

Expansions of k -Schur functions in the affine nilCoxeter algebra

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Abstract

We give a type free formula for the expansion of k -Schur functions indexed by fundamental coweights within the affine nilCoxeter algebra. Explicit combinatorics are developed in affine type C .

1 Introduction

In [1], Berg, Bergeron, Thomas and Zabrocki gave several formulas for the expansion of certain k -Schur functions (indexed by fundamental weights) inside the affine nilCoxeter algebra of type A . In particular, they gave an explicit combinatorial description for the reduced words which appear in the expansion.

These coefficients have been studied extensively; they are the coefficients which appear in the product of two k -Schur functions. These functions have been identified with representing the homology of the affine Grassmannian in type A .

They verified their formula by identifying terms in the expansion of a k -Schur function with pseudo-translations (elements of the nilCoxeter algebra which act by translating alcoves in prescribed directions). This generalized Proposition 4.5 of Lam [4], where he gave formulas for k -Schur functions indexed by root translations.

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Since then, Lam and Shimozono [6] have discovered a type free analogue of this fact for k -Schur functions indexed by coweights. The main goal of this paper is to combine the new result of Lam and Shimozono with the techniques of [1] to give descriptions of the corresponding reduced words appearing in the decomposition of these k -Schur functions, with an emphasis on combinatorics.

Section 2 develops a type free formula for k -Schur functions indexed by special Grassmannian permutations, Section 3 focuses on the specific combinatorics of affine type C , and Section 4 discusses a few examples of the combinatorics in affine types B and D .

1.1 A brief introduction to root systems

1.1.1 Root systems

Let (I, A) be a Cartan datum, i.e., a finite index set I and a generalized Cartan matrix $A = (a_{ij} \mid i, j \in I)$ such that $a_{ii} = 2$ for all $i \in I$, $a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. If the corank of A is 1, then A is of *affine type*; in this case, we write $(I_{\text{af}}, A_{\text{af}})$, and let $I_{\text{af}} = \{0, 1, \dots, n\}$. From a Cartan datum of affine type we may recover a corresponding Cartan datum $(I_{\text{fin}}, A_{\text{fin}})$ of *finite type* by considering $I_{\text{fin}} = I_{\text{af}} \setminus \{0\}$. In general, we denote affine root system data with an “af” subscript, and finite root system data with a “fin” subscript. Root system data of arbitrary type has no subscript.

Also associated to a Cartan datum we have a root datum, which consists of a free \mathbb{Z} -module \mathfrak{h} , its dual lattice $\mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{Z})$, a pairing $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \rightarrow \mathbb{Z}$ given by $\langle \mu, \lambda \rangle = \lambda(\mu)$, and sets of linearly independent elements $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. The α_i are known as *simple roots*, and the α_i^\vee are *simple coroots*. The spaces $\mathfrak{h}_{\mathbb{R}} = \mathfrak{h} \otimes \mathbb{R}$ and $\mathfrak{h}_{\mathbb{R}}^* = \mathfrak{h}^* \otimes \mathbb{R}$ are the coroot and root spaces, respectively.

1.1.2 The affine Weyl group

Associated to a Cartan datum we have the *Weyl group* W , with generators s_i for $i \in I$, and relations $s_i^2 = 1$ and

$$\underbrace{s_i s_j s_i s_j \cdots}_{m(i,j)} = \underbrace{s_j s_i s_j s_i \cdots}_{m(i,j)},$$

where $m(i, j) = 2, 3, 4, 6$ or ∞ as $a_{ij}a_{ji} = 0, 1, 2, 3$ or ≥ 4 , respectively. An element of the Weyl group may be expressed as a word in the generators s_i ; given the relations above, an element of the Weyl group may have multiple *reduced words*, words of minimal length that express that element. The length of any reduced word of w is the *length* of w , denoted $\ell(w)$. The *Bruhat order* on Weyl group elements is a partial order where $v < w$ if there is a reduced word for v that is a subword of a reduced word for w . If $v < w$ and $\ell(v) = \ell(w) - 1$, we write $v \triangleleft w$.

If j is in I_{af} , we denote by W_j the subgroup of W generated by the elements s_i with $i \neq j$. We denote by W^j a set of minimal length representatives of the cosets W/W_j . The elements of W^0 will be referred to as *Grassmannian elements*.

1.1.3 Weyl group actions

Given a simple root α_i , there is an action \star of W on $\mathfrak{h}_{\mathbb{R}}$ or $\mathfrak{h}_{\mathbb{R}}^*$, defined by the action of the generators of W as

$$s_i \star \lambda = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i \quad \text{for } i \in I, \lambda \in \mathfrak{h}_{\mathbb{R}}^* \quad (1)$$

$$s_i \star \mu = \mu - \langle \mu, \alpha_i \rangle \alpha_i^\vee \quad \text{for } i \in I, \mu \in \mathfrak{h}_{\mathbb{R}}. \quad (2)$$

This action by W satisfies $\langle w \star \mu, w \star \lambda \rangle = \langle \mu, \lambda \rangle$.

The set of *real roots* is $\Phi_{\text{re}} = W \star \{\alpha_i \mid i \in I\}$. Given a real root $\alpha = w \star \alpha_i$, we have an *associated coroot* $\alpha^\vee = w \star \alpha_i^\vee$ and an *associated reflection* $s_\alpha = w s_i w^{-1}$ (these are well-defined, and independent of choice of w and i).

The action by W preserves the *root lattice* $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ and *coroot lattice* $Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee$. The *fundamental weights* are $\{\Lambda_i \in \mathfrak{h}_{\mathbb{R}}^* \mid \Lambda_i(\alpha_j^\vee) = \delta_{ij} \text{ for } i, j \in I\}$, and the *fundamental coweights* are $\{\Lambda_i^\vee \in \mathfrak{h}_{\mathbb{R}} \mid \alpha_i(\Lambda_j^\vee) = \delta_{ij} \text{ for } i, j \in I\}$. These generate the *weight lattice* $P = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i$ and *coweight lattice* $P^\vee = \bigoplus_{i \in I} \mathbb{Z} \Lambda_i^\vee$.

We let $\mathfrak{h}_{\text{fin}}$ denote the linear span of $\{\alpha_i^\vee \mid i \neq 0\}$ and $\mathfrak{h}_{\text{fin}}^*$ denote the span of $\{\alpha_i \mid i \neq 0\}$. Then there is another action \diamond of W on $\mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$, called the level one action in [16], which is defined by:

$$s_i \diamond \mu = \begin{cases} s_i \star \mu & \text{if } i \neq 0 \\ s_0 \star \mu - \alpha_0^\vee & \text{if } i = 0 \end{cases}$$

where α_0^\vee is interpreted as $\alpha_0^\vee = -\sum_{i \in I_{\text{fin}}} \alpha_i^\vee$.

In addition to reflections s_α , we have the translation endomorphisms of $\mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$ given by

$$t_\gamma \diamond \mu = \mu + \gamma \quad (3)$$

for $\gamma \in \mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$. One can show that $t_\mu t_\gamma = t_{\mu+\gamma}$ and that $t_{w(\mu)} = w t_\mu w^{-1}$ for $w \in W_{\text{fin}}$, $\gamma, \mu \in \mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$.

If by abuse of notation we let $Q_{\text{fin}}^\vee = \{t_{\alpha^\vee} \mid \alpha^\vee \in Q_{\text{fin}}^\vee\}$, then the affine Weyl group has an alternate presentation as

$$W_{\text{af}} = W_{\text{fin}} \ltimes Q_{\text{fin}}^\vee.$$

Remark 1. Elements of W_{af} corresponding to translations act trivially via the \star action, i.e. $t_\gamma \star \mu = \mu$.

