

Polyhedral embeddings of snarks with arbitrary nonorientable genera

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Abstract

Mohar and Vodopivec [Combinatorics, Probability and Computing (2006) 15, 877-893] proved that for every integer k ($k \geq 1$ and $k \neq 2$), there exists a snark which polyhedrally embeds in \mathbb{N}_k and presented the problem: Is there a snark that has a polyhedral embedding in the Klein bottle? In the paper, we give a positive solution of the problem and strengthen Mohar and Vodopivec's result. We prove that for every integer k ($k \geq 2$), there exists an infinite family of snarks with nonorientable genus k which polyhedrally embed in \mathbb{N}_k . Furthermore, for every integer k ($k > 0$), there exists a snark with nonorientable genus k which polyhedrally embeds in \mathbb{N}_k .

Keywords: polyhedral embedding; snark; nonorientable surface; nonorientable genus; Euler genus

1 Introduction

During a conference in 1968, Grünbaum [5] conjectured that each cubic graph with a polyhedral embedding in an orientable surface is 3-edge-colourable. A positive solution

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of this conjecture would generalize the dual form of the Four-Color-Theorem to every orientable surface. The conjecture holds for the sphere from the results of Tait [16] and Apple and Haken [1]. In [9], Kochol disproved the conjecture by showing that there exist infinitely many snarks with polyhedral embeddings in \mathbb{S}_k ($k \geq 5$). The smallest of the counterexamples found by Kochol is a snark of order 74. With the aid of computer, Mohar and Vodopivec [14] proved that for every cubic graph with fewer than 30 vertices, the conjecture holds true. Furthermore, they proved that for every integer k ($k \geq 1$ and $k \neq 2$), there exists a snark which polyhedrally embeds in \mathbb{N}_k , and proposed the following problem:

Problem 1. (*Problem 5.3 of [14]*) *Is there a snark that has a polyhedral embedding in the Klein bottle?*

In the paper, we give a positive solution of Problem 1 and strengthen Mohar and Vodopivec's result. Actually, we prove that for every integer k ($k \geq 2$), there exists an infinite family of snarks with nonorientable genus k which polyhedrally embed in \mathbb{N}_k . Furthermore, for every integer k ($k > 0$), there exists a snark with nonorientable genus k which polyhedrally embeds in \mathbb{N}_k .

2 Preliminary

All graphs considered in this paper are connected. For some terminologies without description here, we may refer the reader to [4, 6, 13].

A *surface* is a compact closed 2-dimensional manifold without boundary. In topology, surfaces are classified into the *orientable surface* \mathbb{S}_m , with m handles ($m \geq 0$) and the *nonorientable surface* \mathbb{N}_k , with k crosscaps ($k > 0$). A graph embedding into a surface means a *cellular embedding*, so that every face of the embedding is an open topological disk. The *orientable genus* $\gamma(G)$ of a graph G is the smallest integer k such that G cellularly embeds into \mathbb{S}_k . Similarly, the *nonorientable genus* $\tilde{\gamma}(G)$ of a graph G is the smallest integer k such that G cellularly embeds into \mathbb{N}_k . The *Euler genus* $\overline{\gamma}(G)$ of a graph G is defined as $\min\{2\gamma(G), \tilde{\gamma}(G)\}$. Also note that $\overline{\gamma}(G) = \min\{2 - \chi(\mathbb{S}) \mid G \text{ cellularly embeds into a surface } \mathbb{S}\}$, where $\chi(\mathbb{S})$ denotes the Euler characteristic of a surface \mathbb{S} . By the well-known formula $\tilde{\gamma}(G) \leq 2\gamma(G) + 1$, either $\tilde{\gamma}(G) = \overline{\gamma}(G)$ or $\tilde{\gamma}(G) = \overline{\gamma}(G) + 1$.

A graph G is called a *k-amalgamation* of two graphs G_1 and G_2 , denoted by $G = G_1 \bigcup_k G_2$, if $G = G_1 \bigcup G_2$ and $G_1 \cap G_2$ is a set of k vertices. In [2], Archdeacon proved the following theorem:

Lemma 2. ([2], Theorem 1.1)

$$2 - 2k \leq \overline{\gamma}(G_1) + \overline{\gamma}(G_2) - \overline{\gamma}(G_1 \bigcup_k G_2) \leq k^2 - 4.$$

An embedding of a graph G is called *polyhedral* if all facial walks are cycles, and any two of them are either disjoint, intersect in one vertex, or intersect in one edge. If G is

a cubic graph, then any two facial walks are either disjoint or intersect in precisely one edge.

