mostly forgotten.

To this day, Sachs' result is the only constructive proof of the existence of  $(k, g)$ -graphs for any set of parameters  $k \geq 2, g \geq 3$  that is completely non-algebraic.

An algebraic proof of the existence of  $(k, g)$ -cages can be found in Biggs [20, 21]. In Section A.2 we provide a proof that is a generalization of Biggs' proof in [23].

**Theorem 3 ([23])** *For every  $k \geq 3$ ,  $g \geq 3$ , there is  $k$ -regular graph  $G$  whose girth is at least  $g$ .*

The existence of  $k$ -regular graphs of girth precisely  $g$ , for any  $k, g \geq 3$ , follows from the above theorem and the following theorem of Sachs [44].

**Theorem 4 ([44])** *Let  $G$  have the minimum number of vertices for a  $k$ -regular graph with girth at least  $g$ . Then the girth of  $G$  is exactly  $g$ .*

This implies that  $n(k, g)$  increases monotonically with  $g$ . In this form, the result was also proved later in [64] and [53].

Another early result of Sachs tying together consecutive odd and even girths appears in [44] and asserts the following:

**Theorem 5 ([44])** *For every  $k \geq 3$ , and odd  $g \geq 3$ ,*

$$n(k, g + 1) \leq 2n(k, g)$$

The easiest proof of this result takes advantage of the voltage graph construction we describe in Section A.1. This bound has recently been improved by Balbuena, González-Moreno and Montellano-Ballesteros:

**Theorem 6 ([12])** *Let  $k \geq 2$  and  $g \geq 5$ , with  $g$  odd. Then*

$$n(k, g + 1) \leq \begin{cases} 2n(k, g) - 2 \left( \frac{k(k-1)^{(g-3)/4-2}}{k-2} \right), & g \equiv 3 \pmod{4} \\ 2n(k, g) - 4 \left( \frac{(k-1)^{(g-1)/4-1}}{k-2} \right), & \text{otherwise.} \end{cases}$$

The upper bound from Theorem 2 was further improved by Sauer [102]. His bound is now commonly referred to as the *Sauer bound* and is given in the following theorem. Sauer's proof is constructive and a part of it is similar to the original proof of Erdős. Once again, he constructs a graph of degree  $k$ , girth  $g$ , and order  $m$ , for every  $m$  greater than or equal to his bound.

**Theorem 7 ([102])** *For every  $k \geq 2$ ,  $g \geq 3$ ,*

$$n(k, g) \leq \begin{cases} 2(k-2)^{g-2}, & g \text{ odd,} \\ 4(k-1)^{g-3}, & g \text{ even.} \end{cases}$$

The series of papers [102] also contains the following bounds on the order of trivalent cages and two further monotonicity results.

**Theorem 8 ([102])** *For every  $g \geq 3$ ,*

$$n(3, g) \leq \begin{cases} \frac{29}{12}2^{g-2} + \frac{2}{3}, & g \text{ odd,} \\ \frac{29}{12}2^{g-2} + \frac{4}{3}, & g \text{ even.} \end{cases}$$

**Theorem 9 ([102])** *For every  $k \geq 2$ ,  $g \geq 3$ ,*

$$n(k, g) < n(k, g + 1),$$

*and, for even  $k$ ,*

$$n(k, g) \leq n(k + 2, g).$$

The first inequality of Theorem 9 is proved by sharpening Theorem 4. The proof of the second inequality takes advantage of the existence of 2-factors for regular graphs of even degree.

To avoid trivialities, henceforth we will assume that the graphs under consideration have degree at least 3 and girth at least 5.

## 2 Known Cages

Recall that unless there exists a Moore graph, we know that  $n(k, g)$  is strictly greater than the Moore bound. Thus, in order to prove that a specific graph is a  $(k, g)$ -cage, the non-existence of a smaller  $(k, g)$ -graph has to be established. These lower bound proofs are in general very difficult, and consequently, in addition to the Moore graphs, very few cages are known.

In this section, we describe all the known cages. These include three infinite families of geometric graphs, and a finite number of small examples. The latter group includes cages of degree 3 for girths up to 12, cages of girth 5 for degrees up to 7, the  $(7, 6)$ -cage, and the  $(4, 7)$ -cage.

### 2.1 Small Examples

The case of  $k = 3$  has received the most attention, and the value of  $n(3, g)$  is known for all  $g$  up to 12. These values are given in Table 1.

The  $(3, 5)$ ,  $(3, 6)$ ,  $(3, 8)$ , and  $(3, 12)$ -cages are Moore graphs. Showing that the remaining cases in the table are indeed cages requires additional arguments:

There is no Moore graph of girth 7, where the Moore bound is 22, so the lower bound of  $n(3, 7) \geq 24$  follows immediately. The proof for girth 10 was computer assisted [91], while the proofs for girth 9 in [32] and girth 11 in [84] involved extensive computer searches.

$g$	5	6	7	8	9	10	11	12
$n(3, g)$	10	14	24	30	58	70	112	126
number of cages	1	1	1	1	18	3	1	1

**Table 1.** Known trivalent cages.

The cages for girth five are known for degrees up to 7 and are listed in Table 2. The  $(3, 5)$ -cage, the Petersen graph, and the  $(7, 5)$ -cage, the Hoffman-Singleton graph, are Moore graphs. The remaining cases were resolved by a combination of counting arguments and case analysis [99, 89, 111].

$k$	3	4	5	6	7
$n(k, 5)$	10	19	30	40	50
number of cages	1	1	4	1	1

**Table 2.** Known cages of girth 5.

The case of the  $(7, 6)$ -cage was settled in [90], and the value of  $n(4, 7)$  was recently determined in [50].

Next, we provide brief descriptions of the small cages.

### 2.1.1 (3,5)-Cage: Petersen Graph

The Petersen graph [94] is the  $(3, 5)$ -cage and has order 10. It can be constructed as the complement of the line graph of  $K_5$ , from which it follows that the automorphism group is isomorphic to  $Sym(5)$ . It is vertex-transitive, edge-transitive, 3-connected, neither planar nor Hamiltonian, and is the subject of an entire book [64]. It is shown in Figure 1.

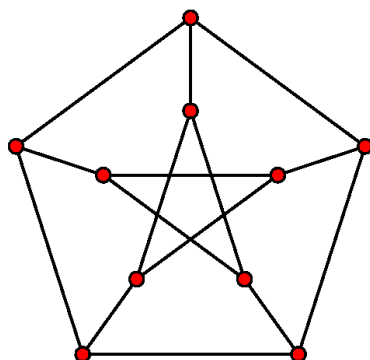


Figure 1: The Petersen graph

### 2.1.2 (3,6)-Cage: Heawood Graph

The Heawood graph is the  $(3, 6)$ -cage and has order 14. It is the point-line incidence graph of the projective plane of order 2 (see 2.2.1). It is vertex-transitive, edge-transitive, and the full automorphism group has order 336 (and is isomorphic to  $PGL(2, 7)$ ). The usual drawing is shown in Figure 2.

### 2.1.3 (3,7)-Cage: McGee Graph

The McGee graph is the  $(3, 7)$ -cage and has order 24. It is the first trivalent cage that is not a Moore graph. Its order exceeds the Moore bound by two. It is also the smallest of

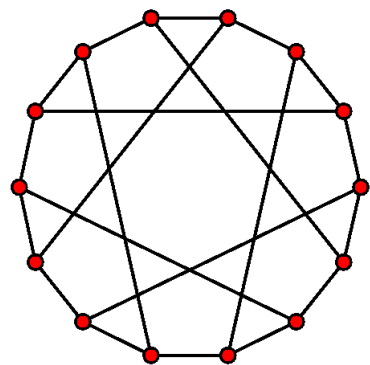


Figure 2: The Heawood graph



the trivalent cages that is not vertex-transitive. There are two vertex orbits of lengths 8 and 16. The full automorphism group has order 32. The standard drawing is shown in Figure 3. In the figure, the vertices colored red are in one orbit and the vertices in green in the other.

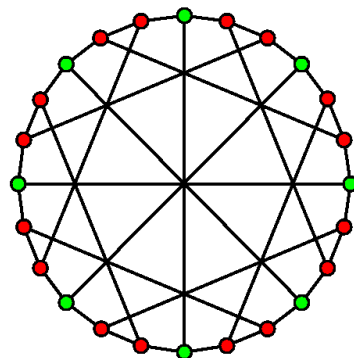


Figure 3: The McGee graph

#### 2.1.4 (3,8)-Cage: Tutte-Coxeter Graph

The Tutte-Coxeter graph (sometimes called Tutte's cage) is the  $(3, 8)$ -cage and has order 30. It is the point-line incidence graph of the generalized quadrangle of order 2 (see 2.2.2). It is vertex-transitive and 4-arc transitive. The full automorphism group has order 1440. The graph is shown in Figure 4.

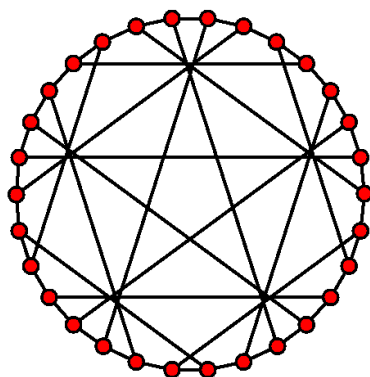


Figure 4: The Tutte-Coxeter graph

#### 2.1.5 (3,9)-Cages

There are 18 different  $(3, 9)$ -cages, each of order 58. The first of these graphs was discovered by Biggs and Hoare [25]. The list of 18 graphs was shown to be complete by Brinkmann, McKay and Saager in 1995 [32]. The orders of the automorphism groups range from 1 to 24.

### 2.1.6 (3,10)-Cages

There are three (3,10)-cages of order 70. The first of these was discovered by Balaban [11]. The other two were found by O’Keefe and Wong [91]. The completeness of the set was established by Wong [115] and, in another context, by McKay [83]. None of the graphs are vertex-transitive. The orders of the automorphism groups are 24, 80, and 120.

### 2.1.7 (3,11)-Cage: Balaban Graph

A (3,11)-graph on 112 vertices was first constructed by Balaban [10] in 1973. It can be obtained from the (3,12)-cage by excision (see 4.1.4). The graph was shown to be the unique cage by McKay, Myrvold and Nadon [84]. It is not vertex-transitive, and its automorphism group has order 64.

### 2.1.8 (3,12)-Cage: Benson Graph

A (3,12)-graph on 126 vertices was first constructed by Benson [16] in 1966. It is the incidence graph of the generalized hexagon of order 2 (see 2.2.3). The graph is vertex-transitive and edge-transitive. Its automorphism group has order 12096 and is a  $\mathbb{Z}_2$  extension of  $PSU(3, 3)$ .

### 2.1.9 (4,5)-Cage: Robertson Graph

The Robertson graph is the unique (4,5)-cage of order 19 (see [99]). It is not vertex-transitive and the full automorphism group is isomorphic to the dihedral group of order 24. It is shown in Figure 5, wherein the three colored vertices on the right are adjacent to the four vertices on the 12-cycle with the corresponding color.

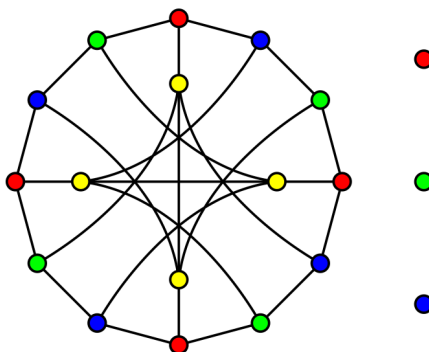


Figure 5: The Robertson graph

### 2.1.10 (5,5)-Cages

The four (5,5)-cages have order 30 [71, 34]. Their automorphism groups have orders 20, 30, 96, and 120. The first of these is a subgraph of the Hoffman-Singleton graph (see

2.1.12). The last is known as the Robertson-Wegner graph [111]. It can be constructed as follows.

Begin with a regular dodecahedron,  $D$ . The vertices of  $D$  determine five (regular) cubes. Furthermore, each of these cubes determines two regular tetrahedra. The vertices of the Robertson-Wegner graph are the 20 vertices of  $D$ , plus one vertex for each of the 10 tetrahedra. Each of the tetrahedral vertices is adjacent to its four determining vertices. In addition, two tetrahedral vertices are adjacent if they are contained in the same cube.

### 2.1.11 (6,5)-Cage

The  $(6, 5)$ -cage is unique and has order 40. It was first presented in [89] and was proved to be minimal in [113]. It can be constructed by removing the vertices of a Petersen graph from the Hoffman-Singleton graph (see 2.1.12). It is vertex-transitive with an automorphism group of order 480.

### 2.1.12 (7,5)-Cage: Hoffman-Singleton Graph

The Hoffman-Singleton graph is the unique  $(7, 5)$ -cage [63]. The graph was first considered at least as long ago as 1956 by Mesner [86] (further details can be found in [68, 104]). There are several constructions known (see, for example, [58]).

The standard construction, Robertson's *pentagons and pentagrams* [17], begins with five pentagons  $P_i$  and five pentagrams  $Q_j$ ,  $0 \leq i, j \leq 4$ , obtained by labeling the vertices so that vertex  $k$  of  $P_i$  is adjacent to vertices  $k - 1$  and  $k + 1$  of  $P_i$  and vertex  $k$  of  $Q_j$  is adjacent to vertices  $k - 2$  and  $k + 2$  of  $Q_j$  (all subscript arithmetic is done modulo 5). The graph is completed by joining vertex  $k$  of  $P_i$  to vertex  $ij + k$  of  $Q_j$  (so that each  $P_i$  together with each  $Q_j$  induce a Petersen graph).

One can obtain the  $(6, 5)$ -cage and one of the  $(5, 5)$ -cages from this construction. To get the  $(6, 5)$ -cage, simply delete a pentagon and the corresponding pentagram. To obtain a  $(5, 5)$ -cage delete two of the pentagons and the corresponding pentagrams.

The graph is vertex-transitive and edge-transitive. Its full automorphism group has order 252000 and is isomorphic to a  $\mathbb{Z}_2$  extension of  $PSU(3, 5)$ .

### 2.1.13 (7,6)-Cage

This case was settled by O'Keefe and Wong [90], who showed that  $n(7, 6) = 90$ . The cage is the incidence graph of an elliptic semiplane discovered some years earlier by Baker [9]. The graph is vertex-transitive and its full automorphism group has order 15120.

### 2.1.14 (4,7)-Cage

Recently, Exoo, McKay, Myrvold and Nadon [50] showed that  $n(4, 7) = 67$ . They exhibited one  $(4, 7)$ -cage on 67 vertices whose automorphism group has order 4. It is unknown whether other  $(4, 7)$ -cages exist.

## 2.2 Geometric Graphs

Geometric graphs are based on generalized polygons whose incidence graphs form three infinite families of cages (girths 6, 8 and 12).

We begin with the definition of a generalized polygon (or  $n$ -gon). Let  $P$  (the set of points) and  $B$  (the set of lines) be disjoint non-empty sets, and let  $I$  (the point-line incidence relation) be a subset of  $P \times B$ . Let  $\mathcal{I} = (P, B, I)$ , and let  $G(\mathcal{I})$  be the associated bipartite *incidence graph* on  $P \cup B$  with edges joining the points from  $P$  to their incident lines in  $B$  ( $p \in P$  is adjacent to  $\ell \in B$  whenever  $(p, \ell) \in I$ ).