1.1.4 The extended affine Weyl group

We can define the *extended affine Weyl group* W_{ext} by

$$W_{\text{ext}} = W_{\text{fin}} \ltimes P_{\text{fin}}^\vee.$$

W_{ext} also has an action on $\mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$ and $\mathfrak{h}_{\text{fin}}^* \otimes \mathbb{R}$ via the translation formula (3). Translations in the extended affine Weyl group also act trivially under the \star action on $\mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$.

1.1.5 Affine hyperplanes and alcoves

In $\mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$, let $H_{\alpha,k} = \{\mu \mid \langle \mu, \alpha \rangle = k\}$, where α is a finite root and $k \in \mathbb{Z}$. Reflection over the hyperplane $H_{\alpha,k}$ is equivalent to $t_{k\alpha^\vee} s_\alpha$ acting by the \diamond action. Each hyperplane $H_{\alpha,k}$ is stabilized by the action of W_{af} and the set of hyperplanes $\mathcal{H} = \cup_{\alpha,k} H_{\alpha,k}$ is stabilized by the action of W_{ext} .

The *fundamental alcove* is the polytope bounded by $H_{\alpha_i,0}$ for $i \in I_{\text{fin}}$ and $H_{\theta,1}$, where θ is the *highest root*. It is a fundamental domain for the \diamond action of W on $\mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$. Therefore, we may identify alcoves with affine Weyl group elements; we define \mathcal{A}_w to be the alcove $w^{-1} \diamond \mathcal{A}_\emptyset$, where \mathcal{A}_\emptyset is the fundamental alcove. Additionally, we may identify alcoves with their *centroids*, i.e., the average of the vertices of the alcove.

1.1.6 Dynkin diagram automorphisms

The length of an element $w \in W$, defined earlier in terms of reduced words, may equivalently be defined to be the number of hyperplanes $H_{\alpha,k}$ that lie between the alcoves \mathcal{A}_w and \mathcal{A}_\emptyset . We can similarly define the length of an element $w \in W_{\text{ext}}$ to be the number of hyperplanes that lie between \mathcal{A}_w and \mathcal{A}_\emptyset . This definition of length implies that there are non-trivial elements of W_{ext} of length 0. In fact, it is known [9] that

$$\Omega := \{u \in W_{\text{ext}} \mid \ell(u) = 0\} \cong \text{Aut}(D) \cong P_{\text{fin}}^\vee / Q_{\text{fin}}^\vee,$$

where $\text{Aut}(D)$ is the set of Dynkin diagram automorphisms.

The first of the above isomorphisms can be viewed concretely as follows. We let $\Omega = \{u \in W_{\text{ext}} \mid \ell(u) = 0\}$. Let $J = \{i \in I_{\text{fin}} \mid \tau(0) = i, \tau \in \text{Aut}(D)\}$ be the set of *cominiscule coweights*. Define x_λ to be a minimal length representative of the coset $t_\lambda W_{\text{fin}}$ for $\lambda \in P_{\text{fin}}^\vee$. It can be shown that $x_\lambda = t_\lambda v_\lambda^{-1}$, where $v_\lambda \in W_{\text{fin}}$ is shortest element of W_{fin} such that $v_\lambda(\lambda) = \lambda_-$, and λ_- is the unique antidominant element of the W_{fin} -orbit of λ . Then $\Omega = \{x_{\Lambda_i} \mid i \in J\}$, and the element x_{Λ_i} corresponds to the Dynkin diagram automorphism sending the node 0 to the node i . Under this map and the action of W_{ext} on P_{af}^\vee given above, an element $\tau \in \text{Aut}(D)$ acts on the coweight lattice P_{af}^\vee via $\tau \star \alpha_i^\vee = \alpha_{\tau(i)}^\vee$. Furthermore,

$$\tau s_i = s_{\tau(i)} \tau \tag{4}$$

for $i \in I_{\text{af}}, \tau \in \Omega$. Finally, for $\tau \in \Omega$, $u = s_{i_1} s_{i_2} \cdots s_{i_k} \in W$, we define $\tau(u) = s_{\tau(i_1)} s_{\tau(i_2)} \cdots s_{\tau(i_k)}$.

The extended affine Weyl group can be realized as a semi-direct product of the affine Weyl group and Ω :

$$W_{\text{ext}} = W_{\text{af}} \ltimes \Omega.$$

The relation (4) describes how elements commute in this realization of W_{ext} .

1.2 k -Schur functions for general type

Let $\mathbb{F} = \mathbb{C}((t))$ and $\mathbb{O} = \mathbb{C}[[t]]$. The *affine Grassmannian* is defined as $\mathrm{Gr}_G := G(\mathbb{F})/G(\mathbb{O})$. Gr_G can be decomposed into *Schubert cells* $\Omega_w = \mathcal{B}wG(\mathbb{O}) \subset G(\mathbb{F})/G(\mathbb{O})$, where \mathcal{B} denotes the Iwahori subgroup and $w \in W^0$, the set of Grassmannian elements in the associated affine Weyl group. The Schubert varieties, denoted X_w , are the closures of Ω_w , and we have $\mathrm{Gr}_G = \sqcup \Omega_w = \cup X_w$, for $w \in W^0$. The homology $H_*(\mathrm{Gr}_G)$ and cohomology $H^*(\mathrm{Gr}_G)$ of the affine Grassmannian have corresponding Schubert bases, $\{\xi_w\}$ and $\{\xi^w\}$, respectively, also indexed by Grassmannian elements. It is well-known that Gr_G is homotopy-equivalent to the space ΩK of based loops in K (due to Quillen, see [13, §8] or [10]). The group structure of ΩK gives $H_*(\mathrm{Gr}_G)$ and $H^*(\mathrm{Gr}_G)$ the structure of dual Hopf algebras over \mathbb{Z} .

The *nilCoxeter algebra* \mathbb{A}_0 may be defined via generators and relations from any Cartan datum, with generators \mathbf{u}_i for $i \in I$, and relations $\mathbf{u}_i^2 = 0$ and

$$\underbrace{\mathbf{u}_i \mathbf{u}_j \mathbf{u}_i \mathbf{u}_j \cdots}_{m(i,j)} = \underbrace{\mathbf{u}_j \mathbf{u}_i \mathbf{u}_j \mathbf{u}_i \cdots}_{m(i,j)},$$

where $m(i, j) = 2, 3, 4, 6$ or ∞ as $a_{ij}a_{ji} = 0, 1, 2, 3$ or ≥ 4 , respectively.

Since the braid relations are exactly those of the corresponding Weyl group, we may index nilCoxeter elements by elements of the Weyl group, e.g., $\mathbf{u}(w) = \mathbf{u}_{i_1} \mathbf{u}_{i_2} \cdots \mathbf{u}_{i_k}$, whenever $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reduced word for w .

By work of Peterson [11], there is an injective ring homomorphism $j_0 : H_*(\mathrm{Gr}_G) \hookrightarrow \mathbb{A}_0$. This map is an isomorphism on its image (actually a Hopf algebra isomorphism) $j_0 : H_*(\mathrm{Gr}_G) \rightarrow \mathbb{B}$, where \mathbb{B} is known as the *affine Fomin-Stanley subalgebra*.

Definition 2. For W of affine type X and $w \in W^0$ we define the non-commutative k -Schur function \mathfrak{s}_w^X of affine type X to be the image of the Schubert class ξ_w under the isomorphism j_0 , so $\mathfrak{s}_w^X = j_0(\xi_w)$. When obvious from context, we will simply write \mathfrak{s}_w , omitting the type. This definition comes from the realization of k -Schur functions identified with the homology of the affine Grassmannian in [4]. In type C this was first properly developed in [5], and in types B and D this was first developed in [12].