A *snark* is a cyclically 4-edge-connected cubic graph of girth at least 5 with no 3-edge-coloring. A graph is called a *4-snark* if it does not admit a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow. It is well known that a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ -flow in a cubic graph G corresponds to a 3-edge-coloring of G . Thus snarks form a proper subclass of 4-snarks. Kochol [8] introduced a general method to construct a 4-snark. It is based on the following two steps.

Suppose v is a vertex of a graph G and a graph G' is obtained from G by the following process. Replace v by a graph H_v so that each edge e of G having one end v now has one end from H_v . If e is a loop incident with v , then both ends of e will now be from H_v . We call G' a *vertex superposition* of G .

Suppose e is an edge of G with ends u and v and a graph G' is constructed from G as follows: replace e by a graph H_e having at least two vertices. In other words, we delete e from G , pick two distinct vertices u', v' of H_e , and identify u' with u and v' with v . We call G' an *edge superposition* of G . If H_e is a 4-snark, then G' is called a *4-strong edge superposition* of G .

A graph G' is called a *(4-strong) superposition* of a graph G if G' is obtained from G by some vertex and (4-strong) edge superpositions. Kochol [8] proved the following lemma:

Lemma 3. ([8], Lemma 4.4) *Let G be a 4-strong superposition of a 4-snark, then G is a 4-snark.*

3 Main theorem

The Petersen graph P is the smallest snark and has a polyhedral embedding in the projective plane, indicated in part (b) of Figure 1.

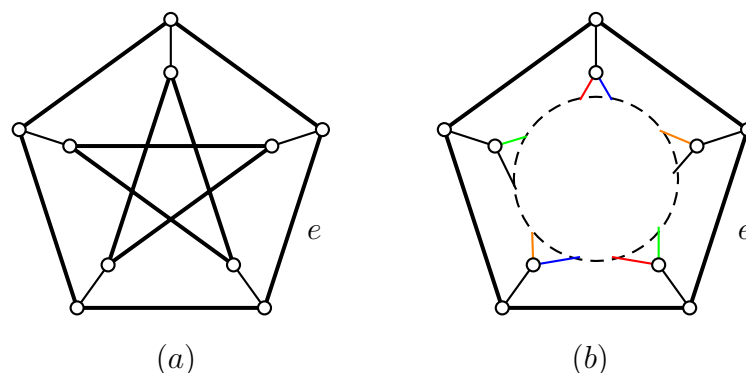


Figure 1: (a) the Petersen graph P and (b) the Petersen graph P polyhedrally embeds in the projective plane

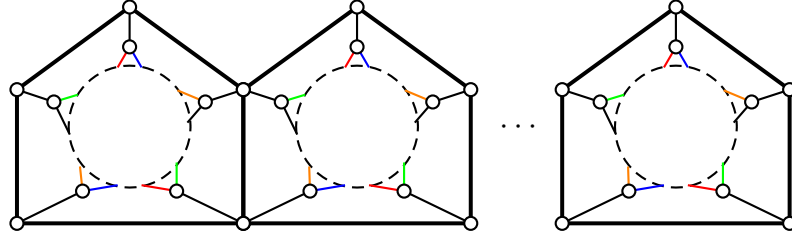


Figure 2: The graph G_{8k+2} resulting from k copies of the Petersen graph polyhedrally embeds in \mathbb{N}_k

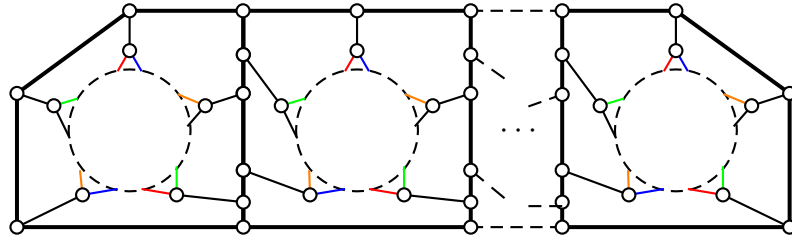


Figure 3: The snark S_{12k-2} polyhedrally embeds in \mathbb{N}_k

Theorem 4. *For every integer k ($k \geq 2$), there exists an infinite family of snarks with nonorientable genus k which polyhedrally embed in \mathbb{N}_k .*

Proof. The proof comprises the following two parts:

(a). For every integer k ($k \geq 2$), we construct an infinite family of snarks with polyhedral embeddings in \mathbb{N}_k ;

(b). For every integer k ($k \geq 2$), we prove that the snarks which are constructed in part (a) have nonorientable genus k .