The ordered triple  $(P, B, I)$  is said to be a *generalized  $n$ -gon* subject to the following four regularity conditions:

*GP1: There exist  $s \geq 1$  and  $t \geq 1$  such that every line is incident to exactly  $s + 1$  points and every point is incident to exactly  $t + 1$  lines.*

*GP2: Any two distinct lines intersect in at most one point and there is at most one line through any two distinct points.*

*GP3: The diameter of the incidence graph  $G(\mathcal{I})$  is  $n$ .*

*GP4: The girth of  $G(\mathcal{I})$  is  $2n$ .*

While the trivial case  $s = t = 1$  leads to two-dimensional polygons, a well-known result of Feit and Higman [52] asserts that if both  $s$  and  $t$  are integers larger than 1, then  $n$  equals 2, 3, 4, 6 or 8; with the parameters 3, 4, 6 and 8 corresponding to the *projective planes*, *generalized quadrangles*, *generalized hexagons*, and *generalized octagons*, respectively. Note that the incidence graphs of generalized octagons are not regular, and so they cannot be cages.

### 2.2.1 The Incidence Graphs of Projective Planes

As mentioned above, finite projective planes are generalized triangles (or 3-gons). In this case,  $s = t$ , and projective planes are known to exist whenever the order  $s$  is a prime power  $q = p^k$ . If  $s$  is not a prime power,  $s \equiv 1, 2 \pmod{4}$ , and  $s$  is not the sum of two integer squares, then no plane exists [33]. The first case not covered by the above is the case  $n = 10$ , for which it has been shown [73] that no plane exists. All remaining cases are unsettled.

A finite projective plane of order  $q$  has  $q^2 + q + 1$  points and  $q^2 + q + 1$  lines, and satisfies the following properties.

*PP1: Any two points determine a line.*

*PP2: Any two lines determine a point.*

*PP3: Every point is incident with  $q + 1$  lines.*

*PP4: Every line is incident with  $q + 1$  points.*

The incidence graph of a projective plane of order  $q$  is regular of degree  $q + 1$ , has  $2(q^2 + q + 1)$  vertices, diameter 3, and girth 6. Since the Moore bound for degree  $q + 1$  and girth 6 is equal to the orders of these graphs, the incidence graphs of projective planes are  $(q + 1, 6)$ -cages. For example, the  $(3, 6)$ -cage Heawood graph (see 2.1.2) is the incidence graph of the plane of order 2. The plane is shown in Figure 6.

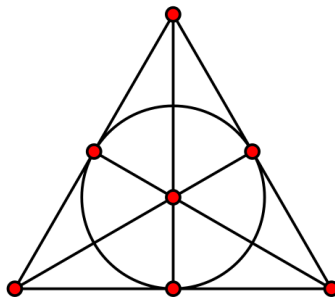


Figure 6: The projective plane of order 2.

### 2.2.2 The Incidence Graphs of Generalized Quadrangles

A generalized quadrangle is an incidence structure with  $s + 1$  points on each line,  $t + 1$  lines through each point. It is said to have order  $(s, t)$ . A generalized quadrangle of order  $(s, t)$  has  $(s + 1)(st + 1)$  points and  $(t + 1)(st + 1)$  lines, and has the following properties.

*GQ1: Any two points lie on at most one line.*

*GQ2: Any two lines intersect in at most one point.*

*GQ3: Every line is incident with  $s + 1$  points.*

*GQ4: Every point is incident with  $t + 1$  lines.*

*GQ5: For any point  $p \in P$  and line  $\ell \in B$ , where  $(p, \ell) \notin I$ , there is exactly one line incident with  $p$  and intersecting  $\ell$ .*

The incidence graph of a generalized quadrangle of order  $(q, q)$  has  $2(q + 1)(q^2 + 1)$  vertices and is regular of degree  $q + 1$ , diameter 4, and girth 8. The orders of these graphs match the Moore bound for degree  $q + 1$  and girth 8, and are therefore cages. Graphs with these parameters are known to exist whenever  $q$  is a prime power. For example, the  $(3, 8)$ -cage Tutte-Coxeter graph (see 2.1.4) is the incidence graph of the generalized quadrangle of order  $(2, 2)$  shown in Figure 7. The 15 lines of the quadrangle are represented by the five sides of the pentagon, the five diagonals, and the five partial circles.

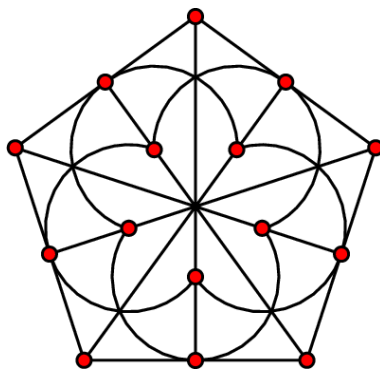


Figure 7: The generalized quadrangle of order  $(2,2)$ .

### 2.2.3 The Incidence Graphs of Generalized Hexagons

Generalized hexagons satisfy the following conditions.

*GH1: Any two points lie on at most one line.*

*GH2: Any two lines intersect in at most one point.*

*GH3: Every line is incident with  $s + 1$  points.*

*GH4: Every point is incident with  $t + 1$  lines.*

*GH5: For any point  $p \in P$  and line  $\ell \in B$ , where  $(p, \ell) \notin I$ , there is a unique shortest path from  $p$  to  $\ell$  of length 3 or 5.*

Once again, the incidence graph of a generalized hexagon of order  $(q, q)$  is regular of degree  $q + 1$ , diameter 6, and girth 12. The order of every such graph matches the Moore bound, which in this case is  $2(q^3 + 1)(q^2 + q + 1)$ . Graphs with these parameters are known to exist whenever  $q$  is a prime power. The  $(3, 12)$ -cage Benson graph (see 2.1.8) is the incidence graph of the generalized hexagon of order  $(2, 2)$ .

## 3 Lower Bounds

Outside of the cases where Theorem 1 asserts the existence of a Moore graph, the obvious lower bound for the order of a  $(k, g)$ -cage is the value of the Moore bound plus one,  $M(k, g) + 1$ , when  $k$  is even, and the value of the Moore bound plus two,  $M(k, g) + 2$ , when  $k$  is odd.

These improved lower bounds do not differ significantly from the Moore bound,  $M(k, g)$ . However the Moore bound values are widely believed to be well below the actual orders  $n(k, g)$  of the  $(k, g)$ -cages. An inspection of the lists of the smallest known  $(k, g)$ -graphs included in Section 4 suggests a significant gap between the orders of the best known graphs and the corresponding Moore bounds. The difference  $n(k, g) - M(k, g)$  is a closely

studied quantity, usually denoted by  $\epsilon(k, g)$ , and given the name *excess*. Despite any evidence that one may find in the tables, the order of magnitude of the excess  $\epsilon(k, g)$  is an open problem. While the best general lower bounds (listed in the following paragraphs) add at most a very small constant to the Moore bound, computational evidence suggests the existence of parameters for which the excess might in fact be significantly larger than the Moore bound.

The only theoretical lower bound that provides at least some evidence as to the nature of the growth of the excess is due to Biggs and deals exclusively with vertex-transitive graphs.

**Theorem 10 ([19])** *For each odd integer  $k \geq 3$  there is an infinite sequence of values of  $g$  such that the excess  $e$  of any vertex-transitive graph with valency  $k$  and girth  $g$  satisfies  $e > g/k$ .*

As the orders of the smallest vertex-transitive  $(k, g)$ -graphs often significantly differ from the orders of the corresponding  $(k, g)$ -cages (as evidenced in Table 7), the applicability of the above result with regard to general  $(k, g)$ -cages remains unclear. In the next paragraphs, we list all the known lower bounds on  $n(k, g)$ .

In [36], Brown showed that  $n(k, 5)$  is never equal to  $M(k, 5) + 1$ . This was further improved by Kovács [72], who showed that  $n(k, 5)$  is not equal to  $M(k, 5) + 2$  when  $k$  is odd and cannot be written in the form  $\ell^2 + \ell - 1$ , for  $\ell$  an integer. Eroh and Schwenk [45] also showed that  $n(k, 5)$  is not equal to  $M(k, 5) + 2$  for  $5 \leq k \leq 11$ . For girth 7, Eroh and Schwenk [45] showed the non-existence of  $k$ -regular graphs of girth 7 and order  $M(k, 7) + 1$ , and finally Bannai and Ito [15] showed the non-existence of  $k$ -regular graphs of odd girth  $g \geq 5$  and order  $M(k, g) + 1$  for all degrees  $k \geq 3$ . Note that in this case, the McGee graph 2.1.3 achieves the lower bound  $M(3, 7) + 2$ , hence is a cage. The only other known cage for girth 7 is the  $(4, 7)$ -cage [50] of order  $67 = M(4, 7) + 14$ .

All cages of even girth have been conjectured to be bipartite by Wong in [114]. If the conjecture were true,  $n(k, g) > M(k, g) + 1$  would automatically follow for all  $k \geq 3$  and even  $g \geq 4$  for which there does not exist a Moore graph. The following result of Biggs and Ito proves Wong's conjecture under the following very special circumstances:

**Theorem 11 ([27])** *Let  $G$  be a  $(k, g)$ -cage of girth  $g = 2m \geq 6$  and excess  $\epsilon$ . If  $\epsilon \leq k - 2$ , then  $G$  is bipartite and its diameter is  $m + 1$ .*

Consequently,  $n(k, g) \geq M(k, g) + 2$  for all  $k \geq 3$  and even  $g \geq 6$  for which there does not exist a Moore graph. In the very same paper the authors further strengthen this result as follows:

**Theorem 12 ([27])** *Let  $G$  be a  $(k, g)$ -cage of girth  $g = 2m \geq 6$  and excess 2. Then  $g = 6$ ,  $G$  is a double-cover of the incidence graph of a symmetric  $(v, k, 2)$ -design  $D(k, 2)$ , and  $k$  is not congruent to 5 or 7 (mod 8).*

In all other cases,  $n(k, g) \geq M(k, g) + 3$ . The symmetric design  $D(k, 2)$  is also called a biplane. The authors observe in their abstract "it is a remarkable fact that we get a

double-cover of  $D(k, 2)$  in the case when  $\epsilon = 2$ , whereas we get the incidence graph of a projective plane in the case  $\epsilon = 0$  and  $g = 6$ .”

The cases  $n(5, 5)$ ,  $n(6, 5)$ ,  $n(7, 6)$ , and  $n(3, 10)$  were resolved by a combination of counting arguments and case analysis [99, 89, 90, 91, 111].

All the remaining improvements on the lower bounds for cages are based on computer searches (see B). There are three cases where extensive computer searches produced the correct lower bound (and the cages are known). Namely, the cases  $n(3, 9)$ ,  $n(3, 11)$ , and  $n(4, 7)$ , [32], [84], and [50], respectively. In the case  $n(3, 13)$ , the lower bound was improved to  $202 = M(3, 13) + 12$  [84], and in the case  $n(3, 14)$  the lower bound was improved to  $258 = M(3, 14) + 4$  [84].

## 4 Upper Bounds

In the preceding sections, we have listed and described all the currently known cages; graphs whose orders  $n(k, g)$  are provably the smallest, and as such, will permanently stay on the list.

In what follows, we list graphs whose orders, denoted by  $rec(k, g)$ , are the smallest currently known. Although some of these graphs may actually be cages, the majority will most likely be eventually replaced by smaller graphs.

We adopt a somewhat arbitrary division of the current record holders into two groups:

*General constructions* – constructions that produce graphs with arbitrarily large values of girth or degree (Section 4.1).

*Individual constructions* – constructions that work for specific values of girth and degree, and may have been introduced for other purposes, or have been found by the use of computers (Section 4.2).

In the next section, we present the general constructions that have produced the best known asymptotic bounds on  $n(k, g)$ . These are followed by constructions that are useful for only a limited number of specific values of  $n(k, g)$ .

### 4.1 General Constructions

#### 4.1.1 Constructions for Large Girth

In this section we describe constructions for regular graphs with arbitrarily large girth. Included are the construction of Sachs [101], the trivalent sextet, hexagon and triplet graphs [26, 62], as well as the higher degree constructions of Lubotzky, Phillips and Sarnak [76], and Lazebnik, Ustimenko and Woldar [74]. We also discuss the techniques of Chandran [38] and of Bray, Parker and Rowley [30].

Biggs observes in [23] that the Moore bound implies that minimal  $k$ -regular graphs of girth  $g$  have approximately  $(k - 1)^{g/2}$  vertices. Thus, when considering infinite families



of  $k$ -regular graphs  $\{G_i\}$  of increasing girth  $g_i$ , we compare their orders  $v_i$  to the Moore bound. We say that  $\{G_i\}$  is a *family with large girth* if there exists  $\gamma > 0$  such that

$$g_i \geq \gamma \log_{k-1}(v_i).$$

It follows from the Moore bound that  $\gamma$  is at most 2, but there are no known families with  $\gamma$  close to 2. The results of Erdős, Sachs [44] and Sauer [102] showed the existence of infinite families with  $\gamma = 1$ . The first explicit constructions go back to Margulis [80] who achieved  $\gamma = \frac{4}{9} (\approx 0.44)$  for some infinite families with arbitrary large degree and  $\gamma \approx 0.83$  for degree 4. These were followed by the results of Imrich [65], who produced infinite families of large degree with  $\gamma \approx 0.48$  and a family of trivalent graphs with  $\gamma \approx 0.96$ . The (trivalent) *sextet graphs* of Biggs and Hoare were shown to satisfy  $\gamma \geq 4/3$  by Weiss [112], and the *Ramanujan graphs* of Lubotzky, Phillips and Sarnak [76] were shown to satisfy  $\gamma \geq 4/3$  (with arbitrary large degree) by Biggs and Boshier [24]. The (current) best results (for arbitrary large degree) are due to Lazebnik, Ustimenko and Woldar [74] who have constructed infinite families  $CD(n, q)$  with  $\gamma \geq 4/3 \log_q(q-1)$ ,  $q$  a power of a prime.

### Construction I. Sachs

A truncation of a map is a well-known construction from topological graph theory in which the vertices of the original map are replaced by cycles attached to the dangling edges of the removed vertices. The original construction of Sachs from [101] is a special case of a truncation construction. We present a slight generalization of his construction.

Let  $G$  be a finite  $k$ -regular graph, and let  $D(G)$  denote the set of *darts* of  $G$  obtained by associating each edge of  $G$  with two opposing directed edges. A *vertex-neighborhood labeling* of  $G$  is a function  $\rho$  from the set  $D(G)$  into the set  $\{1, 2, \dots, k\}$  that maps the darts emanating from a vertex  $v \in V(G)$  bijectively onto  $\{1, 2, \dots, k\}$ . Let  $H$  be a graph of order  $k$ ,  $V(H) = \{u_1, u_2, \dots, u_k\}$ . The *generalized truncation of a  $k$ -regular graph  $G$  with a vertex-neighborhood labeling  $\rho$  by the graph  $H$*  (of order  $k$ ) is the graph  $T(G, \rho; H)$  obtained from  $G$  by replacing the vertices of  $G$  by copies of  $H$  as follows: each vertex  $v$  of  $G$  is replaced by the graph  $H$  attached to the dangling darts originally emanating from  $v$  according to the rule that  $u_i$  is attached to the dart labeled by  $i$ . See Figure 8.