Example 3. In type $A_n^{(1)}$, the elements \mathfrak{s}_w^A are the non-commutative k -Schur functions defined in [4]. One can define a further isomorphism between the affine Fomin-Stanley subalgebra and the ring of symmetric functions generated by the homogeneous symmetric functions h_λ with $\lambda_1 \leq n - 1$. Under this isomorphism, the non-commutative k -Schur functions are conjectured to correspond to the $t = 1$ specializations of the k -Schur functions of Lapointe, Lascoux and Morse [7] indexed by a k -bounded partition corresponding to the element w and are isomorphic to the k -Schur functions of Lapointe and Morse [8].

2 A type-free formula

Given an element $t = w\tau \in W_{\mathrm{ext}}$ with $w \in W_{\mathrm{af}}$ and $\tau \in \Omega$, we denote by $\bar{t} = w$ the image of t modulo Ω . For $\lambda \in P_{\mathrm{fin}}^\vee$ recall that $t_\lambda \in W_{\mathrm{ext}}$ is the translation which acts on $\mathfrak{h}_{\mathrm{af}}$

according to (3). We let $z_\lambda = \bar{t}_\lambda^{-1}$. In [1], the z_λ were called *pseudo-translations*. For a coweight γ , we let $\Gamma_\gamma = W_{\text{fin}}\gamma$.

Independently from Lam and Shimozono [6], we have simultaneously discovered a generalization of [1] and [4] which gives a formula for the k -Schur functions indexed by coweight translations. Rather than include our long proof, we will rely on their result.

Proposition 4 (Lam, Shimozono [6]). *For a dominant coweight γ ,*

$$\mathfrak{s}_{z_\gamma} = \sum_{\eta \in \Gamma_\gamma} \mathbf{u}(z_\eta).$$

Proposition 4 is the starting point for a type-free combinatorial formula generalizing the one that appears in [1]. It should be noted though, that this formula does not give reduced words for the terms z_η ; they are defined only as the image of translations in prescribed directions. In Theorem 18, we will give a combinatorial description of the explicit reduced words which appear in this sum.

2.1 Commutation for k -Schur functions

Theorem 5.1 of [1] gives a nice commutation relation for k -Schur functions in type A and a generator of the affine nil-Coxeter algebra. In this section we will deduce a similar commutation property in Theorem 8 which will allow us to provide more explicit formulas for \mathfrak{s}_w .

Definition 5. We now fix some notation. γ will denote the j^{th} fundamental coweight Λ_j^\vee . If $t_\gamma = z_\gamma^{-1}\tau_\gamma$ then we let $t = t_\gamma$, $z = z_\gamma$, and $\tau = \tau_\gamma$.

Lemma 6. *For a coweight γ , z is the unique element of W which satisfies $\mathcal{A}_z = \mathcal{A}_\emptyset + \gamma$.*

Proof. The alcove $\mathcal{A}_z = z^{-1} \diamond \mathcal{A}_\emptyset$ and $z = \bar{t}^{-1}$, where the action of t corresponds to translation by γ . The uniqueness follows from the fact that W is in bijection with the set of alcoves. \square

Proposition 7. *For γ , z , τ as in Definition 5 and $w \in W$,*

$$z_{w\star\gamma}w = \tau(w)z.$$

Proof. Let $w \in W$. In W_{ext} , we have $wt_{w^{-1}\star\gamma} = tw$. Let $t_{w^{-1}\star\gamma} = z_{w^{-1}\star\gamma}^{-1}\tau'$ for some $\tau' \in \Omega$. Then we have

$$wz_{w^{-1}\star\gamma}^{-1}\tau' = z^{-1}\tau w, \tag{5}$$

$$wz_{w^{-1}\star\gamma}^{-1}\tau' = z^{-1}\tau(w)\tau, \tag{6}$$

with the last equality coming from Equation (4). Therefore, we must have $\tau' = \tau$, and

$$wz_{w^{-1}\star\gamma}^{-1} = z^{-1}\tau(w).$$

Inverting both sides and replacing w^{-1} with w gives the desired result. \square

The following theorem is a generalization of the commutation property for rectangular k -Schur functions found in [1].

Theorem 8. *Let γ be a fundamental coweight, and let $w \in W$. Then*

$$\mathfrak{s}_z \mathbf{u}(w) = \mathbf{u}(\tau(w)) \mathfrak{s}_z.$$

Proof. This follows from Proposition 7 and Proposition 4. \square

2.2 An algebraic formula

We let $W_{0,j}$ denote the subgroup of W generated by the simple reflections s_i with $i \neq 0, j$ and let W_0^j denote the set of minimal length coset representatives of $W_0/W_{0,j}$. This subsection provides another formula for the k -Schur functions which correspond to fundamental coweights.

Remark 9. *Let γ denote the j^{th} fundamental coweight, as in Definition 5. Then Γ_γ is naturally identified with W_0^j . We can construct a bijection between W_0^j and Γ_γ as follows. First we give a map from W_0 to Γ_γ : for $v \in W_0$, we define a map $v \rightarrow v(\gamma)$. This map is clearly onto; Γ_γ is defined to be the image of this map. From equation (2), we see that $s_i \star \gamma = \gamma$ for $i \neq 0, j$. Therefore, $W_0/W_{0,j}$ is in bijection with Γ_γ .*

Lemma 10. *Let $w \in W$ and $\mu, \nu \in \mathfrak{h}_{af}$. The two actions \star and \diamond are related by:*

$$w \diamond (\mu + \nu) = w \diamond \mu + w \star \nu.$$

Proof. We prove this on the generators s_i . If i is not zero, then s_i is linear and the two actions agree, so there is nothing to prove. If $i = 0$, then

$$s_0 \diamond (\mu + \nu) = s_0 \star (\mu + \nu) - \alpha_0^\vee = (s_0 \star \mu - \alpha_0^\vee) + s_0 \star \nu = s_0 \diamond \mu + s_0 \star \nu. \quad \square$$

The following proposition is a stepping stone to proving our main theorem; It is used to connect Proposition 4 to Theorem 18.

Proposition 11. *Let γ be a fundamental coweight as in Def 5. Then*

$$\mathfrak{s}_z = \sum_{v \in W_0^j} \mathbf{u}(\tau(v)zv^{-1}).$$

Proof. We will use Proposition 4; we show that each $\tau(v)zv^{-1}$ is in fact $z_{v\star\gamma}$.

Let $w = \tau(v)zv^{-1}$. We compute

$$\mathcal{A}_{wv} = \mathcal{A}_{\tau(v)z} = \mathcal{A}_{z_{v\star\gamma}v} = v^{-1}z_{v\star\gamma}^{-1} \diamond \mathcal{A}_\emptyset = v^{-1} \diamond (\mathcal{A}_\emptyset + v \star \gamma) = \mathcal{A}_v + \gamma,$$

where the second equivalence comes from Proposition 7 and the last two use Lemma 10. Applying v to the left and right of the equation above yields $\mathcal{A}_w = \mathcal{A}_\emptyset + v \star \gamma$. By Lemma 6, $w = z_{v\star\gamma}$. Combined with Remark 9, this concludes the proof. \square

2.3 Towards a general combinatorial formula

We will outline in this section how to build a combinatorial formula for the k -Schur functions indexed by a fundamental coweight. Section 3 will give more explicit formulas in affine type C .