The Petersen graph P has a polyhedral embedding in \mathbb{N}_1 as shown in part (b) of Figure 1. By applying 4-strong edge superposition ($k - 1$) times so that the edge e is recurrently replaced by the copy of P , we get the graph G_{8k+2} of order $8k + 2$, shown in Figure 2. Replacing all vertices of degree 5 in G_{8k+2} by the paths of order 3, we obtain the graph S_{12k-2} of order $12k - 2$ (see Figure 3). By Lemma 3, S_{12k-2} is a 4-snark. Then S_{12k-2} has no 3-edge-coloring. Since S_{12k-2} is a cubic graph, it has a 4-edge-coloring by Vizing theorem. Because the Petersen graph is cyclically 4-edge-connected and has girth 5, S_{12k-2} is cyclically 4-edge-connected and has girth 5. So S_{12k-2} is a snark. It has a polyhedral embedding in \mathbb{N}_k , indicated in Figure 3.

In G_{8k+2} , replace all vertices of degree 5 by paths of order 3 or by graphs $C_{i,5}$ ($i \geq 1$), drawn in Figure 4. Dashed lines indicate the edges incident with a vertex of degree 5 in G_{8k+2} . The resulting graphs are vertex superpositions of G_{8k+2} and are snarks by Lemma 3. Thus we construct an infinite family of snarks with polyhedral embeddings in \mathbb{N}_k .

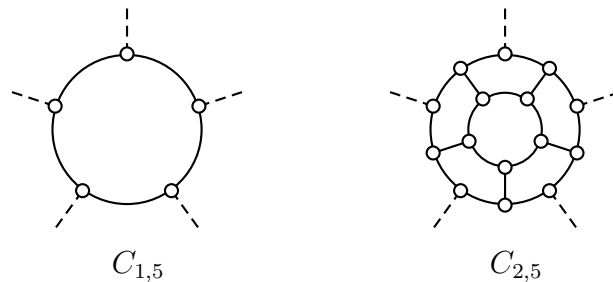


Figure 4: The graph $C_{i,5}$ ($i = 1, 2$)

Now, we prove part (b). Let $k = 2$ in Lemma 2, we get

$$\overline{\gamma}(G_1 \bigcup_2 G_2) \geq \overline{\gamma}(G_1) + \overline{\gamma}(G_2). \quad (1)$$

Since the Petersen graph P has orientable genus 1 and nonorientable genus 1, $\overline{\gamma}(P) = 1$ from the Euler genus definition. Recurrently computing $\overline{\gamma}(G_{8i+2})$ from $i = 2$ to $i = k$ (see Figure 2), we can deduce $\overline{\gamma}(G_{8k+2}) \geq k$ according to the inequality (1). For every graph G , $\tilde{\gamma}(G) = \overline{\gamma}(G)$ or $\tilde{\gamma}(G) = \overline{\gamma}(G) + 1$. Thus we get $\tilde{\gamma}(G_{8k+2}) \geq k$. Since G_{8k+2} has an embedding in \mathbb{N}_k (see Figure 2), $\tilde{\gamma}(G_{8k+2}) = k$ is deduced.

When replacing all vertices of degree 5 in G_{8k+2} by paths of order 3 or by graphs $C_{i,5}$ ($i \geq 1$), the resulting graphs are snarks according to the previous argument. It is clear that the obtained snarks have nonorientable genus k because $\tilde{\gamma}(G_{8k+2}) = k$ and paths of order 3 or graphs $C_{i,5}$ ($i \geq 1$) are all planar graphs. □

Corollary 5. *For every integer k ($k > 0$), there exists a snark with nonorientable genus k which polyhedrally embeds in \mathbb{N}_k .*

Proof. This is directly deduced from Theorem 4 and the fact that the Petersen graph has a polyhedral embedding in the projective plane. □

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