The importance of the generalized truncation for the construction of  $(k, g)$ -graphs becomes clear from the following theorem.

**Theorem 13 ([28])** *Let  $G$  be a finite  $(k, g)$ -graph with a vertex-neighborhood labeling  $\rho$ , and let  $H$  be a  $(k', g')$ -graph of order  $k$ . The generalized truncation graph  $T(G, \rho; H)$  is a  $(k' + 1)$ -regular graph of girth not smaller than  $\min\{2g, g'\}$ , and if  $g' \leq 2g$ , then  $g'$  is the exact girth of  $T(G, \rho; H)$ .*

The proof of this theorem relies on a careful inspection of the effects of the truncation on the original cycles of  $G$ .

As mentioned in our introduction, the first proof of the existence of graphs for any pair of parameters  $k$  and  $g$  can be found in [101]. The construction used in this proof is recursive and based on generalized truncation. It proceeds as follows.

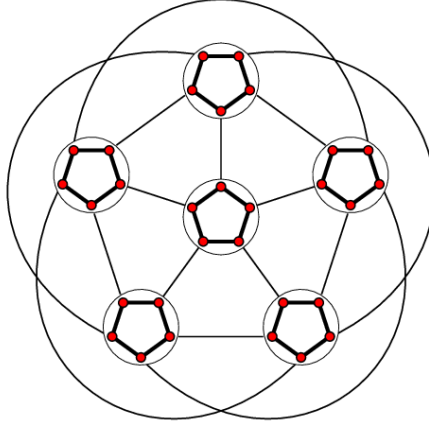


Figure 8: A generalized truncation of  $K_6$  by  $C_5$ .

**Theorem 14 ([101])** *For every pair of parameters  $k \geq 2$  and  $g \geq 3$ , there exists a finite  $k$ -regular graph of girth  $g$ .*

**Proof.** We proceed by induction on  $k$ . If  $k = 2$ , the  $g$ -cycle  $C_g$  is a  $(2, g)$ -cage for all  $g \geq 3$ .

For the induction step, assume the existence of a  $(k, g)$ -graph  $H$  for some  $k \geq 2$  and  $g \geq 3$  of order  $n$ . Let  $G$  be an  $n$ -regular graph of girth at least  $\lceil \frac{g}{2} \rceil$  guaranteed by the construction from Theorem 3, and let  $\rho$  be any vertex-neighborhood labeling of  $G$ . Theorem 13 asserts that the truncated graph  $T(G, \rho; H)$  is a  $(k+1)$ -regular graph of girth  $g$ .  $\square$

The construction of Sachs has more recently reappeared as the *zig-zag construction*, which has successfully been used in the context of expander graphs [97].

## Construction II. Sextet Graphs

Sextet graphs are trivalent graphs introduced by Biggs and Hoare in [26]. Let  $q$  be an odd prime power. A *duet*,  $ab$ , is any unordered pair of elements from  $\mathbb{F}_q \cup \{\infty\}$ , the points of the projective line  $PG(1, q)$ . A *quartet* is an unordered pair of duets,  $ab$  and  $cd$ , satisfying the equality

$$\frac{(a-c)(b-d)}{(a-d)(b-c)} = -1.$$

If one of the vertices is infinity, then  $\{\infty, b \mid c, d\}$  is a quartet if

$$\frac{(b-d)}{(b-c)} = -1.$$

A *sextet* is an unordered triple of duets such that every pair of duets from the triple forms a quartet.

Assume that  $q \equiv 1 \pmod{8}$ . Any quartet uniquely determines a sextet. The group  $PGL(2, q)$  of projective linear transformations of  $PG(1, q)$  preserves and acts transitively

on quartets. In addition, given a quartet  $\{a, b \mid c, d\}$  there is a unique involution in  $PGL(2, q)$  that interchanges  $a$  with  $c$  and  $b$  with  $d$ , and whose fixed points constitute a duet  $ef$  such that  $\{a, b \mid c, d \mid e, f\}$  is a sextet.

Next we define adjacency on the sextets. A sextet  $s = \{a, b \mid c, d \mid e, f\}$  is adjacent to three other sextets, each having a different duet in common with  $s$ . For example,  $s$  is adjacent to  $\{a, b \mid c', d' \mid e', f'\}$  where the duet  $c'd'$  is the pair of points fixed by the involution mapping  $c$  to  $e$  and  $d$  to  $f$ , and the duet  $e'f'$  is the pair of points fixed by the involution mapping  $c$  to  $f$  and  $d$  to  $e$ .

It can be shown that there exist exactly  $q(q^2 - 1)/24$  sextets. The set of all sextets under the above adjacency relation defines the trivalent graph  $\Sigma(q)$ . In general,  $\Sigma(q)$  is not connected.

If  $p$  is an odd prime, the *sextet graph*  $S(p)$  is any connected component of  $\Sigma(p)$ , for  $p \equiv 1 \pmod{8}$ , and its order is  $\frac{1}{48}p(p^2 - 1)$ .

The graph  $S(73)$  is the smallest known trivalent graph of girth 22.

### Construction III. Hexagons

Hexagon graphs are trivalent graphs introduced by Hoare in [62]. Let  $p$  be a prime,  $p > 3$ , and let  $q = p$  if  $p \equiv 1 \pmod{4}$  and  $q = p^2$  otherwise. Consider the complete graph  $K_{q+1}$  with vertices labeled by elements of  $\mathbb{F}_q \cup \{\infty\}$ , the points of the projective line  $PG(1, q)$ . The concepts of duet, quartet, and sextet are defined as above.

For any 6-cycle  $C$  in  $K_{q+1}$ , define a *short diagonal* to be a pair of vertices whose distance in  $C$  is 2, and a *long diagonal* to be a pair of vertices whose distance in  $C$  is 3. A 6-cycle  $C$  in  $K_{q+1}$  is a *hexagon* if any four of its vertices,  $v_1, v_2, v_3, v_4$ , such that  $v_1, v_2$  form a short diagonal and  $v_3, v_4$  form a long diagonal determine a quartet  $\{v_1, v_2 \mid v_3, v_4\}$ .

This set of hexagons is the vertex set of a trivalent graph denoted  $H(q)$ , wherein adjacency is defined as follows:

Let  $H = v_1, v_2, v_3, v_4, v_5, v_6$  be a hexagon. Each of its three long diagonals determines one neighbor of  $H$ . For example, the long diagonal  $v_1, v_4$  together with the short diagonals  $v_2, v_6$  and  $v_3, v_5$  determines unique sextets  $\{v_1, v_4 \mid v_2, v_6 \mid v_3, v_5\}$  and  $\{v_1, v_4 \mid v_3, v_5 \mid v_2, v_6\}$ , which in turn determine the adjacent hexagon  $v_1, v_7, v_9, v_4, v_8, v_{10}$ .

The graph  $H(47)$  is the smallest known trivalent graph of girth 19.

### Construction IV. Triplets

Let  $p$  be an odd prime. The vertex set of the trivalent *triplet graph*  $T(p)$  is the set of all 3-subsets of the points of  $PG(1, p)$ . Two 3-subsets  $\{a, b, c\}$  and  $\{a, b, d\}$  are adjacent if and only if  $\{a, b \mid c, d\}$  is a quartet.

It was shown in [62] that if  $p \equiv 1 \pmod{4}$  then  $T(p)$  has two connected components of size  $p(p^2 - 1)/12$ , and if  $p \equiv 3 \pmod{4}$  then  $T(p)$  is connected of order  $p(p^2 - 1)/6$ .

The girth of  $T(p)$  is greater than  $\log_\phi(p)$ , where  $\phi$  is the golden ratio  $(1 + \sqrt{5})/2$ . Finally, it may be interesting to note that the connected components of  $T(5)$  are Petersen graphs.

### Construction V. Lubotzky, Phillips, Sarnak

The graphs obtained from this construction are Cayley graphs (see A.2) of projective linear groups. They belong to the family of *Ramanujan graphs*, which are  $k$ -regular graphs whose second largest (in absolute value) eigenvalue  $\lambda_2$  satisfies the inequality  $\lambda_2 \leq 2\sqrt{k-1}$ .

Let  $p$  and  $q$  be distinct primes such that  $p, q \equiv 1 \pmod{4}$ , and let  $i$  be an integer satisfying  $i^2 \equiv -1 \pmod{q}$ . Then there are  $8(p+1)$  solutions  $\alpha = (a_0, a_1, a_2, a_3)$  satisfying  $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$ . Exactly  $p+1$  of these 4-tuples  $\alpha$  are such that their first coordinate  $a_0$  is positive and odd and the rest of the coordinates  $a_1, a_2, a_3$  are all even. Associate each such  $\alpha$  with the matrix  $\tilde{\alpha} \in PGL(2, q)$  defined by

$$\tilde{\alpha} = \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{pmatrix}.$$

Let  $\Lambda$  denote the set of the  $p+1$  matrices obtained in this way.

The graphs  $X^{p,q}$  are Cayley graphs of degree  $p+1$  defined in two different ways depending on the sign of the Legendre symbol  $\left(\frac{p}{q}\right)$ :

$$X^{p,q} = \begin{cases} C(PSL(2, q), \Lambda) & \text{if } \left(\frac{p}{q}\right) = 1, \\ C(PGL(2, q), \Lambda) & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$

The orders of the graphs are the orders of the linear groups, which are  $\frac{q(q^2-1)}{2}$  and  $q(q^2-1)$ , respectively. The latter graph is bipartite.

In order to state their result precisely, we need their concept of a *good integer*, which is one that cannot be expressed in the form  $4^\alpha(8\beta+7)$  for nonnegative integers  $\alpha, \beta$ .

### Theorem 15 ([76])

$$g(X^{p,q}) = \begin{cases} 2\lceil 2\log_p q \rceil & \text{if } p^{\lceil 2\log_p q \rceil} - q^2 \text{ is good,} \\ 2\lceil 2\log_p q + \log_p 2 \rceil & \text{otherwise.} \end{cases}$$

Hence, the girths of the resulting graphs are asymptotically  $\frac{4}{3}\log_p(n)$ , where  $n$  is the order of the graph [76].

### Construction VI. Lazebnik, Ustimenko, Woldar

Let  $q$  be a prime power, and let  $P$  (points) and  $L$  (lines) be two copies of the set of infinite sequences of elements from the finite field  $\mathbb{F}_q$ . We adopt the convention that the points in  $P$  will be denoted by

$$(p) = \{p_1, p_{1,1}, p_{1,2}, p_{2,1}, p_{2,2}, p'_{2,2}, p_{2,3}, \dots, p_{i,i}, p'_{i,i}, p_{i,i+1}, p_{i+1,i}, \dots\},$$

and the lines in  $L$  will be denoted by

$$[l] = \{l_1, l_{1,1}, l_{1,2}, l_{2,1}, l_{2,2}, l'_{2,2}, l_{2,3}, \dots, l_{i,i}, l'_{i,i}, l_{i,i+1}, l_{i+1,i}, \dots\}.$$

A point ( $p$ ) is incident to the line [ $l$ ] subject to the following:

$$\begin{array}{rcl}
l_{1,1} & - & p_{1,1} = l_1 p_1 \\
l_{1,2} & - & p_{1,2} = l_{1,1} p_1 \\
l_{2,1} & - & p_{2,1} = l_1 p_{1,1} \\
l_{i,i} & - & p_{i,i} = l_1 p_{i-1,i} \\
l'_{i,i} & - & p'_{i,i} = l_{i,i-1} p_1 \\
l_{i,i+1} & - & p_{i,i+1} = l_{i,i} p_1 \\
l_{i+1,i} & - & p_{i+1,i} = l_1 p'_{i,i},
\end{array}$$

with the last four equations defined for  $i \geq 2$ .

For each positive  $n \geq 2$ , the first  $n - 1$  equations define an incidence relation on  $P_n, L_n$ , two copies of  $(\mathbb{F}_q)^n$ , thought of as the projections of the infinite sequences from  $P$  and  $L$  onto their  $n$  first coordinates. The graph  $D(n, q)$  is the bipartite incidence graph corresponding to the incidence structure induced on  $P_n$  and  $L_n$ .

For all  $n > 1$ , the graphs  $D(n, q)$  are  $q$ -regular graphs of order  $2q^n$  and girth  $g \geq n + 4$  when  $n$  is even, and girth  $g \geq n + 5$  when  $n$  is odd. The automorphism groups of these graphs are transitive on points, lines, and edges. Moreover, for  $n \geq 6$ , the graphs  $D(n, q)$  are disconnected with all of their connectivity components  $CD(n, q)$  mutually isomorphic. Hence, for  $n \geq 6$  and  $q$  a prime power, the graphs  $CD(n, q)$  are bipartite, connected,  $q$ -regular graphs of order  $\leq 2q^{n-\lfloor \frac{n+2}{4} \rfloor + 1}$ , girth  $g \geq n + 4$ , and have point, line, and edge-transitive automorphism groups [74, 75]. The smallest non-trivial graph in their family is  $CD(2, 3)$ , also known as the Pappus graph, shown in Figure 9.

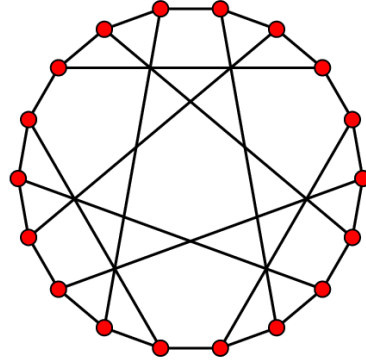


Figure 9: The Pappus graph

Using this construction, in conjunction with the constructions of Lubotzky, Phillips, and Sarnak [76], and of Füredi, Lazebnik, Seress, Ustimenko, and Woldar [54], one obtains the following bound.

**Theorem 16 ([75])** *Let  $k \geq 2$  and  $g \geq 5$  be integers, and let  $q$  denote the smallest odd prime power for which  $k \leq q$ . Then*

$$n(k, g) \leq 2kq^{\frac{3}{4}g-a}, \quad (2)$$

where  $a = 4, 11/4, 7/2, 13/4$  for  $g \equiv 0, 1, 2, 3 \pmod{4}$ , respectively.

**Construction VII. Chandran**

Chandran [38] devised a simple algorithm that constructs a graph of order  $n$  with degree set a subset of  $\{k-1, k, k+1\}$ . He begins with either a matching or a Hamiltonian cycle, and adds edges one at a time based on distance and degree. He is able to show that the resulting graph has  $\log(n)$  girth.

**Construction VIII. Bray, Parker, Rowley**

Historically, highly symmetric graphs (see A) repeatedly proved useful in constructions of relatively small (trivalent) graphs of specific girth. A number of the early best constructions were Cayley graphs (see A.2). The symmetry of Cayley graphs makes the girth computations more efficient than for asymmetric graphs of the same order. When working with Cayley graphs, it is important to choose groups in which the group operation can be computed quickly. Hence, groups which can be represented as groups of small matrices are a natural choice. Several of the early records constructed in this way can be found in [39].