Definition 12. A set of combinatorial objects \mathcal{R} will be called a *combinatorial affine Grassmannian set* for W if:

- There is a transitive action of W on \mathcal{R} .
- There exists an element $\emptyset \in \mathcal{R}$ which satisfies $W_0\emptyset = \{\emptyset\}$.
- The map $W^0 \rightarrow \mathcal{R}$ defined by $w \rightarrow w\emptyset$ is a bijection.

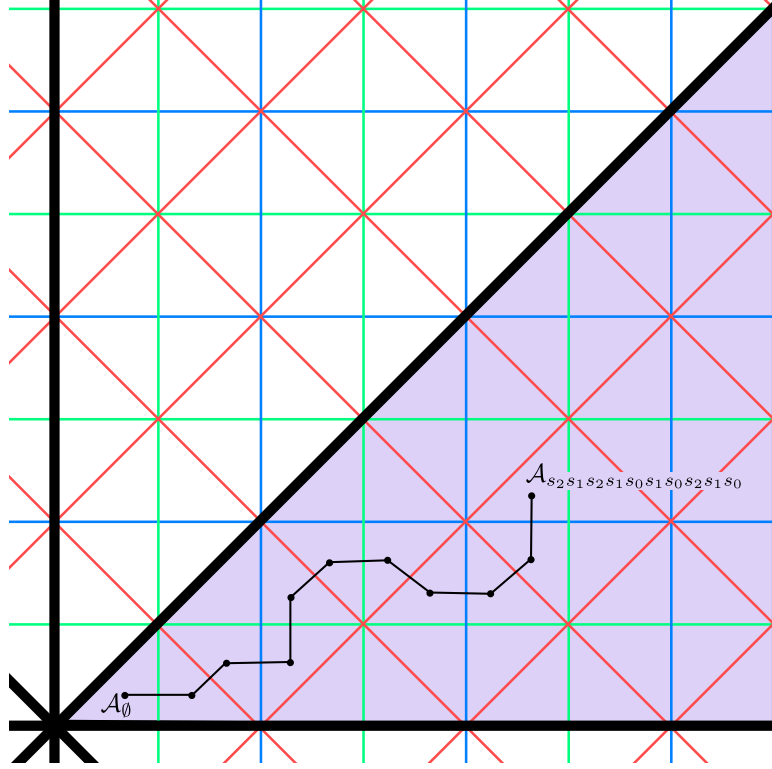
Given a combinatorial affine Grassmannian set \mathcal{R} , $\mu \in \mathcal{R}$, and the above bijection, we define $w_\mu \in W^0$ by $w_\mu\emptyset = \mu$.

Remark 13. *There is another way of calculating the location of an alcove \mathcal{A}_w given a reduced word of the element $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ that we picture as an alcove walk. Given a word $w = s_{i_1}s_{i_2}\cdots s_{i_r}$, the location of $\mathbf{u}(w)$ is calculated by a path starting at \mathcal{A}_\emptyset followed by the alcove $\mathcal{A}_{s_{i_r}}$, then*

$$\mathcal{A}_{s_{i_{r-1}}s_{i_r}}, \mathcal{A}_{s_{i_{r-2}}s_{i_{r-1}}s_{i_r}}, \dots, \mathcal{A}_{s_{i_1}s_{i_2}\cdots s_{i_{r-1}}s_{i_r}}.$$

Each of these alcoves is adjacent (see [1, Proposition 1.1]) and the word for w determines a path which travels from the fundamental alcove to \mathcal{A}_w traversing a single hyperplane for each simple reflection in the word.

Example 14. A walk that corresponds to the reduced word $s_2s_1s_2s_1s_0s_1s_0s_2s_1s_0$ appears below. Each hyperplane is colored according to the simple reflection that corresponds to a crossing of that hyperplane; e.g., crossing a green hyperplane corresponds to an s_0 , a red hyperplane corresponds to an s_1 , and a blue hyperplane corresponds to an s_2 .



In the diagram above, the path represents a particular reduced word for the element of W^0 of type C_2 . The vertices of this path are in correspondence with the sequence of alcoves: $\mathcal{A}_\emptyset \rightarrow \mathcal{A}_{s_0} \rightarrow \mathcal{A}_{s_1 s_0} \rightarrow \mathcal{A}_{s_2 s_1 s_0} \rightarrow \mathcal{A}_{s_0 s_2 s_1 s_0} \rightarrow \mathcal{A}_{s_1 s_0 s_2 s_1 s_0} \rightarrow \mathcal{A}_{s_0 s_1 s_0 s_2 s_1 s_0} \rightarrow \mathcal{A}_{s_1 s_0 s_1 s_0 s_2 s_1 s_0} \rightarrow \mathcal{A}_{s_2 s_1 s_0 s_1 s_0 s_2 s_1 s_0} \rightarrow \mathcal{A}_{s_1 s_2 s_1 s_0 s_1 s_0 s_2 s_1 s_0} \rightarrow \mathcal{A}_{s_2 s_1 s_2 s_1 s_0 s_1 s_0 s_2 s_1 s_0}$.

We can define $x \in \mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$ to be on the *positive* or *negative* side of the hyperplane $H_j := H_{\alpha_j, 0}$ by $\langle x, \alpha_j \rangle > 0$ or $\langle x, \alpha_j \rangle < 0$, respectively.

Lemma 15. (see for instance [17]) *Minimal length expressions of $w \in W$ correspond to alcove walks which do not cross the same affine hyperplane twice.*

Lemma 16. [See for instance [2]] *Let $j \in \{1, 2, \dots, k\}$. Then w has a right j descent ($ws_j < w$) if and only if the alcove \mathcal{A}_w is on the negative side of the hyperplane H_j .*

Lemma 17. *For all $v \in W_0^j$, $\tau(v)z \in W^0$.*

Proof. By Proposition 7, $\tau(v)z = z_{v \star \gamma} v$. Therefore, the alcove

$$\begin{aligned} \mathcal{A}_{\tau(v)z} &= \mathcal{A}_{z_{v \star \gamma} v} = (z_{v \star \gamma} v)^{-1} \diamond \mathcal{A}_\emptyset = \\ v^{-1} z_{v \star \gamma}^{-1} \diamond \mathcal{A}_\emptyset &= v^{-1} \diamond (\mathcal{A}_\emptyset + v \star \gamma) = \mathcal{A}_v + \gamma, \end{aligned}$$

by Lemma 10. Since $v \in W_0^j \subset W^j$, the only right descent of v is a j descent, so for $x \in \mathcal{A}_v$ and $i \neq 0, j$ we have $\langle x, \alpha_i \rangle \geq 0$, by Lemma 16. Furthermore, $v \in W_0^j \subset W_0$, so $\langle x, \alpha_j \rangle \geq -1$ for $x \in \mathcal{A}_v$ (as every alcove corresponding to $v \in W_0$ has a vertex at the origin). Combining these two facts, we get that $\langle x + \gamma, \alpha_i \rangle \geq 0$ for all $i \neq 0$ (since $\langle \gamma, \alpha_j \rangle \geq 1$). Therefore, the alcove $\mathcal{A}_v + \gamma$ is dominant, so the corresponding element is Grassmannian, i.e. $\tau(v)z \in W^0$. \square

We let w_0^j be the (unique) maximal length element of W_0^j . The set \mathcal{R} inherits a partial order from W^0 ; for $\mu, \nu \in \mathcal{R}$ we say $\mu \leq \nu$ whenever $w_\mu \leq w_\nu$. For $\mu, \nu \in \mathcal{R}$ with $\nu \leq \mu$, we let $w_{\mu/\nu} := w_\mu w_\nu^{-1}$.

