Notes on Nonrepetitive Graph Colouring

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Abstract

A vertex colouring of a graph is *nonrepetitive on paths* if there is no path $v_1, v_2, \dots, v_{2t}$ such that v_i and v_{t+i} receive the same colour for all $i = 1, 2, \dots, t$. We determine the maximum density of a graph that admits a k-colouring that is nonrepetitive on paths. We prove that every graph has a subdivision that admits a 4-colouring that is nonrepetitive on paths. The best previous bound was 5. We also study colourings that are nonrepetitive on walks, and provide a conjecture that would imply that every graph with maximum degree Δ has a $f(\Delta)$-colouring that is nonrepetitive on walks. We prove that every graph with treewidth k and maximum degree Δ has a $O(k\Delta)$-colouring that is nonrepetitive on paths, and a $O(k\Delta^3)$-colouring that is nonrepetitive on walks.

1 Introduction

We consider simple, finite, undirected graphs G with vertex set $V(G)$, edge set $E(G)$, and maximum degree $\Delta(G)$. Let $[t] := \{1, 2, \dots, t\}$. A *walk* in G is a sequence $v_1, v_2, \dots, v_t$ of vertices of G, such that $v_i v_{i+1} \in E(G)$ for all $i \in [t-1]$. A *k-colouring* of G is a function f that assigns one of k colours to each vertex of G. A walk $v_1, v_2, \dots, v_{2t}$ is *repetitively* coloured by f if $f(v_i) = f(v_{t+i})$ for all $i \in [t]$. A walk $v_1, v_2, \dots, v_{2t}$ is *boring* if $v_i = v_{t+i}$ for all $i \in [t]$. Of course, a boring walk is repetitively coloured by every colouring. We say a colouring f is *nonrepetitive on walks* (or *walk-nonrepetitive*) if the only walks that

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are repetitively coloured by f are boring. Let $\sigma(G)$ denote the minimum k such that G has a k-colouring that is nonrepetitive on walks.

A walk $v_1, v_2, \dots, v_t$ is a *path* if $v_i \neq v_j$ for all distinct $i, j \in [t]$. A colouring f is *nonrepetitive on paths* (or *path-nonrepetitive*) if no path of G is repetitively coloured by f. Let $\pi(G)$ denote the minimum k such that G has a k-colouring that is nonrepetitive on paths. Observe that a colouring that is path-nonrepetitive is *proper*, in the sense that adjacent vertices receive distinct colours. Moreover, a path-nonrepetitive colouring has no 2-coloured P_4 (a path on four vertices). A proper colouring with no 2-coloured P_4 is called a *star colouring* since each bichromatic subgraph is a star forest; see [1, 8, 17, 18, 25, 28]. The *star chromatic number* $\chi_{\mathrm{st}}(G)$ is the minimum number of colours in a proper colouring of G with no 2-coloured P_4. Thus

$$\chi(G) \leqslant \chi_{\mathrm{st}}(G) \leqslant \pi(G) \leqslant \sigma(G). \tag{1}$$

Path-nonrepetitive colourings are widely studied [2–5, 9, 10, 12, 13, 19, 21, 23, 24]; see the surveys by Grytczuk [20, 22]. Nonrepetitive edge colourings have also been considered [4, 6].

The seminal result in this field is by Thue [27], who in 1906 proved[1] that the n-vertex path P_n satisfies

$$\pi(P_n) = \begin{cases} n & \text{if } n \leqslant 2, \\ 3 & \text{otherwise.} \end{cases} \tag{2}$$

A result by Kündgen and Pelsmajer [23] (see Lemma 3.4) implies

$$\sigma(P_n) \leqslant 4\ . \tag{3}$$

Currie [11] proved that the n-vertex cycle C_n satisfies

$$\pi(C_n) = \begin{cases} 4 & \text{if } n \in \{5, 7, 9, 10, 14, 17\}, \\ 3 & \text{otherwise.} \end{cases} \tag{4}$$

Let $\pi(\Delta)$ and $\sigma(\Delta)$ denote the maximum of $\pi(G)$ and $\sigma(G)$, taken over all graphs G with maximum degree $\Delta(G) \leqslant \Delta$. Now $\pi(2) = 4$ by (2) and (4). In general, Alon et al. [4] proved that

$$\frac{\alpha \Delta^2}{\log \Delta} \leqslant \pi(\Delta) \leqslant \beta \Delta^2, \tag{5}$$

for some constants α and β. The upper bound was proved using the Lovász Local Lemma, and the lower bound is attained by a random graph.

In Section 2 we study whether $\sigma(\Delta)$ is finite, and provide a natural conjecture that would imply an affirmative answer.