Improvements on some of the records obtained using Cayley graphs were made by modifying Cayley graphs. Bray, Parker and Rowley [30] constructed a number of current record holders for degree three by factoring out the 3-cycles in trivalent Cayley graphs. Their construction starts with a trivalent Cayley graph,  $C(G, X)$ , subject to the condition that the generating set  $X$  contains an involution,  $\alpha$ , and two mutually inverse elements of order 3,  $\delta, \delta^{-1}$ , and that the Cayley graph has no cycles of length 4. The graph  $B(G, X)$  is then defined as follows: the vertex set  $\mathcal{T}$  of  $B(G, X)$  is the set of triangles of  $C(G, X)$  with triangle  $T_i$  adjacent to triangle  $T_j$  in  $B(G, X)$  if at least one of the vertices of  $T_i$  is adjacent in  $C(G, X)$  to at least one of the vertices of  $T_j$  via an edge labeled by the involution  $\alpha$ .

**4.1.2 Constructions for Girth 5**

Next we present a series of constructions that produce families of fixed girth 5. We attempt to provide a list that is as complete as possible, including older constructions that may still prove useful, and newer constructions whose efficiency is sometimes hard to compare.

The constructions below produce graphs whose orders are approximately twice the Moore bound, which is  $k^2 + 1$  for degree  $k$  and girth 5.

**Construction IX. Brown**

Brown [36] constructed a family of graphs of girth 5 based on his construction for girth 6 [35].

He begins with a set of points  $P = \{p_0, \dots, p_q\}$  and a set of lines  $L = \{\ell_0, \dots, \ell_q\}$ , such that  $p_0$  is incident with all the lines in  $L$  and  $\ell_0$  is incident with all the points in  $P$ , and removes it from  $PG(2, q)$ . The incidence graph of the resulting geometry is a  $q$ -regular graph of girth 6 and order  $2q^2$ . He then adds  $q$ -cycles to the neighborhoods of the deleted vertices  $\{p_1, \dots, p_q\}$  and  $\{\ell_1, \dots, \ell_q\}$ . The resulting graph has degree  $q+2$ , girth 5, and order  $2q^2$ .

**Construction X. Wegner**

Wegner [111] constructed a family of graphs of prime degree  $k$ , girth 5, and order  $2k^2 - 2k$ .

Let  $p \geq 5$  be a prime. The graph is constructed by connecting the vertices of a  $p^2$ -cycle  $A$  to an independent set  $B$  of size  $p(p-2)$ . Denote the vertices of  $A$  by  $a_0, \dots, a_{p^2-1}$ , with each  $a_i$  adjacent to  $a_{i+1}$ ; denote the vertices of  $B$  by  $b_{s,t}$ , for  $0 \leq s \leq p-3$  and  $0 \leq t \leq p$ . The adjacencies between the vertices of the cycle and the independent set are defined as follows. For each  $a_k$ , write  $k$  as  $ip + j$ , where  $0 \leq i, j < p$ , and make  $a_{ip+j}$  adjacent to  $b_{r,ir+j}$ , for  $0 \leq r \leq p-3$ .

The smallest of Wegner's graphs is of degree  $k = 5$  and has order 40. Note that the  $(5, 5)$ -cage has order 30.

### Construction XI. Parsons

Parsons used finite projective planes to construct infinite families of regular graphs of girth five [93]. The best of these has degree  $k = (q+1)/2$  (for prime power  $q$ ) and has order  $2k^2 - 3k + 1$ .

Recall that the projective plane  $PG(2, q)$  can be constructed from the one and two-dimensional subspaces of a 3-dimensional vector space over the field  $\mathbb{F}_q$ . Parsons defines a graph  $G(q)$  on the points of  $PG(2, q)$ , with two points adjacent if they are distinct and their dot product (as vectors in  $\mathbb{F}_q^3$ ) is zero. The graphs Parsons constructs are induced subgraphs of  $G(q)$ .

To specify the graphs, he partitions the vertex set into three subsets,  $R$ ,  $S$ , and  $T$ , as follows. Let  $R$  be the set of self-orthogonal points (as vectors in  $\mathbb{F}_q^3$ ),  $S$  be the set of points adjacent to some point in  $R$ , and  $T$  be all the remaining points. He proves that if  $q \equiv 3 \pmod{4}$ , the subgraph induced by  $S$  has order  $q(q+1)/2$  and is regular of degree  $(q-1)/2$  and has girth 5, and if  $q \equiv 1 \pmod{4}$ , the subgraph induced by  $T$  has order  $q(q-1)/2$  and is regular of degree  $(q+1)/2$  and has girth 5.

When  $q = 5$ , Parsons construction produces the Petersen graph, and when  $q = 7$ , it produces the Coxeter graph.

### Construction XII. O'Keefe and Wong

A construction based on Latin squares was described by O'Keefe and Wong [92]. It can be viewed as a generalization of Robertson's pentagon-pentagram construction of the Hoffman-Singleton graph 2.1.12. Their construction generates the strongest results when the degree  $k = q + 2$ , for a prime power  $q$ , in which case the order of their graphs is  $2k^2 - 8k + 8$ . This is the case described below. In the general case, when  $3 \leq k \leq q + 1$ , the orders of the graphs are  $2q(k-2)$ .

Let  $q = p^r$  be a prime power. O'Keefe and Wong construct a set of  $q$  mutually orthogonal squares with entries from  $\mathbb{F}_q$ , with the last  $q-1$  of these comprising a complete set of mutually orthogonal Latin squares [109]. The Latin squares are constructed as follows. Let  $a$  be a primitive element of  $\mathbb{F}_q$ , and order the field elements as powers of  $a$ :  $b_0 = 0, b_1 = 1, b_2 = a, \dots, b_{q-1} = a^{q-2}$ . Then the squares  $L_i$ ,  $1 \leq i \leq q-1$ , are defined by setting the  $(i, j)$  entry of  $L_k$  to  $b_k(b_i + b_j)$ . Also let  $L_0$  be the square whose  $(i, j)$  entry is  $j$ . Note that  $L_0$  is not a Latin square, but is orthogonal to each of the Latin squares  $L_1, \dots, L_{q-1}$ .

Next define a graph on the two sets of doubly indexed vertices,  $X = \{x_{i,j}\}$  and  $Y = \{y_{i,j}\}$ ,  $0 \leq i, j \leq q-1$ . Let the  $x_{k,j}$  be adjacent to  $y_{i,t}$ , whenever  $t$  is the  $(i, j)$  entry of  $L_k$ . Since the squares are orthogonal, the resulting graph is a  $q$ -regular bipartite graph of girth 6.

To increase the degree to  $q+2$ , the edges of a 2-regular graph are added in such a way that no cycles of length 3 or 4 are introduced. Define  $X_i = \{x_{i,0}, \dots, x_{i,q-1}\}$  and  $Y_i = \{y_{i,0}, \dots, y_{i,q-1}\}$ . Suppose  $q = p^r$  and  $p \geq 5$ . Add edges between  $x_{i,j}$  and  $x_{i,j+a}$ , and between  $y_{i,j}$  and  $y_{i,j+a^2}$ , for  $0 \leq i, j < q$ . Note that the 2-regular graph induced on  $X_i$  (and  $Y_i$ ) consists of disjoint  $p$ -cycles.

In cases  $p = 2$  and  $p = 3$  the construction is similar in spirit, but is more intricate, and results in the addition of 8-cycles and 9-cycles, respectively.

When  $q = 5$ , this construction yields the Hoffman-Singleton graph.

### Construction XIII. Wang

Wang [110] constructed a family of  $(k, 5)$ -graphs for  $k = 2^s + 1$ . This construction uses a complete set of Latin squares of order  $2^s$ ,  $L_1, \dots, L_{k-2}$ , together with the square  $L_0 = [a_{i,j}]$ , with  $a_{i,j} = i$ , for  $0 \leq j \leq k-2$ .

The construction is based on a tree obtained by taking an edge  $uv$ , and joining  $k-1$  leaves to each of  $u$  and  $v$ . The resulting tree has diameter 3 and  $2k$  vertices.

Begin with  $2^{s-1}$  copies of the above tree,  $T_0, \dots, T_{2^{s-1}-1}$ , and label the two sets of leaves in  $T_i$  by  $\alpha_{2i,j}$  and  $\alpha_{2i+1,j}$ , for  $0 \leq j \leq k-2$ . Next, add  $(k-1)^2$  isolated vertices,  $r_{i,j}$ ,  $0 \leq i, j \leq k-2$ , and join  $r_{i,j}$  to  $\alpha_{h,t}$  if and only if  $\alpha_{h,t}$  is the  $(j, h)$ -entry of the square  $L_i$ . The construction continues by adding edges between each pair of vertices  $r_{i,2j}$  and  $r_{i,2j+1}$ , resulting in  $(k, 5)$ -graphs of order  $2k^2 - 3k + 1$ . Finally, the construction is completed by removing all of the vertices  $r_{0,j}$  and adding edges between  $\alpha_{4h,t}$  and  $\alpha_{4h+3,t}$ , and between  $\alpha_{4h+1,t}$  and  $\alpha_{4h+2,t}$ .

The resulting  $(k, 5)$ -graph has order  $2k^2 - 4k + 2$ .

### Construction XIV. Araujo-Pardo and Montellano-Ballesteros

The authors used finite projective and affine planes to construct an infinite family of regular graphs of girth five and degree  $k$  [6]. Their construction gives the strongest results when the degree  $k = p + 2$ , for a prime  $p$ , which matches the previous result of O'Keefe and Wong of order  $2k^2 - 8k + 8$ .

In the general case, where  $k-2$  is not a prime, their bound is

$$n(k, 5) \leq \begin{cases} 4(k-2)^2, & \text{when } 7 \leq k \leq 3276, \\ 2(k-2)(k-1)(1 + \frac{1}{2 \ln^2(k-1)}), & \text{when } 3276 < k. \end{cases}$$

We present their general construction, an explicit presentation of the original construction of Brown. Consider the case where the degree  $k$  is less than the next prime  $p$ , and let  $A_p$  be the affine plane of order  $p$ . Define a smaller incidence structure  $A_{k,p}$  as follows. The points of  $A_{k,p}$  are the points of the affine plane whose first coordinates are less than  $k-2$ . The lines of  $A_{k,p}$  are those whose slopes are less than  $k-2$ . The incidence graph of this structure has order  $2kp$ , degree  $k-2$  and girth 6. Now add edges joining



pairs of points whose first coordinates are equal and whose second coordinates differ by 1. Similarly join pairs of lines whose slopes are equal and whose  $y$ -intercepts differ by 2.

The resulting graph has girth 5. The precise form of their bound follows by using a new result on the distribution of primes [42].

### Construction XV. Jørgensen

Jørgensen [70] used relative difference sets to construct several infinite families of regular graphs of girth five. The best of these have degree  $k = q + 3$  (for prime power  $q$ ) and order  $2k^2 - 12k + 16$ .

His two general theorems can be summarized as follows.

**Theorem 17 ([70])** *Let  $q$  be a prime power. Then*

$$n(k, 5) \leq \begin{cases} 2(k-1)(q-1), & \text{for } 7 \leq q, \ k \leq q+2, \\ 2(k-2)(q-1), & \text{for } 13 \leq q, \ k \leq q+3, \text{ and } q \text{ odd.} \end{cases}$$

Jørgensen's constructions are based on the concept of a relative difference set. Let  $G$  be a finite group with a normal subgroup  $H \triangleleft G$ . A subset  $D \subset G$  is called a *relative difference set* if any element  $g \in G - H$  can be expressed uniquely as a difference using elements from  $D$ ,  $g = x - y$ ,  $x, y \in D$ , but no element of  $H$  can be expressed in this manner.

Jørgensen's construction starts with a finite group  $G$ , a normal subgroup  $H \triangleleft G$ , a set of coset representatives  $T = \{a_1, \dots, a_k\}$  for  $H$  in  $G$ , and a relative difference set  $D$  for  $H$  in  $G$ . In addition, it also requires two Cayley graphs on  $H$  of girth 5,  $C(H, S_1)$  and  $C(H, S_2)$ , such that  $S_1 \cap S_2 = \emptyset$ .

Once one has all these ingredients, a graph on the vertex set  $G \times \{1, 2\}$  can be constructed by introducing edges of three types.

*Type I.1:  $(ha_i, 1)$  is adjacent to  $(hxa_i, 1)$  for all  $h \in H$ ,  $x \in S_1$ ,  $a_i \in T$ ,*

*Type I.2:  $(ha_i, 2)$  is adjacent to  $(hxa_i, 2)$  for all  $h \in H$ ,  $x \in S_2$ ,  $a_i \in T$ ,*

*Type II:  $(g, 1)$  is adjacent to  $(gy, 2)$  for all  $g \in G$  and  $y \in D$ .*

Note that the first two types create multiple copies of the original Cayley graphs on the cosets of  $H$  in  $G$ .

To obtain the bounds in Theorem 17, one needs to make specific choices for  $G$ ,  $H$ ,  $T$ ,  $S_1$  and  $S_2$ . For the first of the inequalities, Jørgensen took  $G$  to be the cyclic group of order  $(q+1)(q-1)$ ,  $H$  to be the cyclic subgroup of order  $(q-1)$ ,  $T$  to be any complete set of coset representatives for  $H$  in  $G$ , and  $D$  to be a relative difference set of size  $q-1$  whose existence is guaranteed by the results of Bose [29] and Elliot and Butson [43]. The two Cayley graphs in this case are edge disjoint cycles on the vertices of  $H$ .

For example, using this method with the choices  $G = \mathbb{Z}_5 \times \mathbb{Z}_5$ ,  $H = \mathbb{Z}_5 \times \{0\}$ ,  $S_1 = \{(1, 0), (4, 0)\}$ ,  $S_2 = \{(2, 0), (3, 0)\}$ ,  $T = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4)\}$ , and  $D = \{(0, 0), (1, 1), (4, 2), (4, 3), (1, 4)\}$ , results in yet another construction for the Hoffman-Singleton Graph.

For the second inequality in Theorem 17, one proceeds along similar lines using dihedral groups.

### 4.1.3 Constructions for Girth 6, 8, and 12

When  $k - 1$  is a prime power and  $g = 6, 8$ , or  $12$ , the  $(k, g)$ -cage is known, and is the incidence graph of the generalized  $n$ -gon, for  $n = 3, 4$ , or  $6$ , respectively (see 2.2). In this section, we discuss constructions for degrees  $k$  where no generalized  $n$ -gon is known. A number of different constructions for these values are obtained by removing particular sets of vertices from the incidence graphs of the generalized  $n$ -gons. This might be compared to the situation for girth 5 where the cages  $(3, 5)$ ,  $(5, 5)$  and  $(6, 5)$  can be constructed by removing vertices from the Hoffman-Singleton graph (see 2.1.12), which is the  $(7, 5)$ -cage [114].

In [55], Gács and Héger present a unified view of these constructions using the concept of a *t-good structure*. A *t-good structure* in a generalized  $n$ -gon is a pair  $(P, L)$  consisting of a set of points  $P$ , and a set of lines  $L$ , subject to the condition that there are  $t$  lines in  $L$  through any point not in  $P$ , and  $t$  points in  $P$  on any line not in  $L$ . Removing the points and lines of a *t-good structure* from the incidence graph of a generalized  $n$ -gon results in a  $(q + 1 - t)$ -regular graph of girth at least  $2n$ .

#### Construction XVI. Brown

Brown [35] was the first who explicitly considered the case of girth 6, and most subsequent constructions are directly or indirectly derived from his. His construction can be stated in the language of *t-good structures* as follows.