Theorem 18. *Let $R = z\emptyset$ and $S = \tau(w_0^j)z\emptyset$. Then*

$$\mathfrak{s}_z = \sum_{S \leq \lambda \leq R} \mathbf{u}(w_\lambda \tau^{-1}(w_{R/\lambda})).$$

Proof. We construct a map $\Phi : W_0^j \rightarrow \mathcal{R}$ by sending $v \in W_0^j$ to $\Phi(v) = \tau(v)z\emptyset$. By Lemma 17, Φ is injective and hence Φ is a bijection on its image, which is precisely all $\lambda \in \mathcal{R}$ satisfying $S \leq \lambda \leq R$. In other words, $w_\lambda = \tau(v)z$ whenever $\Phi(v) = \lambda$.

Now $w_{R/\lambda} = w_R w_\lambda^{-1}$, so $w_{R/\lambda} = z(\tau(v)z)^{-1} = \tau(v^{-1})$. Therefore $w_\lambda \tau^{-1}(w_{R/\lambda}) = \tau(v)z\tau^{-1}(\tau(v^{-1})) = \tau(v)zv^{-1}$. By Proposition 11, the theorem follows. \square

Remark 19. *It should be noted that Theorem 18 gives the reduced words which appear in the expansion of \mathfrak{s}_z ; they are precisely the reduced words which correspond to objects from \mathcal{R} . Once the bijection between W^0 and \mathcal{R} is understood, the terms in Theorem 18 are as well.*

In this sense the theorem is stronger than Proposition 4, although its proof relies entirely on the proposition. In particular, this theorem generalizes Definition 2.1 of [1] to all affine types.

3 Type C combinatorics

As an application of Theorem 18, we use this section to develop the combinatorics of affine type C .

3.1 Type C root system background

Fix an integer $k > 1$. We recall some facts about roots and weights in affine type C (see [2] for more details). We let $\epsilon_1, \dots, \epsilon_k$ denote an orthonormal basis for $V := \mathbb{R}^k \equiv \mathfrak{h}_{\text{fin}} \otimes \mathbb{R}$. We realize $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3, \dots, \alpha_{k-1} = \epsilon_{k-1} - \epsilon_k, \alpha_k = 2\epsilon_k$ as the simple roots of finite type C_k .

The fundamental weights are realized as $\Lambda_i := \epsilon_1 + \dots + \epsilon_i$ for $i = 1, \dots, k$. The fundamental coweights are $\Lambda_i^\vee = 2\Lambda_i$ for $i \neq k$ and $\Lambda_k^\vee = \Lambda_k$.

The fundamental coweights $\Lambda_1^\vee, \dots, \Lambda_{k-1}^\vee$ also belong to the coroot lattice Q_{fin}^\vee . The elements $t_{\Lambda_i^\vee}$ actually equal $z_{\Lambda_i^\vee}^{-1}$ (for $i \neq k$) in W_{ext} , i.e. these elements have trivial Dynkin diagram automorphisms (as compared to type A , where all fundamental coweights correspond to distinct non-trivial Dynkin diagram automorphisms).

Since Λ_k^\vee is not in Q_{fin}^\vee , $t_{\Lambda_k^\vee}$ corresponds to a non-trivial Dynkin diagram automorphism. In affine type C there is only one such automorphism, which we will denote τ . It is defined by $\tau(i) = k - i$ for all $i \in \{0, 1, \dots, k\}$.

We let W denote the affine Coxeter group of type C . Recall it is generated by s_0, s_1, \dots, s_k subject to the relations:

$$\begin{aligned} s_i^2 &= 1 \text{ for } i \in \{0, 1, \dots, k\}, \\ s_i s_j &= s_j s_i \text{ if } i - j \neq \pm 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ for } i \in \{1, \dots, k-2\}, \\ s_i s_{i+1} s_i s_{i+1} &= s_{i+1} s_i s_{i+1} s_i \text{ for } i \in \{0, k-1\}. \end{aligned}$$

3.2 Bijection between Grassmannian elements and symmetric $2k$ -cores

Definition 20. The *hook length* of a cell x in the Young diagram of a partition λ is the number of cells of the Young diagram of λ to the right of x and above x , including the box x . A partition λ is called an n -core if for every cell x in the Young diagram of λ , n does not divide the hook length of x .

In [3], Hanusa and Jones give a construction for a combinatorial affine Grassmannian set for W for all classical affine W (the affine Grassmannians corresponding to $B_k^{(1)}/B_k, C_k^{(1)}/C_k, D_k^{(1)}/D_k, B_k^{(1)}/D_k$).

In affine type C , the set \mathcal{R} of combinatorial affine Grassmannian elements they give are symmetric $2k$ -core partitions (symmetry is with respect to transposing the partition). We give a short outline of the action of W on \mathcal{R} as follows:

Let the residue of a cell (i, j) of a Young diagram be:

$$res(i, j) = \begin{cases} j - i \bmod 2k & \text{if } 0 \leq (j - i) \bmod 2k \leq k \\ 2k - ((j - i) \bmod 2k) & \text{if } k < (j - i) \bmod 2k < 2k \end{cases}$$

We can then define an action on symmetric $2k$ -core partitions by letting $s_i \lambda =$

$$\begin{cases} \lambda \cup \{\text{residue } i \text{ cells}\} & \text{if } \lambda \text{ has addable cell of residue } i \\ \lambda \setminus \{\text{residue } i \text{ cells}\} & \text{if } \lambda \text{ has removable cell of residue } i \\ \lambda & \text{else} \end{cases}$$

Theorem 21 (Hanusa, Jones [3]). *The action of W on \mathcal{R} described above makes \mathcal{R} into a combinatorial affine Grassmannian set for W .*

Example 22. Let $k = 3$ and let $w = s_1 s_2 s_3 s_2 s_0 s_1 s_0 \in W^0$. Then w corresponds to the symmetric 6-core $(6, 3, 2, 1, 1, 1)$.

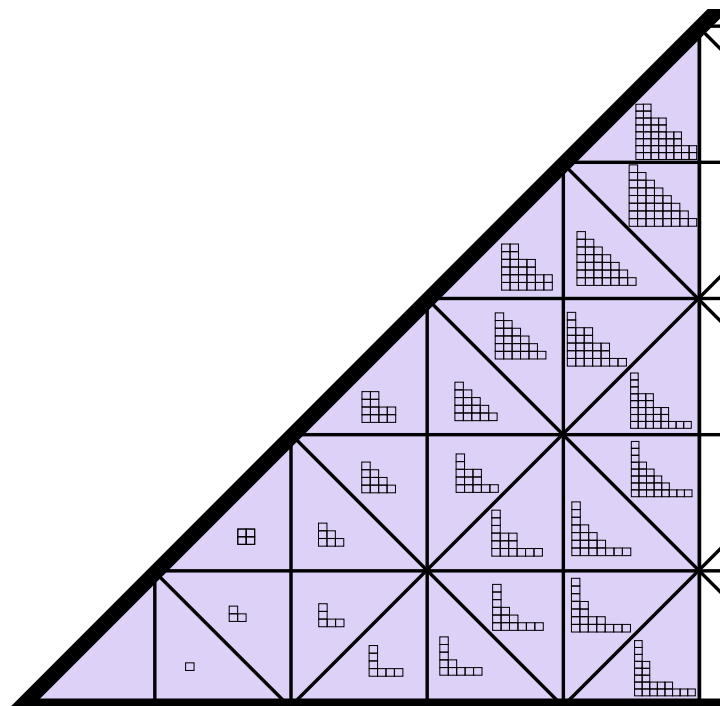
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1	0	1			
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Remark 23. *Symmetric $2k$ -core partitions have extraneous data. Half of the partition is determined from the other, so we will sometimes think of a symmetric $2k$ -core as a diagram with boxes (i, j) for $j \geq i$. We call such a diagram a shifted diagram.*

Example 24. Let $k = 3$ and $w = s_1 s_2 s_3 s_2 s_0 s_1 s_0$ as above. Then the shifted diagram for the 6-core is:

	0	1				
0	1	2	3	2	1	

Example 25. A portion of the lattice of symmetric 4-cores coming from the action of the affine Coxeter group of type $C_2^{(1)}$ is pictured below.