[1] The nonrepetitive 3-colouring of P_n by Thue [27] is obtained as follows. Given a nonrepetitive sequence over $\{1, 2, 3\}$, replace each 1 by the sequence 12312, replace each 2 by the sequence 131232, and replace each 3 by the sequence 1323132. Thue [27] proved that the new sequence is nonrepetitive. Thus arbitrarily long paths can be nonrepetitively 3-coloured.

In Section 3 we study path- and walk-nonrepetitive colourings of graphs of bounded treewidth[2]. Kündgen and Pelsmajer [23] and Barát and Varjú [5] independently proved that graphs of bounded treewidth have bounded π. The best bound is due to Kündgen and Pelsmajer [23] who proved that $\pi(G) \leqslant 4^k$ for every graph G with treewidth at most k. Whether there is a polynomial bound on π for graphs of treewidth k is an open question. We answer this problem in the affirmative under the additional assumption of bounded degree. In particular, we prove a $\mathcal{O}(k\Delta)$ upper bound on π, and a $\mathcal{O}(k\Delta^3)$ upper bound on σ.

In Section 4 we will prove that every graph has a subdivision that admits a path-nonrepetitive 4-colouring; the best previous bound was 5. In Section 5 we determine the maximum density of a graph that admits a path-nonrepetitive k-colouring, and prove bounds on the maximum density for walk-nonrepetitive k-colourings.

2 Is $\sigma(\Delta)$ bounded?

Consider the following elementary lower bound on σ, where G^2 is the *square* graph of G. That is, $V(G^2) = V(G)$, and $vw \in E(G^2)$ if and only if the distance between v and w in G is at most 2. A proper colouring of G^2 is called a *distance*-2 colouring of G.

Lemma 2.1. *Every walk-nonrepetitive colouring of a graph G is distance-2. Thus $\sigma(G) \geqslant \chi(G^2) \geqslant \Delta(G) + 1$.*

Proof. Consider a walk-nonrepetitive colouring of G. Adjacent vertices v and w receive distinct colours, as otherwise v, w would be a repetitively coloured path. If u, v, w is a path, and u and w receive the same colour, then the non-boring walk u, v, w, v is repetitively coloured. Thus vertices at distance at most 2 receive distinct colours. Hence $\sigma(G) \geqslant \chi(G^2)$. In a distance-2 colouring, each vertex and its neighbours all receive distinct colours. Thus $\chi(G^2) \geqslant \Delta(G) + 1$. □

Hence $\Delta(G)$ is a lower bound on $\sigma(G)$. Whether high degree is the only obstruction for bounded σ is an open problem.

Open Problem 2.2. Is there a function f such that $\sigma(\Delta) \leqslant f(\Delta)$?

First we answer Open Problem 2.2 in the affirmative for $\Delta = 2$. The following lemma will be useful.

Lemma 2.3. *Fix a distance-2 colouring of a graph G. If $W = (v_1, v_2, \dots, v_{2t})$ is a repetitively coloured non-boring walk in G, then $v_i \neq v_{t+i}$ for all $i \in [t]$.*

Proof. Suppose on the contrary that $v_i = v_{t+i}$ for some $i \in [t-1]$. Since W is repetitively coloured, $c(v_{i+1}) = c(v_{t+i+1})$. Each neighbour of v_i receives a distinct colour. Thus $v_{i+1} = v_{t+i+1}$. By induction, $v_j = v_{t+j}$ for all $j \in [i, t]$. By the same argument, $v_j = v_{t+j}$ for all $j \in [1, i]$. Thus W is boring, which is the desired contradiction. □

[2] The *treewidth* of a graph G can be defined to be the minimum integer k such that G is a subgraph of a chordal graph with no clique on $k+2$ vertices. Treewidth is an important graph parameter, especially in structural graph theory and algorithmic graph theory; see the surveys [7, 26].

Proposition 2.4. $\sigma(2) \leqslant 5$.

Proof. A result by Kündgen and Pelsmajer [23] implies that $\sigma(P_n) \leqslant 4$ (see Lemma 3.4). Thus it suffices to prove that $\sigma(C_n) \leqslant 5$. Fix a walk-nonrepetitive 4-colouring of the path $(v_1, v_2, \dots, v_{2n-4})$. Thus for some $i \in [1, n-2]$, the vertices v_i and v_{n+i-2} receive distinct colours. Create a cycle C_n from the sub-path $v_i, v_{i+1}, \dots, v_{n+i-2}$ by adding one vertex x adjacent to v_i and v_{n+i-2}. Colour x with a fifth colour. Observe that since v_i and v_{n+i-2} receive distinct colours, the colouring of C_n is distance-2. Suppose on the contrary that C_n has a repetitively coloured walk $W = y_1, y_2, \dots, y_{2t}$. If x is not in W, then W is a repetitively coloured walk in the starting path, which is a contradiction. Thus $x = y_i$ for some $i \in [t]$ (with loss of generality, by considering the reverse of W). Since x is the only vertex receiving the fifth colour and W is repetitive, $x = y_{t+i}$. By Lemma 2.3, W is boring. Hence the 5-colouring of C_n is walk-nonrepetitive. □

