Combinatorial Logarithm and Point-Determining Cographs

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Submitted: May 5, 2012; Accepted: July 10, 2012; Published: July 19, 2012
Mathematics Subject Classifications: 05A15, 05A19

Abstract

We obtain a reduced form of the "combinatorial logarithm" Ω by looking at bijections related to connected point-determining cographs and connected co-point-determining graphs.

1 Introduction

The virtual species Ω, referred to as the "combinatorial logarithm" by Bergeron, Labelle, and Leroux in [12, p. 131], is one that uniquely satisfies the combinatorial equality $1 + X = \mathcal{E} \circ \Omega$, where $\mathcal{E}$ is the species of sets. The associated series of Ω are given on [12, p. 131]. In [7] Labelle gave a formula for computing the molecular expression of Ω. Theoretically [12, Proposition 7, p. 122], every virtual species can be written uniquely as the difference between two "real" species $\Phi = \Phi^+ - \Phi^-$, called the *reduced form* of Φ, where the molecular decompositions of Φ^+ and Φ^- have disjoint terms. To find such two species is generally hard.

In our previous paper [8, (2.8)], we expressed Ω as the difference between the species $\mathcal{Q}^c$ of connected co-point-determining graphs and the species $\mathcal{P}^c_{\geqslant 2}$ of connected point-determining graphs with more than one vertex. Unfortunately, this is not a reduced form for the obvious reason that the non-zero species $\mathcal{B}^c$ of connected bi-point-determining graphs is a subspecies of both $\mathcal{Q}^c$ and $\mathcal{P}^c$.

The aim of this paper is to further reduce formula [8, (2.8)] by studying the species of point-determining cographs and the connected ones. In Section 2 we review some terminologies in the combinatorial theory of species. Section 3 lists the main results

*This work is supported in part by the National Science Foundation Grant Number DUE-0634532.

of [8] and illustrates the key methods used in [8]. Section 4 discusses point-determining cographs and a further reduced formula (Corollary 10) for expressing the virtual species Ω as the difference between two subspecies of cographs.

In the end, a list of notations for species mentioned in this paper is given in Section 5.

2 Combinatorial Theory of Species

The combinatorial theory of species was initiated by Joyal in [4]. A *species* is a functor from the category of finite sets with bijections to itself. A species F generates for each finite set U the set $F[U]$ of F*-structures* on U, and for each bijection $\sigma : U \to V$ a bijection $F[\sigma] : F[U] \to F[V]$, called the *transport of F-structures along σ*. The symmetric group $\mathfrak{S}_n$ acts on the set $F[n] = F[\{1, 2, \dots, n\}]$ by transport of structures. The $\mathfrak{S}_n$-orbits under this action are called *unlabeled* F-structures of order n. Each species F is associated with three generating series, the *exponential generating series* $F(x) = \sum_{n\geqslant 0} |F[n]|x^n/n!$, the *type generating series* $\widetilde{F}(x) = \sum_{n\geqslant 0} f_n\, x^n$, where f_n is the number of unlabeled F-structures of order n, and the *cycle index*

$$Z_F = Z_F\,(p_1, p_2, \dots) = \sum_{n\geqslant 0} \left(\sum_{\lambda \vdash n} \text{fix } F[\lambda]\, \frac{p_\lambda}{z_\lambda} \right),$$

where fix $F[\lambda]$ denotes the number of F-structures on $[n] = \{1, 2, \dots, n\}$ fixed by $F[\sigma]$ for some σ that is a permutation of $[n]$ with cycle type λ, z_λ is the number of permutations in $\mathfrak{S}_n$ that commute with a permutation of cycle type λ, and p_λ is the *power sum symmetric function* (see Stanley [3, p. 297]) indexed by the partition λ of n. The following identities [12, p. 18] illustrate the importance of the cycle index in the theory of species.

$$\begin{aligned} F(x) &= Z_F(x, 0, 0, \dots), \\ \widetilde{F}(x) &= Z_F(x, x^2, x^3, \dots). \end{aligned}$$

We apply operations on species [12] to generate new species, and the operations of species translate into operations of the generating series of species systematically. The species operations that are frequently used in this paper are the *sum* $\Phi + \Psi$, the *product* $\Phi\Psi$ or $\Phi \cdot \Psi$, and the *composition* $\Phi(\Psi)$ or $\Phi \circ \Psi$ of species Φ and Ψ.