The pseudo-translation $z_{\Lambda_1^\vee}$ corresponding to the fundamental coweight $\Lambda_1^\vee = 2\epsilon_1$ takes the fundamental alcove to the alcove indexed by the symmetric 4-core $(4, 1, 1, 1)$ and the pseudo-translation $z_{\Lambda_2^\vee}$ corresponding to the fundamental coweight $\Lambda_2^\vee = \epsilon_1 + \epsilon_2$ takes the fundamental alcove to the alcove indexed by the symmetric 4-core $(2, 2)$.

3.3 Words and cores corresponding to fundamental coweights

Each s_i acts on V by reflecting across the hyperplane corresponding to the simple root α_i for $i \neq 0$ and reflecting across the affine hyperplane $H_{\theta,1} = \{v \in V : \langle v, \theta \rangle = 1\}$, where θ is the highest root, for $i = 0$. Specifically, if we let $(a_1, \dots, a_k) \in V$ represent $\sum_i a_i \epsilon_i$, then:

$$s_i \diamond (a_1, \dots, a_k) = \begin{cases} (a_1, \dots, a_{i+1}, a_i, \dots, a_k) & \text{for } i = 1, \dots, k-1; \\ (a_1, \dots, a_{k-1}, -a_k) & \text{for } i = k; \\ (2 - a_1, \dots, a_k) & \text{for } i = 0. \end{cases}$$

For $i \leq k+1$ we let $w_i := s_{i-1}s_{i-2} \cdots s_1s_0 \in W$.

Lemma 26. For $i \leq k$, the element w_i acts on $v = (a_1, \dots, a_k) \in V$ by:

$$w_i \diamond v = (a_2, a_3, \dots, a_i, 2 - a_1, a_{i+1}, \dots, a_k).$$

Also,

$$w_{k+1} \diamond v = (a_2, a_3, \dots, a_k, a_1 - 2)$$

Proof. Simple calculation using Weyl group action described above. \square

Lemma 27. $w_{k+1}^{-1}w_kw_j^{-1} \diamond (a_1, \dots, a_k) = (a_j - 2, a_1, a_2, \dots, \widehat{a_j}, \dots, a_k).$

Proof. Simple calculation using Lemma 26. \square

If G_\emptyset is the centroid of \mathcal{A}_\emptyset , then

$$G_\emptyset = \frac{1}{k+1} \sum_i \Lambda_i = \left(\frac{k}{k+1}, \frac{k-1}{k+1}, \dots, \frac{1}{k+1} \right).$$

Recall that for a fixed j we let γ denote the coweight Λ_j^\vee .

Lemma 28. For $j \neq k$, $z_\gamma = (w_jw_k^{-1}w_{k+1})^j$.

Proof. Let $w = w_jw_k^{-1}w_{k+1}$. We compute the centroid of the alcove $G_{w^j} = w^{-j} \diamond G_\emptyset = G_\emptyset - \underbrace{(2, 2, \dots, 2, 0, 0, \dots, 0)}_j$ by Lemma 27. Therefore $w^j = z_\gamma$ by Lemma 6. \square

Corollary 29. For $j \neq k$, z_γ corresponds to the symmetric $2k$ -core $\lambda = ((2k)^j, j^{2k-j})$. Equivalently, z_γ corresponds to the shifted partition $(2k, 2k-1, \dots, 2k-j+1)$.

Proof. Let $w = w_jw_k^{-1}w_{k+1}$. The first application of w will add $2k-j+1$ boxes to the shifted diagram. Every subsequent application adds $2k-j+1$ boxes to a new row of the shifted diagram and one box to each previous row. \square

The last case, when $\gamma = \Lambda_k^\vee$, is slightly different. We end this section by describing the corresponding symmetric $2k$ -core in this case.

Lemma 30. If $\gamma = \Lambda_k^\vee$ then $z_\gamma = w_k^{-1}w_{k-1}^{-1} \cdots w_1^{-1}$.

Proof.

$$G_{w_k^{-1}w_{k-1}^{-1} \cdots w_1^{-1}} = (w_k^{-1}w_{k-1}^{-1} \cdots w_1^{-1})^{-1} \diamond G_\emptyset = w_1 \cdots w_{k-1}w_k \diamond G_\emptyset =$$

$$\left(2 - \frac{1}{k+1}, 2 - \frac{2}{k+1}, \dots, 2 - \frac{k}{k+1} \right) = (1, 1, \dots, 1) + G_\emptyset = \gamma + G_\emptyset.$$

By Lemma 6, the statement follows. \square

Lemma 31. *With the action on partitions described above,*

$$w_i^{-1}w_{i-1}^{-1}\cdots w_2^{-1}w_1^{-1}\emptyset = \underbrace{(i, i, \dots, i)}_i.$$

Proof. The proof is by induction. $w_1 = s_0$, and $s_0\emptyset = (1)$. If $w_{i-1}^{-1}\cdots w_1^{-1}\emptyset = (i-1, i-1, \dots, i-1)$, then $w_i^{-1}(i-1, i-1, \dots, i-1) = s_0s_1\cdots s_{i-1}s_i(i-1, \dots, i-1) = s_0s_1\cdots s_{i-1}(i, i-1, \dots, i-1, 1) = s_0s_1\cdots s_{i-2}(i, i, i-1, \dots, i-1, 2) = \cdots = (i, i, \dots, i)$. \square

Corollary 32. $z_{\Lambda_k^\vee}$ corresponds to the symmetric $2k$ -core

$$\underbrace{(k, k, \dots, k)}_k.$$

Equivalently, this corresponds to the shifted partition $(k, k-1, \dots, 2, 1)$.

Proof. Follows from Lemma 30 and Lemma 31. \square

3.4 Subcores and a combinatorial formula

We now illustrate our formulas for $k = 3$. We first introduce the shorthand notation $\mathbf{u}(i_1i_2\dots i_m)$ to denote $\mathbf{u}(s_{i_1}s_{i_2}\cdots s_{i_m})$. The simplest example is $j = 1$.

Example 33. Let $j = 1$. Then $z = z_{\Lambda_1^\vee} = s_1s_2s_3s_2s_1s_0$. The Dynkin automorphism τ corresponding to z is trivial. w_0^1 is the element $s_1s_2s_3s_2s_1$. Therefore $R = z\emptyset = (6, 1, 1, 1, 1, 1)$ and $S = w_0^1z\emptyset = (1)$. There are 6 symmetric 6-cores between S and R , they are:

$$(1), (2, 1), (3, 1, 1), (4, 1, 1, 1), (5, 1, 1, 1, 1), (6, 1, 1, 1, 1, 1).$$

They correspond respectively to the following shifted diagrams.

$$\begin{array}{cccccc} \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{2} & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{2} & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{2} & \boxed{1} \\ \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{2} & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{2} & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{2} & \boxed{1} \end{array}$$