Below we propose a conjecture that would imply a positive answer to Open Problem 2.2. First consider the following lemma which is a slight generalisation of a result by Barát and Varjú [6]. A walk $v_1, v_2, \dots, v_t$ has *length* t and *order* $|\{v_i : 1 \leqslant i \leqslant t\}|$. That is, the order is the number of distinct vertices in the walk.

Proposition 2.5. *Suppose that in some coloured graph, there is a repetitively coloured non-boring walk. Then there is a repetitively coloured non-boring walk of order k and length at most $2k^2$.*

Proof. Let k be the minimum order of a repetitively coloured non-boring walk. Let $W = v_1, v_2, \dots, v_{2t}$ be a repetitively coloured non-boring walk of order k and with t minimum. If $2t \leqslant 2k^2$, then we are done. Now assume that $t > k^2$. By the pigeonhole principle, there is a vertex x that appears at least $k+1$ times in $v_1, v_2, \dots, v_t$. Thus there is a vertex y that appears at least twice in the set $\{v_{t+i} : v_i = x, i \in [t]\}$. As illustrated in Figure 1, $W = AxBxCA'yB'yC'$ for some walks A, B, C, A', B', C' with $|A| = |A'|$, $|B| = |B'|$, and $|C| = |C'|$. Consider the walk $U := AxCA'yC'$. If U is not boring, then it is a repetitively coloured non-boring walk of order at most k and length less than $2t$, which contradicts the minimality of W. Otherwise U is boring, implying $x = y$, $A = A'$, and $C = C'$. Thus $B \neq B'$ since W is not boring, implying $xBxB'$ is a repetitively coloured non-boring walk of order at most k and length less than $2t$, which contradicts the minimality of W. □

We conjecture the following strengthening of Proposition 2.5.

Conjecture 2.6. *Let G be a graph. Consider a path-nonrepetitive distance-2 colouring of G with c colours, such that G contains a repetitively coloured non-boring walk. Then G contains a repetitively coloured non-boring walk of order k and length at most $h(c) \cdot k$, for some function h that only depends on c.*

Theorem 2.7. *If Conjecture 2.6 is true, then there is a function f for which $\sigma(\Delta) \leqslant f(\Delta)$. That is, every graph G has a walk-nonrepetitive colouring with $f(\Delta(G))$ colours.*

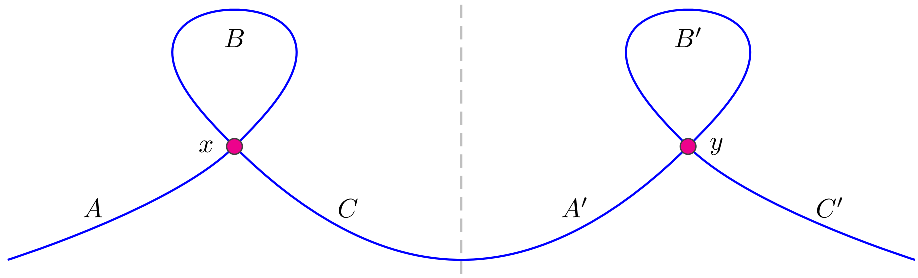

Figure 1: Illustration for the proof of Proposition 2.5.

Theorem 2.7 is proved using the Lovász Local Lemma [16].

Lemma 2.8 ([16])**.** *Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_r$ be a partition of a set of 'bad' events $\mathcal{A}$. Suppose that there are sets of real numbers $\{p_i \in [0, 1) : i \in [r]\}$, $\{x_i \in [0, 1) : i \in [r]\}$, and $\{D_{ij} \geqslant 0 : i, j \in [r]\}$ such that the following conditions are satisfied by every event $A \in \mathcal{A}_i$:*

- *the probability* $\mathbf{P}(A) \ \leqslant \ p_i \ \leqslant \ x_i \prod_{j=1}^{r} (1 - x_j)^{D_{ij}}$ *, and*
- *A is mutually independent of $\mathcal{A} \setminus (\{A\} \cup \mathcal{D}_A)$, for some $\mathcal{D}_A \subseteq \mathcal{A}$ with $|\mathcal{D}_A \cap \mathcal{A}_j| \leqslant D_{ij}$ for all $j \in [r]$.*

Then

$$\mathbf{P}\left(\bigwedge_{A \in \mathcal{A}} \overline{A} \right) \ \geqslant \ \prod_{i=1}^{r} (1 - x_i)^{|\mathcal{A}_i|} \ > \ 0 \ .$$

That is, with positive probability, no event in $\mathcal{A}$ occurs.