He begins with a 1-good structure consisting of a set of points  $P = \{p_0, \dots, p_q\}$  and a set of lines  $L = \{\ell_0, \dots, \ell_q\}$ , such that  $p_0$  is incident with all the lines in  $L$  and  $\ell_0$  is incident with all the points in  $P$ , and removes it from  $PG(2, q)$ . The incidence graph of the resulting geometry is a  $q$ -regular graph of girth 6 and order  $2q^2$ .

For  $t > 1$ , he also removes all the neighbors of  $p_1, \dots, p_{t-1}$  and  $\ell_1, \dots, \ell_{t-1}$  to obtain a geometry whose incidence graph has girth (at least) 6, whose degree is  $q - t + 1$ , and whose order is  $2q(q - t + 1)$ . He then uses Bertrand's postulate that asserts for every  $k > 2$  the existence of a prime  $p$  such that  $k < p < 2k$  [60] to show that  $n(k, 6) < 4k^2$ , for all  $k$ , and other number theoretic results to show that for any  $\epsilon > 0$  there exists and integer  $N$  such that for all  $k > N$ ,  $n(k, 6) < 2(1 + \epsilon)k^2$ . Note that  $M(k, 6) = 2k^2 - 2k + 2$ .

#### Construction XVII. Araujo, González, Montellano-Ballesteros, Serra

In [7], Araujo, González, Montellano-Ballesteros, and Serra apply Brown's construction to generalized quadrangles and hexagons, and obtain the following bounds which are valid for  $g = 6, 8$ , or  $12$ .

$$n(k, g) \leq 2kq^{\frac{g-4}{2}}$$

By using recent results on the distribution of primes they obtain:

$$n(k, g) \leq \begin{cases} 2k(k-1)^{\frac{g-4}{2}} \left(\frac{7}{6}\right)^{\frac{g-4}{2}}, & 7 \leq k \leq 3275 \\ 2k(k-1)^{\frac{g-4}{2}} \left(1 + \frac{1}{2\ln^2(k)}\right)^{\frac{g-4}{2}}, & 3276 \leq k. \end{cases}$$

#### Construction XVIII. Abreu, Funk, Labbate, Napolitano

In [2], Abreu, Funk, Labbate, and Napolitano construct several families of regular graphs of girth 6. Their constructions make use of the addition and multiplication tables of  $\mathbb{F}_q$ . We describe two of them using the language of Gács and Héger [55].

In the first construction, let  $(p_1, \ell_1)$  be a non-incident point-line pair. Choose  $t - 1$  points on  $\ell_1$ :  $p_2, \dots, p_t$ ; and choose  $t - 1$  lines through  $p_1$ :  $\ell_2, \dots, \ell_t$ . The pair  $(P, L)$ , where  $P$  is the set of all points on any of the lines  $\ell_i$ , and  $L$  is the set of all lines through any of the points  $p_i$ , is a  $t$ -good set. Removing  $(P, L)$  from  $PG(2, q)$  leaves a geometry whose incidence graph is  $q - t + 1$ -regular, has girth at least 6, and order  $2(q^2 + (1 - t)q + (t - 2))$ .

The second construction requires the concept of a *Baer subplane*  $(\mathcal{P}', \mathcal{L}')$  of a projective plane  $(\mathcal{P}, \mathcal{L})$ , which is a subplane satisfying the property that any point  $p \notin \mathcal{P}'$  is incident with exactly one line in  $\mathcal{L}'$ , and any line  $\ell \notin \mathcal{L}'$  is incident with exactly one point in  $\mathcal{P}'$ . Baer subplanes of  $PG(2, q)$  exist whenever  $q$  is a square. In fact, it is known that such planes can be partitioned into  $q - \sqrt{q} + 1$  disjoint Baer subplanes [61]. Note that a Baer subplane is a 1-good structure. To obtain a  $t$ -good structure,  $1 \leq t \leq q - \sqrt{q} + 1$ , use  $t$  disjoint Baer subplanes. The resulting graph is  $q - t + 1$ -regular, has girth at least 6, and order  $2(q^2 + (1 - t)q - t\sqrt{q} + (1 - t))$ .

### Construction XIX. Bretto, Gillibert

A construction for  $(k, 6)$  and  $(k, 8)$ -graphs was given by Bretto and Gillibert [31]. They construct graphs  $\Phi(G, S, m)$ , with  $G$  a finite group,  $S$  a non-empty subset of  $G$ , and  $m$  a positive integer. Each  $s \in S$  defines a partition  $G = \cup_{x \in T_s} \langle s \rangle x$ , where  $T_s$  is a complete set of right coset representatives of  $\langle s \rangle$ , the subgroup generated by  $s$ .

The vertices of  $\Phi(G, S, m)$  are the cosets  $\langle s \rangle x$  of the subgroups  $\langle s \rangle$  generated by elements of  $S$ . Two such cosets are adjacent if the cardinality of their intersection is  $m$ .

For example, they choose  $G$  to be the Klein group,  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . The elements of  $G$  are  $\{1, a, b, ab\}$ . Take  $S = \{a, b, ab\}$ . Then the cosets of  $a$  are  $\{1, a\}$  and  $\{b, ab\}$ ; the cosets of  $b$  are  $\{1, b\}$  and  $\{a, ab\}$ ; and the cosets of  $ab$  are  $\{1, ab\}$  and  $\{a, b\}$ . Taking  $m = 1$ , we obtain the octahedral graph.

Using the nonabelian groups of order  $p^3$  in which every nonidentity element has order  $p$ , one can construct  $(p, 6)$ -graphs of order  $2p^2$ . Note that this order is the same as the order obtained by Brown [35].

Similarly, they employed a semidirect product of  $Z_p^3$  by  $Z_p$  to construct  $p$ -regular graphs of girth 8 and order  $2p^3$  in the cases where  $k$  is a prime.

### Construction XX. Gács, Héger

Applying their concept of a  $t$ -good structure to generalized 4-gons and 6-gons, Gács and Héger constructed graphs which establish the following two bounds for prime powers  $q$ .

$$\begin{aligned} n(q, 8) &\leq 2(q^3 - 2q), & q \text{ odd} \\ n(q, 8) &\leq 2(q^3 - 3q - 2), & q \text{ even} \\ n(q, 12) &\leq 2(q^5 - q^3) \end{aligned} \tag{3}$$

### Construction XXI. Balbuena

This construction is based on Latin squares and covers a case for girth 8 not included in the above upper bound of Gács and Héger. She proves that for degree  $k \leq q$ , and  $q$  a prime power  $n(k, 8) \leq 2q(qk - 1)$  [13].

#### 4.1.4 Excision

Several of the above constructions are obtained by removing vertices and incident edges from known small  $(k, g)$ -graphs.

More precisely, let  $G$  be a  $(k, g)$ -graph and let  $H$  be a subgraph of  $G$  with degree set  $\{1, k\}$ . Removing the vertices of  $H$  whose degrees (in  $H$ ) are  $k$ , together with their adjacent edges, leaves a subgraph of  $G$  containing vertices of degree  $k - 1$ . In the case  $k = 3$ , one can suppress the resulting vertices of degree 2. For larger values of  $k$ , the final step involves pairing the degree  $k - 1$  vertices and joining them with edges. The goal is to find subgraphs  $H$  whose removal leaves a graph whose girth is *at least*  $g - 1$ . This process is an example of a class of constructions usually referred to as *excision*.

The pioneering use of this kind of excision was the construction of the  $(3, 11)$ -cage, due to Balaban [10], wherein he removed a small subtree from the  $(3, 12)$ -cage.

Araujo-Pardo [3] applied a type of excision to incidence graphs of generalized polygons to obtain graphs whose girth is one less than that of the original incidence graph. In particular, the examples constructed in her paper give  $rec(5, 11) = 2688$ ,  $rec(9, 7) = 1152$ , and  $rec(9, 11) = 74752$ .

Another Araujo-Pardo and Balbuena girth 6 excision construction [4] applies to any degree  $k$ . Let  $q$  be the smallest prime power greater than  $k$ , then the authors show  $n(k, 6) \leq 2(qk - 2)$ .

Similarly, Araujo-Pardo, Balbuena and Héger apply Brown's excision method to girth 6, 8, and 12 [5]. For girth 12, and  $q$  a prime power larger than 3, they obtain a result for degree  $k \leq q$ .

$$n(k, 12) \leq 2kq^2(q^2 - 1) \tag{4}$$

The  $rec(3, 21)$ ,  $rec(3, 23)$ , and  $rec(3, 25)$ -graphs were also obtained using excision [51], as were the  $rec(3, 29)$ -graph [49] and the  $rec(3, 31)$ -graph [30].

## 4.2 Individual Constructions

### 4.2.1 Individual Constructions for Degree 3

The orders of the trivalent cages of girth 13 and up are all unsettled. The following table shows the current state of knowledge of degree up to 32. Note that for the sake of completeness we have also included the known cages for girths 5 through 12. In cases where the lower and upper bounds were established independently, the authors responsible for the lower bounds are listed first, and separated by a semi-colon from the authors responsible for the upper bound.

Girth $g$	Lower Bound	Upper Bound	# of Cages	Due to
5	10	10	1	Petersen
6	14	14	1	Heawood
7	24	24	1	McGee
8	30	30	1	Tutte
9	58	58	18	Brinkmann-McKay-Saager
10	70	70	3	O’Keefe-Wong
11	112	112	1	McKay-Myrvold; Balaban
12	126	126	1	Benson
13	202	272		McKay-Myrvold; Hoare
14	258	384		McKay; Exoo
15	384	620		Biggs
16	512	960		Exoo
17	768	2176		Exoo
18	1024	2560		Exoo
19	1536	4324		Hoare, H(47)
20	2048	5376		Exoo
21	3072	16028		Exoo
22	4096	16206		Biggs-Hoare, S(73)
23	6144	49326		Exoo
24	8192	49608		Bray-Parker-Rowley
25	12288	108906		Exoo
26	16384	109200		Bray-Parker-Rowley
27	24576	285852		Bray-Parker-Rowley
28	32768	415104		Bray-Parker-Rowley
29	49152	1141484		Exoo-Jajcay
30	65536	1143408		Exoo-Jajcay
31	98304	3649794		Bray-Parker-Rowley
32	131072	3650304		Bray-Parker-Rowley

**Table 3.** Bounds for trivalent cages.

Next, we briefly describe the graphs from the above table that are not cages.

### **Rec(3,13) = 272**

The smallest known trivalent graph of girth 13 has 272 vertices and is a Cayley graph (see A.2) of the group of transformations of the affine plane over  $\mathbb{F}_{17}$ . Biggs reports [21] that the graph was discovered by Hoare. Royle [100] has shown that a smaller (3,13)-graph cannot be a Cayley graph.

### **Rec(3,14) = 384**

The smallest known trivalent graph of girth 14 and order 384 was constructed by Exoo [46]. It is a lift (see A.3) of the multigraph shown in Figure 10.

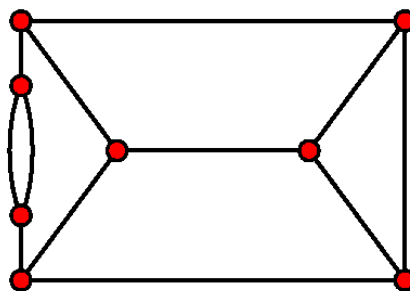


Figure 10: Base graph for girth 14

The voltage group used in the construction is a semidirect product of the cyclic group of order 3 by the generalized quaternion group of order 16, and is  $\text{SmallGroup}(48,18)$  in the Small Group Library in GAP [56]. The automorphism group of this graph has order 96.

### **$\text{Rec}(3,15) = 620$**

The smallest known trivalent graph of girth 15 has 620 vertices and is the sextet graph (see 4.1.1),  $S(31)$ , discovered by Biggs and Hoare [26]. The automorphism group of this graph has order 14880.

### **$\text{Rec}(3,16) = 960$**

The smallest known trivalent graph of girth 16 has 960 vertices and was discovered by Exoo [47]. The graph is a lift (see A.3) of the Petersen graph with voltage assignments in  $\mathbb{Z}_2 \times \mathbb{Z}_{48}$ . In Figure 11 the voltages assigned to each edge are given. Unlabeled edges are assigned the group identity  $(0,0)$ . The automorphism group of this graph has order 96.

### **$\text{Rec}(3,17) = 2176$**

The smallest known trivalent graph of girth 17 has order 2176 and was discovered by Exoo [48]. It is a lift (see A.3) of the multigraph shown in Figure 12. The voltage group is a group of order 272 and is  $\text{SmallGroup}(272,28)$  in the GAP Small Group Library [56]. Note that the group is not the affine group of the same order that was used by Biggs and Hoare in the construction of the  $(3,13)$ -graph discussed above. The automorphism group of this graph has order 544.

### **$\text{Rec}(3,18) = 2560$**

The smallest known trivalent graph of girth 18 has 2560 vertices and was discovered by Exoo [48]. It is a lift (see A.3) of the multigraph in Figure 13. The voltage group has order 320, and is  $\text{SmallGroup}(320,696)$  in the GAP Small Group Library [56]. The maximum

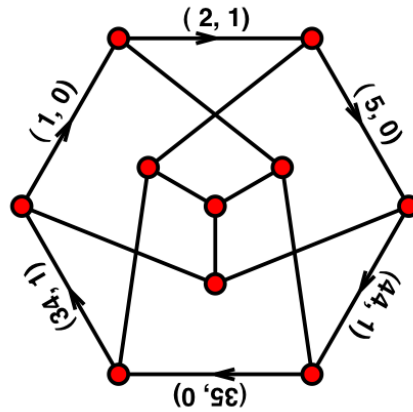


Figure 11: Base graph for girth 16

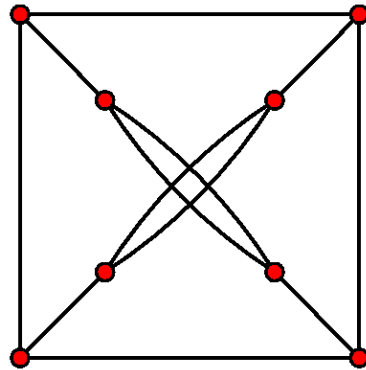


Figure 12: Base graph for girth 17

order among group elements is 20. The Sylow 2-subgroups are nonabelian with exponent 8. The Sylow 5-subgroup is normal, and so the full group is a semi-direct product. The automorphism group of the graph has order 640.

### **Rec(3,19) = 4324**

The smallest known trivalent graph of girth 19 has order 4324. It is the Hexagon graph (see 4.1.1),  $H(47)$ , discovered by Hoare [62]. The automorphism group of the graph has order 51888.

### **Rec(3,20) = 5376**

The smallest known trivalent graph of girth 20 has order 5376 and was discovered by Exoo [48]. It is a lift (see A.3) over the multigraph of order two and size three, with the three edges joining the two vertices. The voltage group is a group of order 2688. The automorphism group of the graph has order 5376 and is transitive on the vertices. The graph turns out to be a Cayley graph, and hence, a graphical regular representation [57].

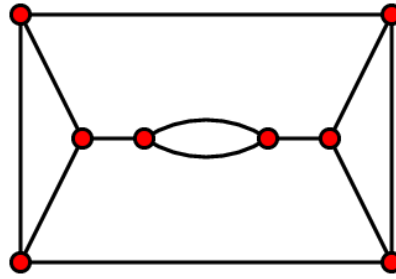


Figure 13: Base graph for girth 18

### **Rec(3,21) = 16028**

The smallest known trivalent graph of girth 21 has order 16028 and was obtained by Exoo [51] using excision on  $S(73)$  [26].