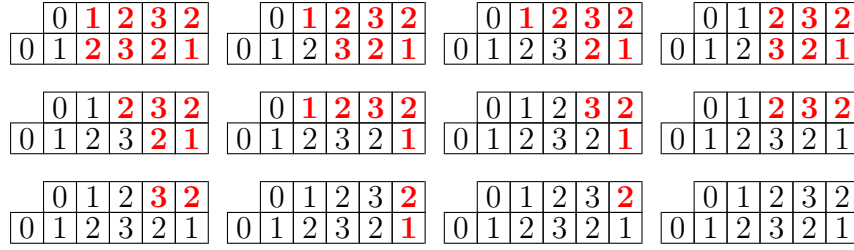
Therefore

$$\begin{aligned} \mathfrak{s}_{z_{\Lambda_1^\vee}}^C &= \mathbf{u}(0\mathbf{12321}) + \mathbf{u}(10\mathbf{1232}) + \mathbf{u}(210\mathbf{123}) \\ &\quad + \mathbf{u}(3210\mathbf{12}) + \mathbf{u}(23210\mathbf{1}) + \mathbf{u}(123210). \end{aligned}$$

Example 34. Let $j = 2$. Then $z = z_{\Lambda_2^\vee} = s_2s_3s_2s_1s_0s_2s_3s_2s_1s_0$. The Dynkin automorphism τ corresponding to z is trivial. $w_0^2 = s_2s_1s_3s_2s_1s_3s_2$. Therefore $R = z\emptyset = (6, 6, 2, 2, 2, 2)$ and $S = w_0^2z\emptyset = (2, 2)$. There are 12 symmetric 6-cores between S and R , they are:

$$\begin{aligned} &(2, 2), (3, 2, 1), (4, 2, 1, 1), (3, 3, 2), \\ &(4, 3, 2, 1), (5, 2, 1, 1, 1), (5, 4, 2, 2, 1), (6, 3, 2, 1, 1, 1), \\ &(6, 4, 2, 2, 1, 1), (5, 5, 2, 2, 2), (6, 5, 2, 2, 2, 1), (6, 6, 2, 2, 2, 2). \end{aligned}$$

They correspond respectively to the following shifted diagrams.



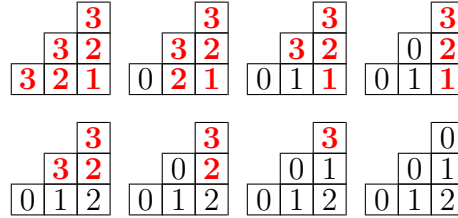
By Theorem 18,

$$\begin{aligned} \mathfrak{s}_{z_{\Lambda_2}^\vee}^C = & \mathbf{u}(010\mathbf{2132132}) + \mathbf{u}(0210\mathbf{232123}) + \mathbf{u}(03210\mathbf{23212}) + \mathbf{u}(10210\mathbf{23123}) \\ & + \mathbf{u}(103210\mathbf{2312}) + \mathbf{u}(023210\mathbf{2321}) + \mathbf{u}(2103210\mathbf{231}) + \mathbf{u}(1023210\mathbf{232}) \\ & + \mathbf{u}(21023210\mathbf{23}) + \mathbf{u}(32103210\mathbf{21}) + \mathbf{u}(321023210\mathbf{2}) + \mathbf{u}(2321023210). \end{aligned}$$

Example 35. Let $j = 3$. The word $z = z_{\Lambda_3}^\vee$ is $s_0s_1s_2s_0s_1s_0$. Then z corresponds to the unique non-trivial Dynkin automorphism defined by $\tau(i) = 3 - i$. The corresponding shifted diagram is $(3, 2, 1)$. Let $R = (3, 3, 3) = z\emptyset$ and $S = \tau(w_0^3)z\emptyset = \emptyset$. There are 8 symmetric 6 cores between S and R . They are

$$\emptyset, (1), (2, 1), (2, 2), (3, 1, 1), (3, 2, 1), (3, 3, 2), (3, 3, 3).$$

These correspond respectively to the following shifted diagrams, where the bold letters correspond to elements not in λ which have τ^{-1} applied to them.



By Theorem 18,

$$\begin{aligned} \mathfrak{s}_{z_{\Lambda_3}^\vee}^C = & \mathbf{u}(\mathbf{321323}) + \mathbf{u}(0\mathbf{32312}) + \mathbf{u}(10\mathbf{3231}) + \mathbf{u}(010\mathbf{321}) \\ & + \mathbf{u}(210\mathbf{323}) + \mathbf{u}(0210\mathbf{32}) + \mathbf{u}(10210\mathbf{3}) + \mathbf{u}(010210). \end{aligned}$$

4 Remaining types

Although Hanusa and Jones [3] did give descriptions of combinatorial affine Grassmannian sets for the type B and D cases, the combinatorics involved are not as nice. It seems plausible that some different collection of elements better suited to describing the terms appearing in expansions of k -Schur functions in these types will arise in the future. Rather than spending a good deal of space here to developing these in full generality, we will include the case of affine B of rank 3 and affine D of rank 4 as examples of what the combinatorics would look like; the compelled reader should easily be able to develop a corresponding expansion in full generality from these examples, the concepts of Section 3, and a full understanding of Hanusa and Jones' combinatorics in these types.

4.1 Affine type B , rank 3

Affine type B has one non-trivial Dynkin diagram automorphism τ , which is defined by permuting the indices 0 and 1, and fixing all other i .

Example 36. The affine Grassmannian element $z = s_0 s_2 s_3 s_2 s_0$ corresponds to translation by the fundamental coweight Λ_1^\vee , which under the identification of Hanusa and Jones corresponds to the even symmetric 6-core $(7, 2, 1, 1, 1, 1)$:

0							
0							
2							
3							
2							
0	0						
0	0	2	3	2	0	0	

This fundamental coweight corresponds to the nontrivial Dynkin automorphism τ . Again, as in type C , the objects involved in the bijection of Hanusa and Jones are symmetric cores, so we will remove half of the diagram, and study the skew partition:

	0						
0	0	2	3	2	0	0	

In this case, $\tau(w_0^1) = z$, so we need to look at all sub-diagrams between $S = \emptyset$ and $R = (7, 1)$. There are six such diagrams:

<div><div>1</div><div>1123211</div></div>	<div><div>0</div><div>0023211</div></div>	<div><div>0</div><div>0023211</div></div>
<div><div>0</div><div>0023211</div></div>	<div><div>0</div><div>0023211</div></div>	<div><div>0</div><div>0023200</div></div>

By Theorem 18,

$$\mathfrak{s}_{z_{\Lambda_1}^\vee}^B = \mathbf{u}(\mathbf{12321}) + \mathbf{u}(0\mathbf{1232}) + \mathbf{u}(20\mathbf{123}) + \mathbf{u}(320\mathbf{12}) + \mathbf{u}(2320\mathbf{1}) + \mathbf{u}(02320).$$

Example 37. The affine Grassmannian element $z = s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_0$ corresponds to translation by the Fundamental coweight Λ_2^\vee , which under the identification of Hanusa and Jones corresponds to the even symmetric 6-core $(6, 6, 2, 2, 2, 2)$:

1	2				
2	3				
3	2				
2	1				
0	0	1	2	3	2
0	0	2	3	2	1

This coweight corresponds to the trivial Dynkin automorphism, and the even symmetric 6-core corresponds to the following skew partition:

	0	1	2	3	2
0	0	2	3	2	1

In this case, $w_0^2 = s_2 s_3 s_1 s_2 s_3 s_1 s_2$, so we need to look at all skew sub-diagrams between $S = w_0^2 z \emptyset = (2, 1)$ and $R = (7, 1)$. There are twelve such diagrams:

<table><tr><td></td><td>0</td><td>1</td><td>2</td><td>3</td><td>2</td></tr><tr><td>0</td><td>0</td><td>2</td><td>3</td><td>2</td><td>1</td></tr></table>		0	1	2	3	2	0	0	2	3	2	1	<table><tr><td></td><td>0</td><td>1</td><td>2</td><td>3</td><td>2</td></tr><tr><td>0</td><td>0</td><td>2</td><td>3</td><td>2</td><td>1</td></tr></table>		0	1	2	3	2	0	0	2	3	2	1	<table><tr><td></td><td>0</td><td>1</td><td>2</td><td>3</td><td>2</td></tr><tr><td>0</td><td>0</td><td>2</td><td>3</td><td>2</td><td>1</td></tr></table>		0	1	2	3	2	0	0	2	3	2	1	<table><tr><td></td><td>0</td><td>1</td><td>2</td><td>3</td><td>2</td></tr><tr><td>0</td><td>0</td><td>2</td><td>3</td><td>2</td><td>1</td></tr></table>		0	1	2	3	2	0	0	2	3	2	1
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0	0	2	3	2	1																																														

By Theorem 18,

$$\begin{aligned} \mathfrak{s}_{z_{\Lambda_2}^\vee}^B &= \mathbf{u}(0\mathbf{2132132}) + \mathbf{u}(20\mathbf{213231}) + \mathbf{u}(120\mathbf{21323}) + \mathbf{u}(320\mathbf{21321}) \\ &\quad + \mathbf{u}(2320\mathbf{2321}) + \mathbf{u}(12320\mathbf{232}) + \mathbf{u}(3120\mathbf{2132}) + \mathbf{u}(23120\mathbf{231}) \\ &\quad + \mathbf{u}(123120\mathbf{23}) + \mathbf{u}(323120\mathbf{21}) + \mathbf{u}(1323120\mathbf{2}) + \mathbf{u}(21323120). \end{aligned}$$

Example 38. The affine Grassmannian element $z = s_3 s_2 s_3 s_0 s_2 s_3 s_1 s_2 s_0$ corresponds to translation by the fundamental coweight Λ_3^\vee , which under the identification of Hanusa and Jones corresponds to the even symmetric 6-core $(7, 6, 6, 4, 3, 3, 1)$:

0						
0	2	3				
2	3	2				
3	2	0	0			
2	1	0	0	2	3	
0	0	1	2	3	2	
0	0	2	3	2	0	0

This coweight corresponds to the nontrivial Dynkin automorphism τ , and the even symmetric 6-core corresponds to the following skew partition:

				0			
			0	0	2	3	
		0	1	2	3	2	
0	0	2	3	2	0	0	

In this case, $w_0^3 = s_3 s_2 s_1 s_3 s_2 s_3$, so we need to look at all skew sub-diagrams between $S = \tau(w_0^3)z\emptyset = (3, 2)$ and $R = z\emptyset = (7, 5, 4, 1)$. There are eight such diagrams:

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By Theorem 18,

$$\begin{aligned} \mathfrak{s}_{z_{\Lambda_3}^B} = & \mathbf{u}(120\mathbf{323123}) + \mathbf{u}(3120\mathbf{32312}) + \mathbf{u}(23120\mathbf{3231}) + \mathbf{u}(023120\mathbf{323}) \\ & + \mathbf{u}(323120\mathbf{321}) + \mathbf{u}(3023120\mathbf{32}) + \mathbf{u}(23023120\mathbf{3}) + \mathbf{u}(323023120). \end{aligned}$$

4.2 Affine type D , rank 4

All Dynkin automorphisms in affine D rank 4 leave the index 2 fixed. Here we give explicit expansions for two of the fundamental coweights.

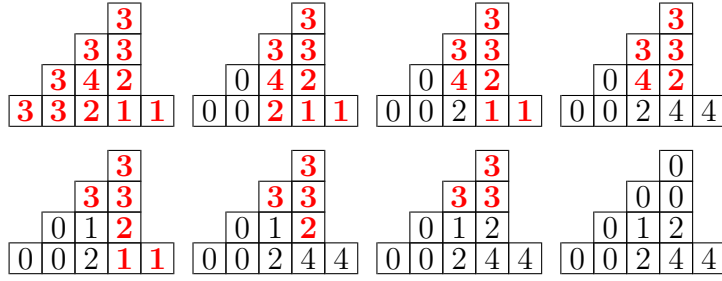
Example 39. The affine Grassmannian element $z = s_0 s_2 s_4 s_1 s_2 s_0$ corresponds to translation by the fundamental coweight Λ_3^\vee , which under the identification of Hanusa and Jones corresponds to the even symmetric 8-core $(5, 4, 4, 4, 1)$:

4					
4	2	0	0		
2	1	0	0		
0	0	1	2		
0	0	2	4	4	

This coweight corresponds to a nontrivial Dynkin automorphism τ which swaps 0 with 3 and 1 with 4, and the even symmetric 8-core corresponds to the following skew partition:

				0	
			0	0	
		0	1	2	
	0	0	2	4	4

In this case, $w_0^3 = s_3 s_2 s_4 s_1 s_2 s_3$, so we need to look at all skew sub-diagrams between $S = \tau(w_0^3)z\emptyset = \emptyset$ and $R = z\emptyset = (5, 3, 2, 1)$. There are eight such diagrams:



By Theorem 18,

$$\mathfrak{s}_{z_{\Lambda_4}^\vee}^D = \mathbf{u}(\mathbf{321423}) + \mathbf{u}(0\mathbf{32142}) + \mathbf{u}(20\mathbf{3241}) + \mathbf{u}(420\mathbf{324}) \\ + \mathbf{u}(120\mathbf{321}) + \mathbf{u}(4120\mathbf{32}) + \mathbf{u}(24120\mathbf{3}) + \mathbf{u}(024120).$$

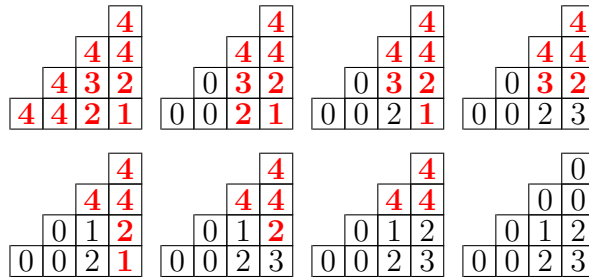
Example 40. The affine Grassmannian element $z = s_0 s_2 s_3 s_1 s_2 s_0$ corresponds to translation by the fundamental coweight Λ_4^\vee , which under the identification of Hanusa and Jones corresponds to the even symmetric 8-core $(4, 4, 4, 4)$:

3	2	0	0
2	1	0	0
0	0	1	2
0	0	2	3

This coweight corresponds to a nontrivial Dynkin automorphism τ which swaps 0 with 4 and 1 with 3, and the even symmetric 8-core corresponds to the following skew partition:

			0
		0	0
	0	1	2
0	0	2	3

In this case, $w_0^4 = s_4 s_2 s_3 s_1 s_2 s_4$, so we need to look at all skew sub-diagrams between $S = \tau(w_0^4)z\emptyset = \emptyset$ and $R = z\emptyset = (4, 3, 2, 1)$. There are eight such diagrams:



By Theorem 18,

$$\mathfrak{s}_{z_{\Lambda_4}^\vee}^D = \mathbf{u}(\mathbf{421324}) + \mathbf{u}(0\mathbf{42132}) + \mathbf{u}(20\mathbf{4231}) + \mathbf{u}(320\mathbf{423}) \\ + \mathbf{u}(120\mathbf{421}) + \mathbf{u}(3120\mathbf{42}) + \mathbf{u}(23120\mathbf{4}) + \mathbf{u}(023120).$$

Acknowledgments

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The authors would like to thank the reviewer for helpful comments.

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