Proof of Theorem 2.7. Let f_1 be a path-nonrepetitive colouring of G with $\pi(G)$ colours. Let f_2 be a distance-2 colouring of G with $\chi(G^2)$ colours. Note that $\pi(G) \leqslant \beta\Delta^2$ for some constant β by Equation (5), and $\chi(G^2) \leqslant \Delta(G^2) + 1 \leqslant \Delta^2 + 1$ by a greedy colouring of G^2. Hence f_1 and f_2 together define a path-nonrepetitive distance-2 colouring of G. The number of colours $\pi(G) \cdot \chi(G^2)$ is bounded by a function solely of $\Delta(G)$. Consider this initial colouring to be fixed. Let c be a positive integer to be specified later. For each vertex v of G, choose a third colour $f_3(v) \in [c]$ independently and randomly. Let f be the colouring defined by $f(v) = (f_1(v), f_2(v), f_3(v))$ for all vertices v.

Let $h := h(\pi(G) \cdot \chi(G^2))$ from Conjecture 2.6. A non-boring walk $v_1, v_2, \dots, v_{2t}$ of order i is *interesting* if its length $2t \leqslant hi$, and $f_1(v_j) = f_1(v_{t+j})$ and $f_2(v_j) = f_2(v_{t+j})$ for all $j \in [t]$. For each interesting walk W, let A_W be the event that W is repetitively coloured by f. Let $\mathcal{A}_i$ be the set of events A_W, where W is an interesting walk of order i. Let $\mathcal{A} = \bigcup_i \mathcal{A}_i$.

We will apply Lemma 2.8 to prove that, with positive probability, no event A_W occurs. This will imply that there exists a colouring f_3 such that no interesting walk is repetitively coloured by f. A non-boring non-interesting walk $v_1, v_2, \dots, v_{2t}$ of order i satisfies (a) $2t > hi$, or (b) $f_1(v_j) \neq f_1(v_{t+j})$ or $f_2(v_j) \neq f_2(v_{t+j})$ for some $j \in [t]$. In case (a), by the assumed truth of Conjecture 2.6, W is not repetitively coloured by f. In case (b), $f(v_j) \neq f(v_{t+j})$ and W is not repetitively coloured by f. Thus no non-boring walk is repetitively coloured by f, as desired.

Consider an interesting walk $W = v_1, v_2, \dots, v_{2t}$ of order i.

We claim that $v_\ell \neq v_{t+\ell}$ for all $\ell \in [t]$. Suppose on the contrary that $v_\ell = v_{t+\ell}$ for some $\ell \in [t]$. Since W is not boring, $v_j \neq v_{t+j}$ for some $j \in [t]$. Thus $v_j = v_{t+j}$ and $v_{j+1} \neq v_{t+j+1}$ for some $j \in [t]$ (where v_{t+t+1} means v_1). Since W is interesting, $f_2(v_{j+1}) = f_2(v_{t+j+1})$, which is a contradiction since v_{j+1} and v_{t+j+1} have a common neighbour v_j $(= v_{t+j})$. Thus $v_j \neq v_{t+j}$ for all $j \in [t]$, as claimed.

This claim implies that for each of the i vertices x in W, there is at least one other vertex y in W, such that $f_3(x) = f_3(y)$ must hold for W to be repetitively coloured. Hence at most $c^{i/2}$ of the c^i possible colourings of W under f_3, lead to repetitive colourings of W under f. Thus the probability $\mathbf{P}(A_W) \leqslant p_i := c^{-i/2}$, and Lemma 2.8 can be applied as long as

$$c^{-i/2} \;\leqslant\; x_i \prod_j (1 - x_j)^{D_{ij}} \quad , \tag{6}$$

Every vertex is in at most $hj\Delta^{hj}$ interesting walks of order j. Thus an interesting walk of order i shares a vertex with at most $hij\Delta^{hj}$ interesting walks of order j. Thus we can take $D_{ij} := hij\Delta^{hj}$. Define $x_i := (2\Delta^h)^{-i}$. Note that $x_i \leqslant \frac{1}{2}$. So $1 - x_i \geqslant \mathbf{e}^{-2x_i}$. Thus to prove (6) it suffices to prove that