If F is a species of structures, we denote by F_n, for nonnegative integers n, the species of F-structures *concentrated on the cardinality* n (see [12, p. 30]), and by $F_{\geqslant n}$ the F-structures of cardinality at least n. Hence $F_{\geqslant n} = F_n + F_{n+1} + \cdots$. We usually write $F_{\geqslant 1}$ as F_+.

A *virtual species* is a formal difference of species (see [12, p. 121]). Proposition 18 of [12, p. 129] asserts that there exists a unique virtual species which we denote by Ω, the virtual species of "connected $(1+X)$-structures" with $1 + X = \mathcal{E} \circ \Omega$, or equivalently, $X = \mathcal{E}_+ \circ \Omega$. Thus Ω is referred to as the "combinatorial logarithm of the species $1 + X$" (see [12, p. 131]), or the compositional inverse of $\mathcal{E}_+$, also written $\mathcal{E}_+^{\langle -1 \rangle}$. The associated series of Ω are given on [12, p. 131]. Every virtual species Φ can be written uniquely in its *reduced form*

$$\Phi = \Phi^+ - \Phi^-,$$

where Φ^+ and Φ^- are species with no molecular components in common [12, Proposition 7, p. 122].

A species M is called a *molecular species* by Yeh [14] if there is only one isomorphism class of M-structures. Thus a molecular species is one that is indecomposable under addition. Every species can be expressed uniquely as the sum of molecular species, and this expression is called its *molecular decomposition* (see [12, p. 141]). For example, the molecular decomposition of the virtual species Ω starts with

$$\Omega = X - \mathcal{E}_2 + (X\mathcal{E}_2 - \mathcal{E}_3) + (X\mathcal{E}_3 + \mathcal{E}_2 \circ \mathcal{E}_2 - X^2\mathcal{E}_2 - \mathcal{E}_4) + \cdots .$$

We consider in this paper only simple graphs (without loops or multiple edges). A graph G is thought of as an ordered pair (V, E), where $V = V(G)$ is the vertex set of G, and $E = E(G)$ is the edge set of G, a set of 2-subsets of V. Two graphs are called *disjoint* if they have no common vertices. An *unlabeled graph* is formally defined as an isomorphism class of graphs, though we think of an unlabeled graph as simply a graph without vertex labels. A graph with no vertices is called *empty*. The empty graph is not considered as connected. The *empty* species, denoted by 0, is defined by $0[U] = \emptyset$ for all U. The species of the empty graph is denoted by 1. The species of the singleton graph is denoted by X.

We denote by $\mathcal{K}$ the species of *complete graphs*, which are graphs in which each pair of vertices are adjacent. The complement of a complete graph is called an *edgeless graph*. The species of edgeless graphs, which are graphs with isolated vertices, is the same as the species $\mathcal{E}$ of sets. There is a natural transformation α that produces for every finite set U a bijection between $\mathcal{E}[U]$ and $\mathcal{K}[U]$, namely, sending the edgeless graph on U to the complete graph with vertex set U. This bijection is carried through the complementation of graphs. The following diagram commutes for any finite sets U, V and any bijection $\sigma : U \to V$:

$$\begin{array}{ccc} \mathcal{E}[U] & \xrightarrow{\mathcal{E}[\sigma]} & \mathcal{E}[V] \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathcal{K}[U] & \xrightarrow{\mathcal{K}[\sigma]} & \mathcal{K}[V] \end{array}$$