### **Rec(3,22) = 16206**

The smallest known trivalent graph of girth 22 has order 16206 and is the sextet graph (see 4.1.1),  $S(73)$ , discovered by Biggs and Hoare [26].

### **Rec(3,23) = 49326**

The smallest known trivalent graph of girth 23 was discovered by Exoo [51] and was obtained by excision from the graph  $B(PSL(2, 53) \times \mathbb{Z}_2, \{\alpha, \delta, \delta^{-1}\})$  defined next. It is of order 49326.

### **Rec(3,24) = 49608**

The smallest known trivalent graph of girth 24 was discovered by Bray, Parker and Rowley [30]. It is the graph  $B(PSL(2, 53) \times \mathbb{Z}_2, \{\alpha, \delta, \delta^{-1}\})$  of order 49608. The generating permutations  $\alpha$  and  $\delta$  are as follows.

$\alpha$  :

( 1, 4)( 2, 5)( 3, 6)( 7,23)( 8,16)( 9,14)(10,24)(11,46)  
 (12,21)(15,32)(17,47)(18,41)(19,49)(20,55)(22,29)(25,48)  
 (26,36)(27,38)(28,39)(30,40)(31,35)(33,60)(34,51)(37,44)  
 (42,56)(43,59)(45,54)(50,58)(52,57)

$\delta$  :

( 7,49,12)( 8,52,57)( 9,35,44)(10,34,18)(11,53,48)(13,56,55)  
 (14,17,21)(15,31,38)(16,25,51)(19,60,40)(20,54,41)(22,29,45)  
 (23,24,32)(26,50,42)(27,33,30)(28,36,37)(39,43,46)(47,59,58)

### **Rec(3,25) = 108906**



The smallest known trivalent graph of girth 25 is of order 108906 and was obtained by Exoo [51] using excision from  $B((PSL(2, 25) \times 7 : 3) : 2, \{\alpha, \delta, \delta^{-1}\})$  defined next.

### **Rec(3,26) = 109200**

The smallest known trivalent graph of girth 26 is of order 109200. It is the graph  $B((PSL(2, 25) \times 7 : 3) : 2, \{\alpha, \delta, \delta^{-1}\})$  discovered by Bray, Parker and Rowley [30]. The generating permutations  $\alpha$  and  $\delta$  are as follows.

$\alpha :$

( 2, 7)( 3, 6)( 4, 5)( 8, 18)( 9, 20)(10, 17)(11, 12)(13, 32)  
(14, 29)(15, 31)(16, 25)(19, 21)(22, 26)(23, 27)(24, 33)(28, 30)

$\delta :$

( 1, 7, 5)( 3, 4, 6)( 8, 15, 30)( 9, 32, 11)(10, 27, 22)(13, 24, 26)  
(14, 21, 25)(16, 29, 31)(18, 23, 20)(19, 33, 28)

### **Rec(3,27) = 285852**

The smallest known trivalent graph of girth 27 is of order 285852. It is the graph  $B(PSL(2, 83) \times \mathbb{Z}_3, \{\alpha, \delta, \delta^{-1}\})$  discovered by Bray, Parker and Rowley [30]. The generating permutations  $\alpha$  and  $\delta$  are as follows.

$\alpha :$

( 7, 14)( 8, 73)( 9, 24)(10, 89)(11, 23)(12, 60)(13, 87)(15, 76)  
(16, 68)(17, 25)(18, 40)(19, 64)(20, 26)(21, 43)(22, 31)(27, 67)  
(28, 74)(29, 79)(30, 39)(32, 66)(33, 71)(34, 78)(35, 41)(36, 44)  
(37, 52)(38, 50)(42, 81)(45, 77)(46, 69)(47, 54)(48, 58)(49, 85)  
(51, 56)(53, 72)(55, 80)(57, 70)(59, 62)(61, 63)(65, 90)(75, 88)  
(82, 84)(83, 86)

$\delta :$

( 1, 3, 5)( 2, 4, 6)( 7, 38, 65)( 8, 30, 43)( 9, 64, 18)(10, 27, 37)  
(11, 31, 72)(12, 32, 55)(13, 35, 84)(14, 75, 68)(15, 51, 57)(16, 33, 90)  
(17, 49, 19)(20, 80, 42)(21, 66, 40)(22, 87, 61)(23, 85, 47)(24, 78, 26)  
(25, 63, 34)(28, 70, 76)(29, 59, 52)(36, 62, 89)(39, 79, 86)(41, 48, 88)  
(44, 58, 77)(45, 73, 81)(46, 54, 82)(50, 69, 83)(53, 56, 67)(60, 71, 74)

### **Rec(3,28) = 415104**

The smallest known trivalent graph of girth 28 is of order 415104. It is the graph  $B(PGL(2, 47) \times Alt(4), \{\alpha, \delta, \delta^{-1}\})$  discovered by Bray, Parker and Rowley [30]. The generating permutations  $\alpha$  and  $\delta$  are as follows.

$\alpha :$

( 1, 4)( 2, 3)( 5,20)( 6,52)( 7,45)( 8,12)( 9,28)(10,32)  
 (11,33)(13,47)(14,42)(15,34)(16,22)(17,24)(18,41)(19,26)  
 (21,27)(23,38)(25,50)(29,49)(30,44)(31,35)(36,46)(37,39)  
 (43,48)

$\delta$  :

( 1, 4, 2)( 5,33,11)( 6,30,17)( 7,19,21)( 8,38,45)( 9,34,39)  
 (10,37,20)(12,14,26)(13,44,23)(15,35,31)(16,51,27)(18,50,46)  
 (22,48,28)(24,42,47)(25,36,43)(29,32,41)(40,49,52)

### **Rec(3,29) = 1141484**

The smallest known trivalent graph of girth 29 is of order 1141484 and was constructed by Exoo and Jajcay [49]. It was obtained by excision from the girth 30 graph described in the next section. This improves the bound of 1143408 due to Bray, Parker and Rowley [30].

### **Rec(3,30) = 1143408**

The smallest known trivalent graph of girth 30 and order  $83^3 - 83 = 1143408$  was constructed by Exoo and Jajcay [49]. It is a voltage graph lift of the  $\theta$ -graph of order 2 using the group  $SL(2, 83)$  and the voltages 1,  $a$ , and  $b$ , where:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 11 \\ 23 & 5 \end{pmatrix}.$$

The elements  $a$  and  $b$  are conjugates of order 84. The graph is vertex-transitive and the size of its automorphism group is twice the size of the graph. The previous record holder was the sextet graph of Biggs and Hoare [26], S(313), of order 1227666 (see 4.1.1).

### **Rec(3,31) = 3649794**

The smallest known trivalent graph of girth 31 is of order 3649794 and was obtained by Bray, Parker and Rowley [30], using excision, from the graph  $B(PGL(2, 97) \times Alt(4), \{\alpha, \delta, \delta^{-1}\})$  defined next.

### **Rec(3,32) = 3650304**

The smallest known trivalent graph of girth 32 is of order 3650304. It is the graph  $B(PGL(2, 97) \times Alt(4), \{\alpha, \delta, \delta^{-1}\})$  discovered by Bray, Parker and Rowley [30]. The generating permutations  $\alpha$  and  $\delta$  are as follows.

$\alpha$  :

( 1, 3)( 2, 4)( 5, 34)( 6, 91)( 7, 22)( 8, 45)( 9, 66)  
 (10, 41)(11, 95)(12, 88)(13, 67)(14, 80)(15, 19)(16, 17)  
 (18, 68)(20, 30)(21, 26)(23, 31)(24, 61)(25, 71)(27, 65)

(28, 59)(29, 36)(32, 97)(33, 40)(35, 64)(37, 70)(38, 46)  
 (39, 49)(42, 102)(43, 48)(44, 96)(47, 62)(50, 54)(51, 99)  
 (52, 53)(55, 87)(56, 100)(57, 79)(58, 72)(60, 101)(63, 76)  
 (69, 86)(73, 92)(74, 83)(75, 94)(77, 89)(78, 90)(81, 98)  
 (82, 85)(84, 93)

$\delta$  :

( 2, 3, 4)( 5, 62, 69)( 6, 87, 20)( 7, 13, 49)( 8, 25, 79)  
 ( 9, 56, 72)(10, 21, 33)(11, 19, 38)(12, 52, 44)(14, 39, 30)  
 (15, 43, 81)(16, 97, 83)(17, 85, 29)(18, 86, 74)(22, 60, 88)  
 (23, 28, 27)(24, 78, 95)(26, 67, 100)(31, 47, 94)(32, 45, 42)  
 (34, 41, 98)(35, 53, 46)(36, 77, 101)(37, 102, 55)(48, 99, 66)  
 (50, 68, 57)(51, 91, 59)(54, 90, 96)(58, 71, 61)(64, 89, 73)  
 (65, 84, 92)(70, 82, 93)(75, 80, 76)

#### 4.2.2 Individual Constructions for Girth 5

For graphs of degree  $k$  and girth 5, the Moore bound is  $k^2 + 1$ . In this case there are Moore graphs for degrees 3, 7, and perhaps 57. The case  $k = 57$  is unresolved, and has received a lot of attention. Aschbacher showed that there does not exist a rank 3 permutation group with subdegree 57. This means that a Moore graph of degree 57 is not a rank 3 graph and is not distance transitive. In an unpublished work, G. Higman also showed that such a graph is not vertex-transitive [37]. Further restrictions on the automorphism group of a potential Moore graph of degree 57 were obtained by Makhnev and Paduchikh [79], who showed that if the automorphism group of such a graph contains an involution, then the order of the group must be relatively small. Mačaj and Širáň continued in this direction and showed that the order of the automorphism group of such a graph is at most 375 [77].

The best currently known graphs of girth 5 and degree up to 20 are listed in the next table.

Degree $k$	Lower Bound	Upper Bound	Due to
3	10	10	Petersen
4	19	19	Robertson
5	30	30	Robertson-Wegner-Wong
6	40	40	Wong
7	50	50	Hoffman-Singleton
8	67	80	Royle
9	86	96	Jørgensen
10	103	124	Exoo
11	124	154	Exoo
12	147	203	Exoo
13	174	230	Exoo
14	199	288	Jørgensen
15	230	312	Jørgensen
16	259	336	Jørgensen
17	294	448	Schwenk
18	327	480	Schwenk
19	364	512	Schwenk
20	403	576	Jørgensen

**Table 4.** Bounds for girth 5 cages.

Next, we briefly describe the graphs from the above table that are not cages.

### **$\text{Rec}(8,5) = 80$**

The smallest known  $(8,5)$ -graph is of order 80. It was discovered by Royle [100], and is a Cayley graph. It can be constructed using either  $\text{SmallGroup}(80,32)$  or  $\text{SmallGroup}(80,33)$  in the GAP Small Group Library [56].

### **$\text{Rec}(9,5) = 96$**

The smallest known  $(9,5)$ -graph, of order 96, was constructed by Jørgensen [70] using a cyclic relative difference set. The construction is from the first part of Theorem 17, for the case  $q = 7$ . The group used is cyclic of order 48, the subgroup is cyclic of order 6, and the relative difference set has size 6.

### **$\text{Rec}(10,5) = 124$**

The smallest known  $(10,5)$ -graph has order 124 and was discovered by Exoo [51]. It was found by a computer search starting from a subgraph of the incidence graph of a projective plane of order 11. The algorithm was modeled after a method described by Abreu, Araujo-Pardo, Balbuena and Labbate in [1].

**Rec(11,5) = 154**

The smallest known (11, 5)-graph has order 154 and was discovered by Exoo [51].

**Rec(12,5) = 203**

The smallest known (12, 5)-graph has order 203 and was discovered by Exoo [51]. It is a Cayley graph of the semi-direct product  $\mathbb{Z}_{29} \rtimes \mathbb{Z}_7$  (of order 203), whose full automorphism group is the same group again, and hence, it is a graphical regular representation [57].

**Rec(13,5) = 230**

The smallest known (13, 5)-graph has order 230 and was discovered by Exoo [51].

**Rec(14,5) = 288**

The smallest known (14, 5)-graph, of order 288, was constructed by Jørgensen [70] using a dihedral relative difference set. The construction is from the second part of Theorem 17, for the case  $q = 13$  and  $k = 14$ .

**Rec(15,5) = 312**

The smallest known (15, 5)-graph, of order 312, was constructed by Jørgensen [70] using a dihedral relative difference set. The construction is from the second part of Theorem 17, for the case  $q = 13$  and  $k = 15$ .

**Rec(16,5) = 336**

The smallest known (16, 5)-graph, of order 336, was constructed by Jørgensen [70] using a dihedral relative difference set. The construction is from the second part of Theorem 17, for the case  $q = 13$  and  $k = 16$ .

**Rec(17,5) = 448**

The smallest known (17, 5)-graph has order 448, and was constructed by Schwenk [103] from the (19, 5)-graph by removing two copies of the Möbius-Kantor graph from each of the sets  $P$  and  $Q$ .

**Rec(18,5) = 480**

The smallest known (18, 5)-graph has order 480, and was constructed by Schwenk [103] from the (19, 5)-graph by removing one copy of the Möbius-Kantor graph from each of the sets  $P$  and  $Q$ .

**Rec(19,5) = 512**

The smallest known (19, 5)-graph has order 512, and was discovered by Schwenk [103]. The graph is constructed from two sets  $P$  and  $Q$  each consisting of 16 copies, indexed by  $\mathbb{F}_{16}$ , of the Möbius-Kantor graph (see Figure 14). In addition, the vertices of each copy of the graph are labeled by  $\mathbb{F}_{16}$ . The remaining edges join vertices from copies in  $P$  to

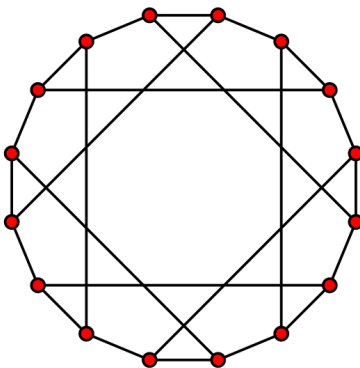


Figure 14: The Möbius-Kantor graph

vertices from copies in  $Q$  according to a rule based on field operations in  $\mathbb{F}_{16}$  analogous to the rule used in Robertson's construction of the Hoffmann-Singleton graph (see 2.1.12).

### **Rec(20,5) = 576**

The smallest known  $(20, 5)$ -graph, of order 576, was constructed by Jørgensen [70] using a dihedral relative difference set. The construction is from the second part of Theorem 17, for the case  $q = 17$  and  $k = 20$ .

#### **4.2.3 Individual Constructions for Girth 6**

When  $k = q + 1$ , for a prime power  $q$ , the  $(k, 6)$ -cage is the incidence graph of a projective plane. Outside these cases, there is only one case where the order of the cage has been established, namely  $k = 7$  [90]. The remaining cases listed in the following table all come from infinite families discussed in Section 4.1.3.