$$\begin{aligned}
& c^{-i/2} &&\leqslant\; x_i \prod_j \mathbf{e}^{-2x_j D_{ij}} \quad , \\
\Longleftarrow\quad & c^{-i/2} &&\leqslant\; (2\Delta^h)^{-i} \prod_j \mathbf{e}^{-2(2\Delta^h)^{-j} hij\Delta^{hj}} \quad , \\
\Longleftarrow\quad & c^{-1/2} &&\leqslant\; (2\Delta^h)^{-1} \prod_j \mathbf{e}^{-2(2)^{-j} hj} \quad , \\
\Longleftarrow\quad & c^{-1/2} &&\leqslant\; (2\Delta^h)^{-1} \mathbf{e}^{-2h\sum_j j2^{-j}} \quad , \\
\Longleftarrow\quad & c^{-1/2} &&\leqslant\; (2\Delta^h)^{-1} \mathbf{e}^{-4h} \quad , \\
\Longleftarrow\quad & c &&\geqslant\; 4(\mathbf{e}^4\Delta)^{2h} \quad .
\end{aligned}$$

Choose c to be the minimum integer that satisfies this inequality, and the lemma is applicable. We obtain a c-colouring f_3 of G such that f is nonrepetitive on walks. The number of colours in f is at most $h\lceil 4(\mathbf{e}^4\Delta)^{2h}\rceil$, which is a function solely of Δ. $\square$

3 Trees and Treewidth

We start this section by considering walk-nonrepetitive colourings of trees.

Theorem 3.1. *Let T be a tree. A colouring c of T is walk-nonrepetitive if and only if c is path-nonrepetitive and distance-2.*

Proof. For every graph, every walk-nonrepetitive colouring is path-nonrepetitive (by definition) and distance-2 (by Lemma 2.1).

Now fix a path-nonrepetitive distance-2 colouring c of T. Suppose on the contrary that T has a repetitively coloured non-boring walk. Let $W = (v_1, v_2, \dots, v_{2t})$ be a repetitively coloured non-boring walk in T of minimum length. Some vertex is repeated in W, as otherwise W would be a repetitively coloured path. By considering the reverse of W, without loss of generality, $v_i = v_j$ for some $i \in [1, t-1]$ and $j \in [i+2, 2t]$. Choose i and j to minimise $j-i$. Thus v_i is not in the sub-walk $(v_{i+1}, v_{i+2}, \dots, v_{j-1})$. Since T is a tree, $v_{i+1} = v_{j-1}$. Thus $i+1 = j-1$, as otherwise $j-i$ is not minimised. That is, $v_i = v_{i+2}$. Assuming $i \neq t-1$, since W is repetitively coloured, $c(v_{t+i}) = c(v_{t+i+2})$, which implies that $v_{t+i} = v_{t+i+2}$ because c is a distance-2 colouring. Thus, even if $i = t-1$, deleting the vertices $v_i, v_{i+1}, v_{t+i}, v_{t+i+1}$ from W, gives a walk $(v_1, v_2, \dots, v_{i-1}, v_{i+2}, \dots, v_{t+i-1}, v_{t+i+2}, \dots, v_{2t})$ that is also repetitively coloured. This contradicts the minimality of the length of W. $\square$

Note that Theorem 3.1 implies that Conjecture 2.6 is vacuously true for trees. Also, since every tree T has a path-nonrepetitive 4-colouring [23] and a distance-2 $(\Delta(T)+1)$-colouring, Theorem 3.1 implies the following result, where the lower bound is Lemma 2.1.

Corollary 3.2. *Every tree T satisfies $\Delta(T) + 1 \leqslant \sigma(T) \leqslant 4(\Delta(T) + 1)$.*

In the remainder of this section we prove the following polynomial upper bounds on π and σ in terms of the treewidth and maximum degree of a graph.

Theorem 3.3. *Every graph G with treewidth k and maximum degree $\Delta \geqslant 1$ satisfies $\pi(G) \leqslant ck\Delta$ and $\sigma(G) \leqslant ck\Delta^3$ for some constant c.*

We prove Theorem 3.3 by a series of lemmas. The first is by Kündgen and Pelsmajer [23][3].

Lemma 3.4 ([23])**.** *Let P^+ be the pseudograph obtained from a path P by adding a loop at each vertex. Then $\sigma(P^+) \leqslant 4$.*

Now we introduce some definitions by Kündgen and Pelsmajer [23]. A *levelling* of a graph G is a function $\lambda : V(G) \to \mathbb{Z}$ such that $|\lambda(v) - \lambda(w)| \leqslant 1$ for every edge $vw \in E(G)$. Let $G_{\lambda=k}$ and $G_{\lambda>k}$ denote the subgraphs of G respectively induced by $\{v \in V(G) : \lambda(v) = k\}$ and $\{v \in V(G) : \lambda(v) > k\}$. The *$k$-shadow* of a subgraph H of G is the set of vertices in $G_{\lambda=k}$ adjacent to some vertex in H. A levelling λ is *shadow-complete* if the k-shadow of every component of $G_{\lambda>k}$ induces a clique. Kündgen and Pelsmajer [23] proved the following lemma for repetitively coloured paths. We show that the same proof works for repetitively coloured walks.