In this case we call these two species *isomorphic* to each other, denoted $\mathcal{E} \simeq \mathcal{K}$. The general definition of two species being isomorphic to each other is similar. The concept of isomorphism is compatible with the transition to counting series of species (see [12, pp. 12–20]). Two isomorphic species essentially possess the "same" combinatorial properties. Henceforth they will be considered as *equal* in the combinatorial algebra. Thus we write $\Phi = \Psi$ in place of $\Phi \simeq \Psi$, and say there is a *combinatorial equality* (see [12, p. 21]) between the species Φ and Ψ.

3 Enumeration of Bi-Point-Determining Graphs

The *neighborhood* of a vertex v is the set of all vertices adjacent to v. A *point-determining graph* (Sumner [13], Bull and Pease [1], also called a *mating-type graph*, *mating graph*, or *M-graph*) is a graph in which no two vertices have the same neighborhood. Point-determining graphs (both labeled and unlabeled) were counted by Read [9]. Complements of point-determining graphs, which we call co-point-determining graphs (they have also been called "point-distinguishing"), are graphs in which no two vertices have the same closed neighborhood. (The closed neighborhood of a vertex is the vertex together with its neighborhood.) Bi-point-determining graphs (also called "totally point-determining" or "totally supercompact") are both point-determining and co-point-determining.

The notion of superimposition as introduced in [8, Definition 1.2] serves as a bridge bringing together the composition of species and the decomposition of graphs. Let $H_1, \ldots, H_m$ be graphs with disjoint vertex sets, and let G be a graph with vertex set $\{V(H_1), \ldots, V(H_m)\}$. We define the *superimposition* $G|_{H_1,\ldots,H_m}$ of G on $\{H_1, \ldots, H_m\}$ to be the graph with vertex set $\bigcup_{i=1}^m V(H_i)$ in which $\{u, v\}$ is an edge if it is an edge of some H_i or if $u \in V(H_i)$ and $v \in V(H_j)$ for some $i \neq j$, and $\{V(H_i), V(H_j)\} \in E(G)$.

Figure 1 illustrates the superimposition of a graph G on a set of graphs $\{H_1, H_2, H_3\}$.

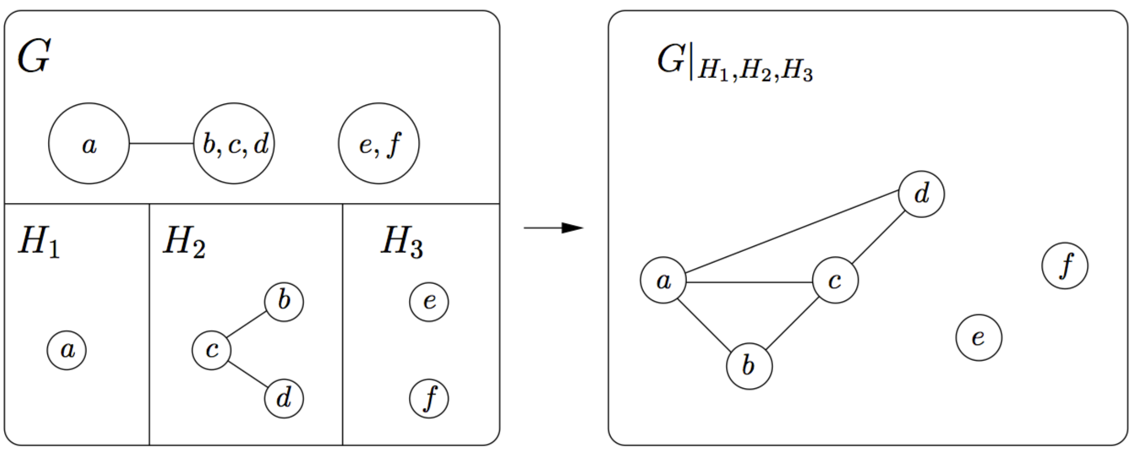

Figure 1: The superimposition $G|_{H_1, H_2, H_3}$.