Degree $k$	Lower Bound	Upper Bound	Due to
3	14	14	Projective Plane
4	26	26	Projective Plane
5	42	42	Projective Plane
6	62	62	Projective Plane
7	90	90	O’Keefe-Wong
8	114	114	Projective Plane
9	146	146	Projective Plane
10	182	182	Projective Plane
11	224	240	Wong
12	266	266	Projective Plane
13	314	336	Abreu-Funk-Labbate-Napolitano
14	366	366	Projective Plane
15	422	462	Abreu-Funk-Labbate-Napolitano
16	482	504	Abreu-Funk-Labbate-Napolitano
17	546	546	Projective Plane
18	614	614	Projective Plane
19	686	720	Abreu-Funk-Labbate-Napolitano
20	762	762	Projective Plane

**Table 5.** Bounds on cages of girth 6.

#### 4.2.4 Summary of Individual Constructions

In the table below, we summarize the best known upper bounds for degrees up to 20 and girths up to 16. Recall that entries for degree 3 and girths greater than 16 can be found in Table 3.

Most of the cases where there is an entry in the table that does not appear in any of the previous tables are for girths 8 and 12. For these girth there is a cage known whenever  $k - 1$  is a prime power. All but two of the other entries for 8 and 12 come from Gács and Héger [55]. The entries, (15, 8) and (15, 12), are due to Balbuena [13] and Araujo-Pardo, Balbuena and Héger [5]. In addition, the entries (5, 11), (9, 7) and (9, 11) are due to Araujo-Pardo [3]. The four entries for girth 7 consist of the two known cages (degrees 3 and 4), and two graphs constructed using excision by McKay and Yuanshen [85]. The remaining three entries, that have not appeared previously, are for degree 4 and girths 9 and 10, and for degree 5 and girth 10 [51].

$k/g$	5	6	7	8	9	10	11	12	13	14	15	16
3	10	14	24	30	58	70	112	126	272	384	620	960
4	19	26	67	80	275	384		728				
5	30	42	152	170		1296	2688	2730				
6	40	62	294	312				7812				
7	50	90		672				32928				
8	80	114		800				39216				
9	96	146	1152	1170			74752	74898				
10	124	182		1640				132860				
11	154	240		2618				319440				
12	203	266		2928				354312				
13	230	336		4342				738192				
14	288	366		4760				804468				
15	312	462		7648				1957376				
16	336	504		8092				2088960				
17	448	546		8738				2236962				
18	480	614		10440				3017196				
19	512	720		13642				4938480				
20	576	762		14480				5227320				

**Table 6.** Summary of upper bounds for  $n(k, g)$ .

## 5 Open Problems

The central problem is that of determining  $n(k, g)$  and finding all the corresponding cages. Since this is a very hard problem in general, in what follows we list some subproblems on which progress might be made.

Since upper bounds can be improved through graph constructions, much more work has been done on the upper bounds for  $n(k, g)$ . Our first problem calls for remedying this imbalance.

1. *Improve the lower bounds on the order of cages in cases where a Moore graph does not exist.* Beside [36, 72, 45], very little progress has been made on lower bounds.

Next, we address the relationship between the growth rate of  $n(k, g)$  and the Moore bound. The following two problems are specific cases of this general problem, one for fixed degree and one for fixed girth.

2. *Find an infinite family of trivalent graphs with large girth  $g$  and order  $2^{cg}$  for  $c < \frac{3}{4}$ .*  
Note that in this case the Moore bound is essentially  $2^{\frac{1}{2}g}$ , whereas the best known constructions [26, 76, 75] are asymptotically of order  $2^{\frac{3}{4}g}$ .



3. *Find infinite families of graphs of girth 10, degree  $k$ , and with orders within a constant multiple of the Moore bound.* For girths 5, 6, 8, and 12, such families are known. Girth 10 appears to be the most approachable candidate for progress.

Progress on obtaining exact values for  $n(k, g)$  for specific values of  $k$  and  $g$  has been slow. A few of these merit special mention.

4. *Settle the existence question for the missing Moore graph, which would be a  $(57, 5)$ -cage of order 3250.* Such a graph cannot be vertex-transitive and its automorphism group can have order at most 375.
5. *Improve the bounds on the value of  $n(3, 13)$ , which have been 202 and 272 for some time.* Already in 1998, Biggs [23] stated that the failure to improve the upper bound of 272 “is becoming an embarrassment”.
6. *Determine the order of the  $(8, 5)$ -cage.* The  $(k, 5)$ -cages, for  $k \leq 7$  have been known for over 30 years. We currently know that  $66 \leq n(8, 5) \leq 80$ , where the upper bound is due to Royle [100].

It has been frequently observed that it is much easier to construct small regular graphs of even girth if one assumes in addition that they are bipartite. In fact, all known even girth cages and record holders are bipartite. In Wong’s 1982 survey [114], he asks if every cage of even girth is bipartite. A positive answer to this question was stated as a conjecture by Frank Harary to the first author of this survey (when he was a graduate student) in the 1970’s.

7. *Is every cage of even girth bipartite?*
8. *Find direct constructions for trivalent graphs of odd girth  $g > 19$ .* Each of the entries in Table 1 for these girths was obtained by excision from the bipartite record graph of even girth  $g + 1$ .

One can consider the cage problem in restricted contexts. Of particular interest are the classes of Cayley graphs and, more generally, of vertex-transitive graphs. The vertex symmetry of these graphs makes the central problem of finding cages more manageable, both for upper and lower bounds. For this reason, many of the current record holders are highly symmetric.

9. *Find the smallest vertex-transitive graphs of given degree  $k$  and girth  $g$ .*

# Appendices

## A Symmetric Graphs

Many of the graphs discussed above exhibit a high level of symmetry. All the known Moore graphs possess an automorphism group that acts transitively on the set of vertices

of the graph. Similarly, the majority of the other graphs discussed in this survey have a large automorphism group with very few orbits. These observations suggest that highly symmetric graphs deserve a careful examination.

In the case of small cages the reason behind the usefulness of symmetric graphs may be due to the fact that small regular graphs are more likely to have a relatively large group of automorphisms, while for larger instances of the cage problem, this may be due to the fact that symmetric graphs are easier to construct.

An *automorphism* of a graph  $G$  is a permutation  $\varphi$  of the vertices of  $G$  that preserves the structure of  $G$ , i.e., any two vertices,  $u$  and  $v$ , are adjacent if and only if  $\varphi(u)$  is adjacent to  $\varphi(v)$ . The set of all automorphisms of  $G$  forms a group,  $\text{Aut}(G)$ , under the operation of composition.

In the following subsections, we review the class of vertex-transitive graphs and its subclass of Cayley graphs.

## A.1 Vertex-Transitive Graphs

A group  $G$  acting on a set  $V$  is said to *act transitively* on  $V$  if for any  $u, v \in V$ , there exists an element  $g \in G$  that maps  $u$  to  $v$ . A graph  $G$  is *vertex-transitive* if  $\text{Aut}(G)$ , the automorphism group of  $G$ , acts transitively on the set of vertices  $V(G)$  ([22]). Thus, vertex-transitive graphs look the same at each vertex, and all the vertices of such graphs lie on the same number of cycles of any particular length. In particular, each vertex lies on a cycle of length  $g$ , the girth of the graph. In fact, the number of cycles of any fixed length through any vertex  $v$  must satisfy additional arithmetic properties related to the order of the graph,  $|V(G)|$ , (see, for example, [66]). Looking at the other end of the cycle spectrum, note that all but four of the known non-trivial vertex-transitive graphs are Hamiltonian; the four exceptions being the Petersen and Coxeter graph, and the two graphs obtained from these by replacing their vertices by triangles.

The existence of vertex-transitive graphs for any degree  $k \geq 2$  and girth  $g \geq 3$  has been proved using voltage graphs and lifts in [88], and similar results can be deduced from the paper of Mačaj, Širáň and Ipolyiová [78].

Of all the known constructions of vertex-transitive graphs, let us mention the most direct one. For the purpose of simplification we restrict ourselves to the case of finite groups acting on finite sets.

Let  $\Gamma$  be a *permutation group acting transitively on a set  $V$*  (i.e.,  $\Gamma \leq \text{Sym}(V)$ , the full symmetric group of all permutations of the set  $V$ ), and let  $v^g$  denote the vertex resulting from the action of  $g \in \Gamma$  on  $v \in V$ . Then  $\Gamma$  has a natural induced action on the Cartesian product  $V \times V$  defined by  $(u, v)^g = (u^g, v^g)$ , for all  $(u, v) \in V \times V$  and  $g \in \Gamma$ , and the  $\Gamma$  orbits in  $V \times V$  are called the *orbitals* of  $\Gamma$ . For each orbital  $\Delta$  of the action of  $\Gamma$  on  $V \times V$ , there is a *paired orbital*  $\Delta^* = \{(v, u) \mid (u, v) \in \Delta\}$ .

Let  $\Omega$  be any set of orbitals of  $\Gamma$  closed under taking paired orbitals, and let  $E$  be the set of (unordered) pairs  $\{u, v\}$  such that  $(u, v)$  belongs to  $\Omega$ . Then the graph  $G = (V, E)$  is called the *orbital graph* of  $\Gamma$  (with respect to  $\Omega$ ). It is easy to see that the orbital graph of any transitive permutation group is a vertex-transitive graph, and moreover, that any

vertex-transitive graph  $G = (V, E)$  is the orbital graph of any of its vertex-transitive automorphism groups  $\Gamma \leq \text{Aut}(G)$  with  $\Omega = \{(u, v)^g \mid \{u, v\} \in E, g \in \Gamma\}$ , i.e., the class of vertex-transitive graphs is the class of orbital graphs of transitive permutation groups (for more details see [96]).

Note in addition that an abstract group  $\Gamma$  has a transitive permutation representation on a set of size  $n$  if and only if  $\Gamma$  has a subgroup  $\Lambda$  of index  $n$ ; in which case  $\Gamma$  can be thought of as acting on the (right) cosets of  $\Lambda$  in  $\Gamma$  via (right) multiplication ([18]). Hence, the class of (finite) vertex-transitive graphs is the class of orbital graphs of (finite) groups acting via left multiplication on the left cosets of their subgroups, and *every* vertex-transitive graph whose automorphism group contains  $\Gamma$  acting transitively on its vertices can be obtained by choosing appropriate  $\Lambda$  and  $\Omega$ .

Although vertex-transitive graphs constitute a significant part of the known cages and record holders, it can be shown that their orders have to be (in at least some cases) bigger than the Moore bound. In addition to Biggs' Theorem 10, which showed that the excess for vertex-transitive graphs can be arbitrarily large, the paper [67] contains additional improvements on the lower bounds for vertex-transitive graphs of given degree and girth.

**Theorem 18 ([67])** *Let  $G$  be a vertex-transitive graph of valency  $k$  and girth  $g = p^r > k$ , where  $p$  is an odd prime and  $r \geq 1$ . If  $G$  is not a Moore graph (that is,  $|V(G)| > M(k, g)$ ), and  $g$  is relatively prime to all the integers in the union*

$$\bigcup_{0 \leq i \leq k} \mathcal{L}(k, g, i),$$

*where  $\mathcal{L}(k, g, 0) = \{M(k, g) + 1, M(k, g) + 2, \dots, M(k, g) + k\}$ , and  $\mathcal{L}(k, g, i) = \{k(k-1)^{(g-1)/2} - ik, k(k-1)^{(g-1)/2} - ik + 1, \dots, k(k-1)^{(g-1)/2} - ik + i - 1\}$ ,  $i > 0$ , then the order of  $G$  is at least  $M(k, g) + k + 1$ .*

## A.2 Cayley Graphs

A vertex-transitive graph  $G$  is Cayley if there exists an automorphism group  $\Gamma$  of  $G$  that *acts regularly* on  $V(G)$ ; i.e., for each  $u, v \in V(G)$ , there exists *exactly one* automorphism  $\varphi \in \Gamma$  such that  $\varphi(u) = v$ .

Cayley graphs constitute a subclass of the class of vertex-transitive graphs that have proved particularly useful in the construction of cages. The following definition is the one used throughout our survey.

Let  $\Gamma$  be an (abstract) finite group with a generating set  $X$  that does not contain the identity of  $\Gamma$  and is closed under taking inverses,  $X = X^{-1}$  (no assumptions are made about the minimality of  $X$ ). The *Cayley graph*  $C(\Gamma, X)$  is the regular graph of degree  $|X|$  that has  $\Gamma$  for its set of vertices and whose adjacency is defined by making each vertex  $g$  of the graph,  $g \in \Gamma$ , adjacent to all the vertices in the set  $g \cdot X = \{g \cdot x \mid x \in X\}$ . Alternatively, for any two vertices  $g, h \in \Gamma$ ,  $h$  is adjacent to  $g$  if and only if  $g^{-1}h \in X$ . Note that the fact that  $X$  is closed under inverses makes the resulting graph undirected.

Besides being useful in cage construction, Cayley graphs have also been useful in proofs. The proof of the following result mentioned in the introduction is a generalization of a proof due to Biggs [21].

**Theorem 19** *Given any  $k, g \geq 3$ , there exists a  $k$ -regular graph  $G$  with girth at least  $g$ .*

**Proof.** The graph we construct is a Cayley graph.

Let  $k, g \geq 3$ ,  $r = \lfloor \frac{g}{2} \rfloor$ , and  $T_{k,r}$  denote the finite tree of radius  $r$  with center  $x$  in which all the vertices, whose distance from  $x$  is less than  $r$ , are of degree  $k$ ; and the vertices at distance  $r$  from  $x$  are leaves of degree 1. Color the edges of  $T_{k,r}$  by the  $k$  colors  $\{1, 2, 3, \dots, k\}$  subject to the edge-coloring rule that no two adjacent edges are of the same color. For each color  $i$ , let  $\alpha_i$  denote the involutory permutation of the vertices of  $T_{k,r}$ :

$$\alpha_i(u) = v \text{ if and only if the edge } \{u, v\} \text{ is colored by } i.$$

Let  $\Gamma = \langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$  be the finite permutation group generated by the involutions  $\alpha_i$ , and take  $X = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ . We claim that the  $k$ -regular graph  $C(\Gamma, X)$  has girth at least  $g$ . First observe that any cycle of length  $s$  in  $C(\Gamma, X)$  corresponds to a reduced word  $w(\alpha_1, \alpha_2, \dots, \alpha_k)$  of length  $s$  that is equal to  $1_\Gamma$ ,  $w(\alpha_1, \alpha_2, \dots, \alpha_k) = 1_\Gamma$  (a word is reduced if no  $\alpha_i$ 's is immediately followed by itself). Clearly,  $w(\alpha_1, \alpha_2, \dots, \alpha_k) = 1_\Gamma$  implies  $w(\alpha_1, \alpha_2, \dots, \alpha_k)(x) = x$ , for all  $x \in X$ , in the action of  $\Gamma$  on the vertices of  $T_{k,r}$ . Consider the effect of  $w(\alpha_1, \alpha_2, \dots, \alpha_k)$  on  $x$ , a neighbor of  $1_\Gamma$ . Initially, each element of the word moves  $x$  one step toward the leaves. In order for the image of  $x$  to return back to  $x$ , an additional  $r + 1$  elements are required. Thus, any reduced word representing the identity must have length at least  $2r + 1$ . Equivalently, the girth of  $C(\Gamma, X)$  is at least  $g$ .  $\square$

Note that not all vertex-transitive graphs are Cayley. The Petersen graph is the smallest exception. Thus, the constructions of vertex-transitive  $(k, g)$ -graphs referenced in Section A.1 do not imply the existence of  $(k, g)$ -Cayley graphs. The existence of Cayley graphs for all pairs  $(k, g)$  has first been proved in [67].