[3]The 4-colouring in Lemma 3.4 is obtained as follows. Given a nonrepetitive sequence on $\{1, 2, 3\}$, insert the symbol 4 between consecutive block of length two. For example, from the sequence 123132123 we obtain 1243143241243.

Lemma 3.5. *For every levelling λ of a graph G, there is a 4-colouring of G, such that every repetitively coloured walk $v_1, v_2, \dots, v_{2t}$ satisfies $\lambda(v_j) = \lambda(v_{t+j})$ for all $j \in [t]$.*

Proof. The levelling λ can be thought of as a homomorphism from G into P^+, for some path P. By Lemma 3.4, P^+ has a 4-colouring that is nonrepetitive on walks. Colour each vertex v of G by the colour assigned to $\lambda(v)$ (thought of as a vertex of P^+). Suppose $v_1, v_2, \dots, v_{2t}$ is a repetitively coloured walk in G. Thus $\lambda(v_1), \lambda(v_2), \dots, \lambda(v_{2t})$ is a repetitively coloured walk in P^+. Since the 4-colouring of P^+ is nonrepetitive on walks, $\lambda(v_1), \lambda(v_2), \dots, \lambda(v_{2t})$ is boring. That is, $\lambda(v_j) = \lambda(v_{t+j})$ for all $j \in [t]$. □

Lemma 3.6 ([23])**.** *If λ is a shadow-complete levelling of a graph G, then*

$$\pi(G) \leqslant 4 \cdot \max_k \pi(G_{\lambda=k}).$$

Now we generalise Lemma 3.6 for walks.

Lemma 3.7. *If H is a subgraph of a graph G, and λ is a shadow-complete levelling of G, then*

$$\sigma(H) \;\leqslant\; 4\,\chi(H^2) \cdot \max_k \sigma(G_{\lambda=k}) \;\leqslant\; 4(\Delta(H)^2+1) \cdot \max_k \sigma(G_{\lambda=k}).$$

Proof. Let c_1 be the 4-colouring of G from Lemma 3.5. Let c_2 be an optimal walk-nonrepetitive colouring of each level $G_{\lambda=k}$. Let c_3 be a proper $\chi(H^2)$-colouring of H^2. The second inequality in the lemma follows from the first since $\chi(H^2) \leqslant \Delta(H)^2 + 1$. Let $c(v) := (c_1(v), c_2(v), c_3(v))$ for each vertex v of H. We claim that c is nonrepetitive on walks in H.

Suppose on the contrary that $W = v_1, \dots, v_{2t}$ is a non-boring walk in H that is repetitively coloured by c. Then W is repetitively coloured by each of c_1, c_2, and c_3. Thus $\lambda(v_i) = \lambda(v_{t+i})$ for all $i \in [t]$ by Lemma 3.5. Let W_k be the sequence (allowing repetitions) of vertices $v_i \in W$ such that $\lambda(v_i) = k$. Since $v_i \in W_k$ if and only if $v_{t+i} \in W_k$, each sequence W_k is repetitively coloured. That is, if $W_k = x_1, \dots, x_{2s}$ then $c(x_i) = c(x_{s+i})$ for all $i \in [s]$.

Let k be the minimum level containing a vertex in W. Let v_i and v_j be consecutive vertices in W_k with $i < j$. If $j = i + 1$ then $v_i v_j$ is an edge of W. Otherwise there is walk from v_i to v_j in $G_{\lambda>k}$ (since k was chosen minimum), implying $v_i v_j$ is an edge of G (since λ is shadow-complete). Thus W_k forms a walk in $G_{\lambda=k}$ that is repetitively coloured by c_2. Hence W_k is boring. In particular, some vertex $v_i = v_{t+i}$ is in W_k. Since W is not boring, $v_j \neq v_{t+j}$ for some $j \in [t]$. Without loss of generality, $i < j$ and $v_\ell = v_{t+\ell}$ for all $\ell \in [i, j-1]$. Thus v_j and v_{t+j} have a common neighbour $v_{j-1} = v_{t+j-1}$ in H, which implies that $c_3(v_j) \neq c_3(v_{t+j})$. But $c(v_j) = c(v_{t+j})$ since W is repetitively coloured, which is the desired contradiction. □

Note that some dependence on $\Delta(H)$ in Lemma 3.7 is unavoidable, since $\sigma(H) \geqslant \chi(H^2) \geqslant \Delta(H) + 1$.