Let n be any positive integer. The *edgeless graph* of order n is a graph with n isolated vertices, denoted E_n. The *complete graph* of order n is a graph with n vertices each pair of which is adjacent to each other, denoted K_n. Let $\{G_1, \ldots, G_n\}$ be a set of nonempty pairwise disjoint graphs. The *union* of $\{G_1, \ldots, G_n\}$ is set to be the superimposition $E_n|_{G_1,\ldots,G_n}$, and the *join* of $\{G_1, \ldots, G_n\}$ is set to be the superimposition $K_n|_{G_1,\ldots,G_n}$, where the vertex set of E_n and K_n is $\{V(G_1), \ldots, V(G_n)\}$.

The operation of superimposition of species of graphs is closely related to composition of species. Let Φ and Ψ be two species of graphs; i.e., for every finite set U, $\Phi[U]$ and $\Psi[U]$ are sets of graphs with vertex set U. The *species of superimposition* $\Phi \diamond \Psi$ is such that $(\Phi \diamond \Psi)[U]$ is the set of all superimpositions $G|_{H_1,\ldots,H_m}$ in which $H_1, \ldots, H_m$ are Ψ-graphs with $\bigcup_{i=1}^{m} V(H_i) = U$ and G is a Φ-graph with vertex set $\{V(H_1), \ldots, V(H_n)\}$.

Lemma 1. *([8, Lemma 1.4]) Let Φ and Ψ be species of graphs such that every $\Phi \diamond \Psi$-graph can be expressed uniquely as a superimposition of a Φ-graph on a set of Ψ-graphs. Then $\Phi \circ \Psi$ is isomorphic to $\Phi \diamond \Psi$.* $\square$

We use the combinatorial theory of species [4, 5, 12] as our framework to enumerate point-determining graphs, connected point-determining graphs, and bi-point-determining graphs [8]. The generating series of the species $\mathcal{G}$ of graphs is known (see [12, p. 79] and Robinson [10, p. 334, Theorem 2]). The following theorem gives a way to enumerate point-determining graphs.

Theorem 2. *([8, Theorem 2.2]) For the species $\mathcal{G}$ of graphs, the species $\mathcal{P}$ of point-determining graphs, the species $\mathcal{E}_+$ of nonempty edgeless graphs, the species $\mathcal{Q}$ of co-point-determining graphs, and the species $\mathcal{K}_+$ of nonempty complete graphs, we have*

$$\mathcal{G} = \mathcal{P} \circ \mathcal{E}_+ = \mathcal{Q} \circ \mathcal{K}_+. \tag{1}$$

The connected point-determining graphs and the connected co-point-deteriming graphs can be enumerated using the following Theorem.

Theorem 3. *([8, Theorem 2.3]) For the species $\mathcal{P}$ of point-determining graphs, $\mathcal{Q}$ of co-point-determining graphs, $\mathcal{P}^c$ of connected point-determining graphs, and $\mathcal{Q}^c$ of connected co-point-determining graphs, we have*

$$\mathcal{P} = \mathcal{Q} = (1 + X) \cdot (\mathcal{E} \circ \mathcal{P}^c_{\geqslant 2}) = \mathcal{E} \circ \mathcal{Q}^c. \tag{2}$$

A consequence of Theorem 3 is

$$(1 + X) \cdot (\mathcal{E} \circ \mathcal{P}^c_{\geqslant 2}) = \mathcal{E} \circ (\Omega + \mathcal{P}^c_{\geqslant 2}) = \mathcal{E} \circ \mathcal{Q}^c.$$

So [8, Lemma 2.4] gives the following.