**Theorem 20 ([67])** *For every pair of parameters  $k \geq 2, g \geq 3$ , there exists a finite Cayley graph  $C(\Gamma, X)$  of valency  $k$  and girth  $g$ .*

In Table 7 below, we compare the orders of the smallest known  $(k, g)$ -graphs, the smallest vertex-transitive  $(k, g)$ -graphs, and the smallest Cayley  $(k, g)$ -graphs. The column labels in Table 7 stand for the current lower bound (*lb*), the order of the current record  $(3, g)$ -graph (*rec*), the order of the the smallest vertex-transitive  $(3, g)$ -graph (*vt*), and the order of the smallest Cayley  $(3, g)$ -graph (*cay*). It is worth emphasizing that in the cases of vertex-transitive and Cayley graphs, the orders listed in the columns *vt* and *cay* are not bounds, but rather exact values [95].

<i>girth</i>	<i>lb</i>	<i>rec</i>	<i>vt</i>	<i>cay</i>
5	10	10	10	50
6	14	14	14	14
7	24	24	26	30
8	30	30	30	42
9	58	58	60	60
10	70	70	80	96
11	112	112	192	192
12	126	126	126	162
13	202	272	272	272
14	258	384	406	406
15	512	620	620	864
16	768	960	1008	1008

**Table 7.** Cubic vertex transitive and Cayley cages.

### A.3 Voltage Graphs

The voltage graph construction originally comes from topology. Its main use in the context of graph theory has been primarily restricted to topological graph theory where it was instrumental in such important achievements as Ringel and Youngs' solution of the Heawood Map Coloring problem [98]. Voltage graphs are a special case of graph covers, distinguished by the existence of an automorphism group that acts regularly on each of the fibers. The main reason for the efficacy of voltage graphs in combinatorial constructions lies in their flexibility with respect to the full symmetry spectrum starting from highly symmetric graphs on one side and ending with graphs with trivial automorphism groups on the other.

Informally, voltage graphs are lifts of base graphs determined by an assignment of group elements to oriented edges of the base graph. A *base graph* is a finite digraph with possible loops and multiple edges. We denote its vertex, edge, and arc (oriented edge) sets by  $V(G)$ ,  $E(G)$ , and  $D(G)$ , respectively. Each edge  $e \in E(G)$  is represented twice in  $D(G)$  (once with each of the two possible orientations) and if  $e \in D(G)$ , we denote the reverse arc by  $e^{-1}$ .

Given a finite group  $\Gamma$ , a *voltage assignment* is a function  $\alpha : D(G) \rightarrow \Gamma$  satisfying the property  $\alpha(e^{-1}) = g^{-1}$  whenever  $\alpha(e) = g$ . The group  $\Gamma$  is called the *voltage group*.

Given a voltage assignment  $\alpha : D(G) \rightarrow \Gamma$ , the *lift* of  $G$ , often called the *derived graph* or the *derived regular cover* of  $G$ , denoted by  $G^\alpha$ , is the graph whose vertex set is  $V(G) \times \Gamma$  with two vertices  $(u, g)$  and  $(v, h)$  adjacent if and only if  $uv \in E(G)$  and  $g \cdot \alpha(uv) = h$ .

We include two simple examples of the derived graph construction.

First, consider the *dumbbell* graph depicted in Figure 15. with a voltage assignment chosen from  $Z_5$ . The derived graph in this example is the Petersen graph.

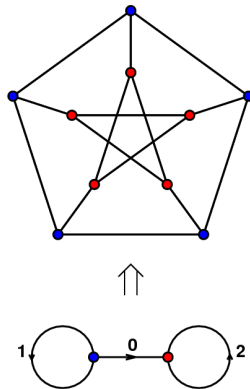


Figure 15: The Petersen graph as a lift

Our second example, shown in Figure 16, gives rise to the Heawood graph. The base graph is a  $\theta$ -graph (dipole) with a voltage assignment from  $\mathbb{Z}_7$ .

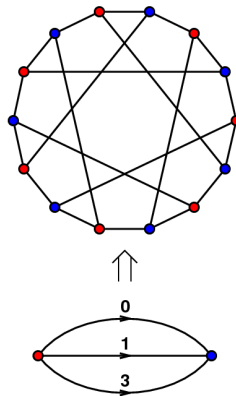


Figure 16: The Heawood graph as a lift

Another example of the use of voltage graphs in cage construction is in the proof of Sachs' result concerning consecutive odd-even girths in Theorem 5. Namely, using the voltage group  $\mathbb{Z}_2$  and assigning the non-identity voltage 1 to all edges of a regular graph  $G$  results in a lift of twice the order of  $G$  with the property that all the odd-length cycles of  $G$  are lifted into cycles of doubled lengths while the lengths of the even-length cycles remain unchanged. Hence,

$$n(k, g + 1) \leq 2n(k, g)$$

for  $g$  odd.

The following result shows that one can use the voltage graph construction to increase the girth of a base graph by an arbitrary multiple.

**Theorem 21** ([49]) *Let  $G$  be a base graph of girth  $g$ , and  $k > 1$  be an integer. Then there exists a voltage graph lift of  $G$  of girth at least  $kg$ .*

## B Computer Methods

Computers have been used in the study of cages since the 1960's. O'Keefe and Wong used them to complete detailed case analyses [89, 91, 90, 113]. More intensive use of computers followed the introduction of McKay's *nauty* package [82], which proved to be particularly useful in work on lower bounds [32, 84, 50]. Similarly, progress on upper bounds followed the development of programs such as GAP [56], which facilitated the use of large groups in Cayley and voltage graph constructions.

### B.1 Lower Bound Proofs and Isomorphism Checking

Computational proofs that have established the correct lower bounds for the orders of some of the cages were made possible by the ability to do fast isomorphism testing [82].

This approach has been used to establish the correct lower bounds for  $n(3, 9)$  [32],  $n(3, 11)$  [84], and  $n(4, 7)$  [50]. Such proofs are organized by splitting the problem into a large number of subproblems, which can then be handled independently, and the work can be done in parallel on many different computers.

The computation can begin by selecting a root vertex and constructing a rooted  $k$ -nary tree of radius  $\frac{g-1}{2}$ . The actual computation proceeds in two phases. First, all non-isomorphic ways to add sets of  $m$  edges to the tree are determined (for some experimentally determined value of  $m$ ). This phase involves extensive isomorphism checking. The second phase is the one that is more easily distributed across a large number of computers. Each of the isomorphism classes found in the first phase becomes an independent starting point for an exhaustive search to determine whether the desired graph can be completed. Of all possible edges that could be added to the graph at this point, those that would violate the degree or girth conditions are eliminated. The order in which the remaining edges are considered is then determined by heuristics.

### B.2 Computer Searches

The great majority of upper bounds presented in our tables were obtained using extensive computer search techniques. The methods differ by the level of symmetry assumed.

On one extreme, one can assume that the desired graph is Cayley. Several investigators have developed fast methods that found small Cayley graphs of large girth [39, 59]. However, most of the current record holders for degree  $k \geq 3$  and girth  $g \geq 14$  were found by relaxing the Cayley graph symmetry requirement. For example, Bray, Parker,

and Rowley [30] constructed their graphs by collapsing cycles in Cayley graphs. All their resulting graphs are vertex-transitive.

Further relaxation of the symmetry requirements can be made by noting that all Cayley graphs are voltage graphs: lifts of a one-vertex bouquet of cycles. As one increases the order of the base graph and decreases the order of the group, the symmetry tends to decrease (and ultimately one can think of all graphs as voltage graphs with a trivial voltage group). Hence, the voltage graph construction is a natural generalization of the Cayley construction allowing one to fine-tune symmetry assumptions. Instead of searching through the space of groups whose order  $n$  equals the order of the desired graph, one can expand the search to include base graphs of order greater than one and voltage groups of order less than  $n$  [46, 47, 48].

Ultimately, one can completely eliminate symmetry assumptions. The largest cases where this type of search has been successful in finding cages, and perhaps the largest cases where it is feasible, are  $n(3, 9) = 58$  [32] and  $n(4, 7) = 67$  [50].

## C The Upper Bound of Erdős

The following is essentially a verbatim translation of the Erdős' original proof in German [44], subject to some notational changes.

**Theorem 22** ([44]) *For every  $k \geq 2$ ,  $l \geq 3$ ,*

$$n(k, g) \leq 4 \sum_{t=1}^{g-2} (k-1)^t \quad (5)$$

Note that for  $k = 2$  as well as  $g = 3$  the bound (5) is sharp, and so from now on we will assume  $k > 2$  and  $g > 3$ .

In order to prove the upper bound, we prove the following slightly stronger claim:

*Let  $k \geq 2$ ,  $g \geq 4$ ,  $m \geq 2 \sum_{t=1}^{g-2} (k-1)^t$ . Then there exists a  $k$ -regular graph  $G^{(2m)}$  of order  $2m$  with the property that all of its cycles are of length at least  $g$ .*

We prove this theorem for a fixed  $g$  using induction on  $k$ . For  $k = 2$ , everything is trivial,  $G^{(2m)}$  is simply the cycle of length  $2m$  ( $2m > g$  is clear). Let us assume now that our theorem holds for  $k-1$ ; we want to prove it for  $k$ . From now on, let  $G^{(n)}(k, g)$  denote a  $k$ -regular graph of girth  $\geq g$  with exactly  $n$  vertices. By our induction hypothesis, there exists a graph  $G^{(2m)}(k-1, g)$ . Let  $G^{(2m)}$  now be a graph with the following three properties:

- (I)  $k-1 \leq v(x) \leq k$  for the degree  $v(x)$  of all the  $2m$  vertices
- (II) all cycles have length at least  $g$
- (III)  $G^{(2m)}$  has the maximal number of edges among all the graphs satisfying (I) and (II).



As  $G^{(2m)}(k-1, g)$  satisfies (I) and (II), it is clear that  $G^{(2m)}$  exists. We intend to show that the degrees of all the vertices of  $G^{(2m)}$  are exactly  $k$ , hence  $G^{(2m)}$  is  $k$ -regular, which will complete the proof.

First, we want to show that  $G^{(2m)}$  contains at most one vertex of degree  $< k$ . Since this proof is not at all easy, let us first show that this result already implies the  $k$ -regularity of  $G^{(2m)}$ . Because of (I), the exceptional vertex must be of degree  $k-1$ . But that is impossible, due to the well-known fact that the number of vertices of odd degree must be even, and an even  $k$  would force the existence of exactly one vertex of odd degree, while an odd  $k$  would force the existence of exactly  $2m-1$  such vertices; it therefore follows that all the vertices are of degree  $k$  and the proof of our theorem is completed.

Hence, it remains to prove that the existence of two vertices  $x_1$  and  $x_2$  in  $G^{(2m)}$  of degree  $< k$  (i.e., of degree  $k-1$ ) leads to a contradiction. Let  $N(x_i, g-2)$  stands for the set of vertices in  $G^{(2m)}$  whose distance from  $x_i$  is at most  $g-2$ . Then we claim:

**Lemma 1** *The set of all the vertices in  $G^{(2m)}$  of degree less than  $k$  is contained in  $N(x_1, g-2) \cap N(x_2, g-2)$ .*

It is enough to show this for  $N(x_1, g-2)$ . If there existed an  $x \notin N(x_1, g-2)$  of degree  $< k$ , the graph  $G^{(2m)} + (x_1, x)$  (i.e.,  $G^{(2m)}$  with an added edge) would obviously satisfy conditions (I) and (II). Property (I) follows because the degree of both  $x_1$  and  $x$  is assumed to be less than  $k$ , and property (II) follows because  $x \notin N(x_1, g-2)$ . This would however contradict the maximality property (III) of  $G^{(2m)}$ , and so Lemma 1 is proved.

**Lemma 2** *Let  $x$  be a vertex of  $G^{(2m)}$  of degree  $< k$ . Then*

$$|N(x, r)| \leq \sum_{t=0}^r (k-1)^t. \quad (6)$$

Since the degree of  $x$  is smaller than  $k$ , (6) follows for  $r = 1$ . The rest of the proof for  $r > 1$  follows from a simple induction on  $r$  using the fact that the degree of all the vertices is at most  $k$ .

Combining Lemmata 1 and 2, we obtain

$$|N(x_1, g-2) \cup N(x_2, g-2)| \leq m, \quad (7)$$

as it follows obviously from (6) and Lemma 1 that

$$\begin{aligned} & |N(x_1, g-2) \cup N(x_2, g-2)| = \\ & = |N(x_1, g-2)| + |N(x_2, g-2)| - |N(x_1, g-2) \cap N(x_2, g-2)| \leq \\ & \leq 2 \sum_{t=0}^{g-2} (k-1)^t - 2 \leq m \end{aligned}$$

(because of Lemma 1 and because the degrees of  $x_1$  and  $x_2$  are smaller than  $k$ ,  $|N(x_1, g-2) \cap N(x_2, g-2)| \geq 2$ ). This proves (7).

Now, let  $x_1, \dots, x_p$  be the vertices contained in  $N(x_1, g-2) \cup N(x_2, g-2)$ , and  $y_1, \dots, y_{2m-p}$  be the remaining vertices of  $G^{(2m)}$ . It follows from (7) that

$$2m - p \geq p \quad (8)$$

We want to show now that at least two of the  $y_j$ 's are joined by an edge. First, because of Lemma 1, the degree of all the vertices  $y_j$  is  $k$ . If no two of the vertices  $y_j$  were connected through an edge, there would have to be  $k(2m-p)$  edges connecting the  $y_j$ 's to the  $x_i$ 's. However, due to (8) and the fact that the degrees of all the  $(2m-p)$  vertices  $y_j$  is  $k$ , this would force the degrees of  $x_i$ 's to be equal to  $k$  as well; a contradiction.

We may assume without loss of generality that  $G^{(2m)}$  contains the edge  $y_1y_2$ . Let us consider now the graph  $(G^{(2m)} - y_1y_2 + x_1y_1 + x_2y_2) = \overline{G}^{(2m)}$  (the edges  $x_1y_1$  and  $x_2y_2$  do not belong to  $G^{(2m)}$ ). We claim that  $\overline{G}^{(2m)}$  satisfies (I) and (II). Clearly, (I) holds true. If  $\overline{G}^{(2m)}$  contained a cycle of length less than  $g$ , this cycle would have to contain one (or both) of the edges  $x_1y_1, x_2y_2$ . Because of the way the vertices  $y_j$  were defined

$$\begin{aligned} e(G^{(2m)}; x_1, y_1) &\geq g-1 \\ e(G^{(2m)}; x_2, y_2) &\geq g-1 \\ e(G^{(2m)} - (y_1, y_2); y_1, y_2) &\geq g-1 \end{aligned}$$

where  $e(G^{(2m)}; x_i, y_i)$  stands for the distance between  $x_i$  and  $y_i$  in  $G^{(2m)}$ , and the last inequality follows from the fact that  $G^{(2m)}$  does not contain cycles of length less than  $g$ .

It follows easily from these inequalities that  $\overline{G}^{(2m)}$  satisfies (II). That, however, contradicts (III), as  $\overline{G}^{(2m)}$  has more edges than  $G^{(2m)}$ .

Hence,  $G^{(2m)}$  can contain at most one vertex of degree  $< k$ , and that completes the proof of Theorem 2.

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