Lemma 3.7 enables the following strengthening of Corollary 3.2.

Lemma 3.8. *Every tree T satisfies $\Delta(T)+1 \leqslant \sigma(T) \leqslant 4\,\Delta(T)$.*

Proof. Let r be a leaf vertex of T. Let $\lambda(v)$ be the distance from r to v in T. Then λ is a shadow-complete levelling of T in which each level is an independent set. A greedy algorithm proves that $\chi(T^2) \leqslant \Delta(T)+1$. Thus Lemma 3.7 implies that $\sigma(T) \leqslant 4\,\Delta(T)+4$. Observe that the proof of Lemma 3.7 only requires $c_3(v) \neq c_3(w)$ whenever v and w are in the same level and have a common parent. Since r is a leaf, each vertex has at most $\Delta(T)-1$ children. Thus a greedy algorithm produces a $\Delta(T)$-colouring with this property. Hence $\sigma(T) \leqslant 4\,\Delta(T)$. □

A *tree-partition* of a graph G is a partition of its vertices into sets (called *bags*) such that the graph obtained from G by identifying the vertices in each bag is a forest (after deleting loops and replacing parallel edges by a single edge)[4].

Lemma 3.9. *Let G be a graph with a tree-partition in which every bag has at most ℓ vertices. Then G is a subgraph of a graph G' that has a shadow-complete levelling in which each level satisfies*

$$\pi(G'_{\lambda=k}) \leqslant \sigma(G'_{\lambda=k}) \leqslant \ell.$$

Proof. Let G' be the graph obtained from G by adding an edge between all pairs of nonadjacent vertices in a common bag. Let F be the forest obtained from G' by identifying the vertices in each bag. Root each component of F. Consider a vertex v of G' that is in the bag that corresponds to node x of F. Let $\lambda(v)$ be the distance between x and the root of the tree component of F that contains x. Clearly λ is a levelling of G'. The k-shadow of each connected component of $G'_{\lambda>k}$ is contained in a single bag, and thus induces a clique on at most ℓ vertices. Hence λ is shadow-complete. By colouring the vertices within each bag with distinct colors, we have $\pi(G'_{\lambda=k}) \leqslant \sigma(G'_{\lambda=k}) \leqslant \ell$. □

Lemmas 3.6, 3.7 and 3.9 imply:

Lemma 3.10. *If a graph G has a tree-partition in which every bag has at most ℓ vertices, then $\pi(G) \leqslant 4\ell$ and $\sigma(G) \leqslant 4\ell(\Delta(G)^2+1)$.*

Wood [30] proved[5] that every graph with treewidth k and maximum degree $\Delta \geqslant 1$ has a tree-partition in which every bag has at most $\frac{5}{2}(k+1)(\frac{7}{2}\Delta-1)$ vertices. With Lemma 3.10 this proves the following quantitative version of Theorem 3.3.

Theorem 3.11. *Every graph G with treewidth k and maximum degree $\Delta \geqslant 1$ satisfies $\pi(G) \leqslant 10(k+1)(\frac{7}{2}\Delta-1)$ and $\sigma(G) \leqslant 10(k+1)(\frac{7}{2}\Delta-1)(\Delta^2+1)$.*

[4]The proof by Kündgen and Pelsmajer [23] that $\pi(G) \leqslant 4^k$ for graphs with treewidth at most k can also be described using tree-partitions; cf. [15, 29].

[5]The proof by Wood [30] is a minor improvement to a similar result by an anonymous referee of the paper by Ding and Oporowski [14].

4 Subdivisions

The results of Thue [27] and Currie [11] imply that every path and every cycle has a subdivision H with $\pi(H) = 3$. Brešar et al. [9] proved that every tree has a subdivision H such that $\pi(H) = 3$. Which graphs have a subdivision H with $\pi(H) = 3$ is an open problem [20]. Grytczuk [20] proved that every graph has a subdivision H with $\pi(H) \leqslant 5$. Here we improve this bound as follows.