Theorem 4. *([8, Equation (2.8)]) We express the virtual species Ω as the difference between the species $\mathcal{Q}^c$ of connected co-point-determing graphs and the species $\mathcal{P}^c_{\geqslant 2}$ of connected point-determining graphs with at least two vertices.*

$$\Omega = \mathcal{Q}^c - \mathcal{P}^c_{\geqslant 2}. \tag{3}$$

Note that $\mathcal{Q}^c - \mathcal{P}^c_{\geqslant 2}$ is not the reduced form of Ω, since $\mathcal{Q}^c$ share the same molecular components as $\mathcal{P}^c_{\geqslant 2}$. Now for any finite set U, the intersection $\mathcal{P}^c[U] \cap \mathcal{Q}^c[U]$ is the set of connected bi-point-determining graphs on U, denoted $\mathcal{B}^c[U]$. In other words, the species $\mathcal{B}^c_{\geqslant 2}$ is a *subspecies* (see [12, p. 120]) of both $\mathcal{P}^c_{\geqslant 2}$ and $\mathcal{Q}^c$. So (3) can be refined into

$$\Omega = (\mathcal{Q}^c - \mathcal{B}^c_{\geqslant 2}) - (\mathcal{P}^c_{\geqslant 2} - \mathcal{B}^c_{\geqslant 2}). \tag{4}$$

However, further examination shows that (4) is still not the reduced form of Ω.

A *cograph*, also called a *complement-reducible graph* is defined recursively as follows [2]:

1. A graph on a single vertex is a cograph.

2. For a set of cographs $\{G_1, \ldots, G_n\}$, their union $E_n|_{G_1,\ldots,G_n}$ is also a cograph.

3. If G is a cograph, then so is its complement.

The species $\mathcal{C}$ of cograghs was enumerated by [8, Lemma 4.2]. The following are unlabeled cographs with no more than four vertices.

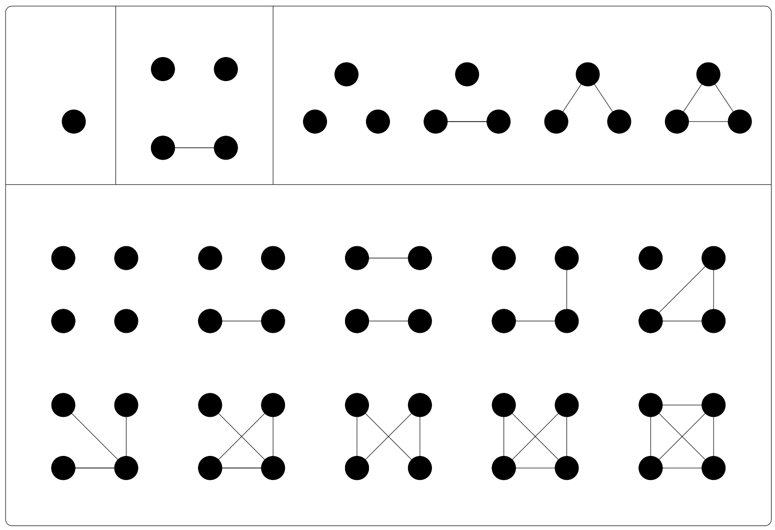

Figure 2: Unlabeled Cographs

The following theorem is the main result of [8].

Theorem 5. *([8, Theorem 4.4]) The species $\mathcal{G}$ of graphs is the composition of the species $\mathcal{B}$ of bi-point-determining graphs and $\mathcal{C}$ of cographs. That is,*

$$\mathcal{G} = \mathcal{B} \circ \mathcal{C}.$$

Theorem 5 states that every graph can be expressed uniquely as a superimposition of a bi-point-determining graph on a set of cographs. More explicitly, the proof of Theorem 5 gives, for any finite set U, a bijection

$$\alpha_U : \mathcal{G}[U] \longrightarrow (\mathcal{B} \diamond \mathcal{C})[U], \tag{5}$$

between the set $\mathcal{G}[U]$ of all graphs on U and the set of all superimpositions $G|_{H_1,\ldots,H_m}$ such that

1. Each H_i is a cograph.

2. The graph G is bi-point-determining.

3. The union of the vertex sets $V(H_i)$ for $i = 1, 2, \ldots, m$ is equal to the set U.

4 Connected Point-Determining Cographs

Now we consider point-determining cographs and co-point-determining cographs. In the literature, the connected point-determining cographs have been enumerated by Moon in [6], where he called them "σ-networks" or "series-networks", which are series-parallel networks with one or more cut-nodes [11, A058385].

Lemma 6. *The species $\mathcal{S}$ of point-determining cographs, $\mathcal{S}^c$ of connected point-determining cographs, $\mathcal{T}$ of co-point-determining cographs, and $\mathcal{T}^c$ of connected co-point-determining cographs are related by*

$$\mathcal{S} = \mathcal{T}, \tag{6}$$

$$\mathcal{S} = \mathcal{S}^c + \mathcal{T}^c - X,$$

$$\mathcal{S} = \mathcal{E}_+ \circ (\mathcal{S}^c - X) + X, \tag{7}$$

$$\mathcal{T} = \mathcal{E}_+ \circ \mathcal{T}^c, \tag{8}$$

Proof. Since a graph is a cograph if and only if its complement is a cograph, the complement of a point-determining cograph is a co-point-determining cograph, and vice versa. The combinatorial equality $\mathcal{S} = \mathcal{T}$ follows straightforwardly.

On the other hand, $\mathcal{S} - (\mathcal{S}^c - X)$ is the species of the point-determining graphs that are disconnected, whose complements are connected co-point-determing graphs. Therefore, $\mathcal{S} = \mathcal{S}^c - X + \mathcal{T}^c$.

The proofs of (7) and (8) are similar to that of Theorem 3 in [8]. □

Lemma 7. *The species $\mathcal{S}^c$ of connected point-determining cographs and $\mathcal{T}^c$ of connected co-point-determining cographs are related by*

$$\mathcal{S}^c = X + \mathcal{E}_{\geqslant 2} \circ \mathcal{T}^c, \tag{9}$$

$$\mathcal{T}^c = X \cdot (\mathcal{E} \circ (\mathcal{S}^c - X)) + \mathcal{E}_{\geqslant 2}(\mathcal{S}^c - X). \tag{10}$$

Proof. Let G be a connected point-determining cograph. Then G is either a graph with one vertex or that the complement of G is the union of a set of at least two connected co-point-determining cographs. Equation (9) follows.

Suppose H is a connected co-point-determining cograph. Then the connected components of the complement of H consists of at most one connected component with one vertex, and the rest have to be connected point-determining graphs with more than one vertex. Equation (10) follows. □

It follows from (9) and (10) that the species $\mathcal{T}^c$ satisfies the following species equation

$$\mathcal{T}^c = X \cdot [\mathcal{E} \circ (\mathcal{E}_{\geqslant 2} \circ \mathcal{T}^c)] + \mathcal{E}_{\geqslant 2}(\mathcal{E}_{\geqslant 2} \circ \mathcal{T}^c), \tag{11}$$

which allows us to find the generating series of $\mathcal{T}^c$ recursively. Figure 3 shows connected co-point-determining graphs with no more than five vertices.

Next, we will see that the uniqueness of expressing any graph as the superimposition of a bi-point-determining graph on a set of cographs provided by Theorem 5 in [8] gives a way

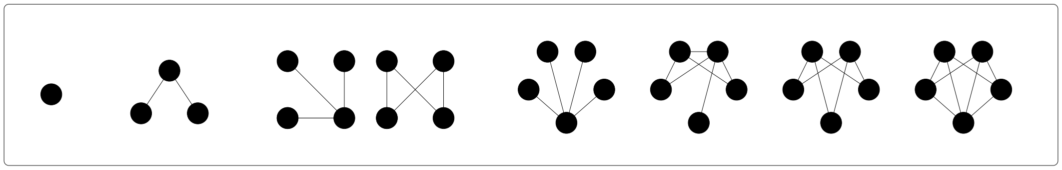

Figure 3: Connected Co-Point-Determining Cographs

to finding new functional equations for connected point-determining graphs and connected co-pointdetermining graphs. We begin with the following Lemma that is straightforward to show.