Theorem 4.1. *Every graph G has a subdivision H with $\pi(H) \leqslant 4$.*

Proof. Without loss of generality G is connected. Say $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$. As illustrated in Figure 2, let H be the subdivision of G obtained by subdividing every edge $v_i v_j \in E(G)$ (with $i < j$) $j - i - 1$ times. The distance of every vertex in H from v_0 defines a levelling of H such that the endpoints of every edge are in consecutive levels. By Lemma 3.5, there is a 4-colouring of H, such that for every repetitively coloured path $x_1, x_2, \dots, x_t, y_1, y_2, \dots, y_t$ in H, x_j and y_j have the same level for all $j \in [t]$. Hence there is some j such that x_{j-1} and x_{j+1} are at the same level. Thus x_j is an original vertex v_i of G. Without loss of generality x_{j-1} and x_{j+1} are at level $i-1$. There is only one original vertex at level i. Thus y_j, which is also at level i, is a division vertex. Now y_j has two neighbours in H, which are at levels $i-1$ and $i+1$. Thus y_{j-1} and y_{j+1} are at levels $i-1$ and $i+1$, which contradicts the fact that x_{j-1} and x_{j+1} are both at level $i-1$. Hence we have a 4-colouring of H that is nonrepetitive on paths. $\square$

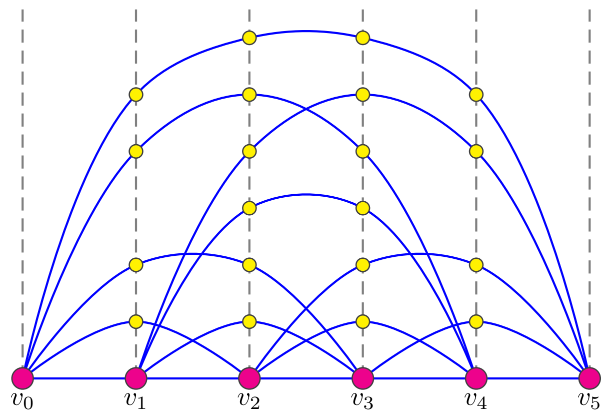

Figure 2: The subdivision H with $G = K_6$.

It is possible that every graph has a subdivision H with $\pi(H) \leqslant 3$. If true, this would provide a striking generalisation of the result of Thue [27] discussed in Section 1.

5 Maximum Density

In this section we study the maximum number of edges in a nonrepetitively coloured graph.

Proposition 5.1. *The maximum number of edges in an n-vertex graph G with $\pi(G) \leqslant c$ is $(c-1)n - \binom{c}{2}$.*

Proof. Say G is an n-vertex graph with $\pi(G) \leqslant c$. Fix a c-colouring of G that is nonrepetitive on paths. Say there are n_i vertices in the i-th colour class. Every cycle receives at least three colours. Thus the subgraph induced by the vertices coloured i and j is a forest, and has at most $n_i + n_j - 1$ edges. Hence the number of edges in G is at most

$$\sum_{1 \leqslant i < j \leqslant c} (n_i + n_j - 1) = \sum_{1 \leqslant i \leqslant c} (c-1)n_i - \binom{c}{2} = (c-1)n - \binom{c}{2}.$$

This bound is attained by the graph consisting of a complete graph K_{c-1} completely connected to an independent set of $n-(c-1)$ vertices, which obviously has a c-colouring that is nonrepetitive on paths. □

Now consider the maximum number of edges in a coloured graph that is nonrepetitive on walks. First note that the example in the proof of Proposition 5.1 is repetitive on walks. Since $\sigma(G) \geqslant \Delta(G) + 1$ and $|E(G)| \leqslant \frac{1}{2}\Delta(G)|V(G)|$, we have the trivial upper bound,

$$|E(G)| \leqslant \tfrac{1}{2}(\sigma(G) - 1)|V(G)|.$$

This bound is tight for $\sigma = 2$ (matchings) and $\sigma = 3$ (cycles), but is not known to be tight for $\sigma \geqslant 4$.

We have the following lower bound.

Proposition 5.2. *For all $p \geqslant 1$, there are infinitely many graphs G with $\sigma(G) \leqslant 4p$ and*

$$|E(G)| \geqslant \tfrac{1}{8}(3\sigma(G) - 4)|V(G)| - \tfrac{1}{9}\sigma(G)^2.$$

Proof. Let G be the lexicographic product of a path and K_p; that is, G is the graph with a levelling λ in which each level induces K_p, and every edge is present between consecutive levels. Let c_1 be the 4-colouring of G from Lemma 3.5. If v is the j-th vertex in its level, where $j \in [p]$, then let $c(v) := (c_1(v), j)$. The number of colours is $4p$. Applying Lemma 3.5, it is easily verified that c is nonrepetitive on walks. Hence $\sigma(G) \leqslant 4p$. Now we count the edges: $|E(G)| = \frac{1}{2}(3p-1)|V(G)| - p^2$. As a lower bound, $\sigma(G) \geqslant \Delta(G)+1 = 3p$. Thus $|E(G)| \geqslant \frac{1}{2}(3\sigma(G)/4 - 1)|V(G)| - (\sigma(G)/3)^2$. □

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