Lemma 8. *The superimposition $G|_{H_1,H_2,\dots,H_m}$ is connected if and only if either G is connected with at least two vertices or G is a single vertex and H is connected.*

Theorem 9. *For the species $\mathcal{P}^c - \mathcal{S}^c$ of connected point-determining graphs that are not cographs and $\mathcal{Q}^c - \mathcal{T}^c$ of connected co-point-determining graphs that are not cographs, we have the following combinatorial equality*

$$\mathcal{P}^c - \mathcal{S}^c = \mathcal{Q}^c - \mathcal{T}^c. \tag{12}$$

Proof. Let U be any finite nonempty set, and let

$$\alpha_U : \mathcal{G}[U] \longrightarrow \mathcal{B} \diamond \mathcal{C}[U]$$

be the bijection in (5) that sends each graph K on U uniquely to a superimposition of a bi-point-determining graph G on a set of cographs $H_1, \dots, H_m$ with $\cup_{i=1}^m V(H_i) = U$.

Note that because G is bi-point-determining, for any pair $i \neq j \in [m]$, and any vertex $v \in H_i$ and $w \in H_j$, v and w can neither have the same neighborhood nor have the same closed neighborhood. Therefore, K is point-determining if and only if each of the H_i is point-determining, and that K is co-point-determining if and only if each of the H_i is co-point-determining.

Obviously, a graph K on U is a cograph if and only if $\alpha_U(K)$ is the superimposition of the form $G|_{H_1}$ where G is the trivial graph and $H_1 = K$. In other words, $\alpha_U(\mathcal{C}[U])$ is isomorphic to $\mathcal{C}[U]$.

Suppose K is a connected point-determining graph on U that is not a cograph. Then Lemma 8 gives that $\alpha_U(K)$ is of the form $G|_{H_1,\dots,H_m}$ where G is a connected bi-point-determining graph with at least two vertices and each of $H_i, i = 1, \dots, m$, is a point-determining cograph. Therefore,

$$\alpha_U\big((\mathcal{P}^c - \mathcal{S}^c)[U]\big) = \mathcal{B}^c_{\geqslant 2} \diamond \mathcal{S}[U].$$

By a similar argument, we have

$$\alpha_U\big((\mathcal{Q}^c - \mathcal{T}^c)[U]\big) = \mathcal{B}^c_{\geqslant 2} \diamond \mathcal{T}[U].$$

On othe other hand, Equation (6) gives that there exists a bijection

$$\beta_U : \mathcal{B}^c \diamond \mathcal{S}[U] \longrightarrow \mathcal{B}^c \diamond \mathcal{T}[U]$$

that is sending each superimposition $G|_{H_1,\ldots,H_m}$ in which each $H_i, i = 1, \ldots, m$, is a point-determining cograph to the superimposition $G|_{H'_1,\ldots,H'_m}$ where each $H'_i, i = 1, \ldots, m$, is the complement of H_i.

Thus, for each nonempty finite set U, we obtain the bijection

$$\tau_U = \alpha_U^{-1} \circ \beta_U \circ \alpha_U : \mathcal{P}^c[U] - \mathcal{S}^c[U] \longrightarrow \mathcal{Q}^c[U] - \mathcal{T}^c[U],$$

that is sending each connected point-determining graph that is not a cograph uniquely to a connected co-point-determining graph that is not a cograph. $\square$

As a consequence of Theorem 4, we obtain a "refined" expression for the virtual species Ω as the difference of two species.

Corollary 10. *The virtual species, so called "combinatorial logarithm" Ω can be expressed as the difference of the species $\mathcal{T}^c$ of connected co-point-determining cographs and the species $\mathcal{S}^c_{\geqslant 2}$ of connected point-determining cographs with at least two vertices. That is,*

$$\Omega = \mathcal{T}^c - \mathcal{S}^c_{\geqslant 2} = X + \mathcal{T}^c - \mathcal{S}^c. \tag{13}$$

The following Proposition shows that for any finite set U with $\#U > 1$, the intersection of the sets $\mathcal{T}^c[U]$ and $\mathcal{S}^c[U]$ is empty. In other words, there exists no bi-point-determining cographs with more than one vertex.

Proposition 11. *The species of bi-point-determining cographs is isomorphic to X.*

Proof. First of all, the trivial graph is a bi-point-determining cograph. In fact, we will show that there is no bi-point-determining graph with more than one vertex.

Suppose the contrary. We pick G be to one such with the least possible number of vertices. We notice that G cannot be a disjoint union of isolated vertices or a complete graph, since such a cograph is not bi-point-determining. Therefore G must be a disjoin union, or the complement of the disjoint union, of bi-point-determining cographs, at least one of which, say, H, has at least two vertices. This contradicts the assumption. $\square$

There are still common terms in the molecular expressions of $\mathcal{T}^c$ and $\mathcal{S}^c$, although these common terms are not bi-point-determining cographs. If we could identify all these common terms, a precise molecular expression of Ω would be obtained after cancellation. For example, in the following molecular decomposition of Ω recursively obtained from identities (9) and (10),

n	$\mathcal{T}^c_n$	$\mathcal{S}^c_n$	$\Omega_n = X + \mathcal{T}^c_n - \mathcal{S}^c$
1	X	X	X
2	0	$\mathcal{E}_2$	$-\mathcal{E}_2$
3	$X\mathcal{E}_2$	$\mathcal{E}_3$	$X\mathcal{E}_2 - \mathcal{E}_3$
4	$X\mathcal{E}_3 + \mathcal{E}_2 \circ \mathcal{E}_2$	$X^2\mathcal{E}_2 + \mathcal{E}_4$	$X\mathcal{E}_3 + \mathcal{E}_2 \circ \mathcal{E}_2 - X^2\mathcal{E}_2 - \mathcal{E}_4$
5	$X\mathcal{E}_4 + X^3\mathcal{E}_2$ $+X\mathcal{E}_2 \circ \mathcal{E}_2 + \mathcal{E}_2\mathcal{E}_3$	$X^2\mathcal{E}_3 + X\mathcal{E}_2 \circ \mathcal{E}_2$ $+X\mathcal{E}_2^2 + \mathcal{E}_5$	$X\mathcal{E}_4 + X^3\mathcal{E}_2 + \mathcal{E}_2\mathcal{E}_3$ $-X^2\mathcal{E}_3 - X\mathcal{E}_2^2 - \mathcal{E}_5$
6	$X^3\mathcal{E}_3 + X^2\mathcal{E}_2 \circ \mathcal{E}_2$ $+2X^2\mathcal{E}_2^2 + X\mathcal{E}_5$ $+X\mathcal{E}_2\mathcal{E}_3 + \mathcal{E}_2\mathcal{E}_4$ $+\mathcal{E}_2 \circ \mathcal{E}_3 + \mathcal{E}_3 \circ \mathcal{E}_2$	$X^4\mathcal{E}_2 + X^2\mathcal{E}_4$ $+X^2\mathcal{E}_2 \circ \mathcal{E}_2 + 3X\mathcal{E}_2\mathcal{E}_3$ $+\mathcal{E}_2 \circ (X\mathcal{E}_2)$ $+\mathcal{E}_2(\mathcal{E}_2 \circ \mathcal{E}_2) + \mathcal{E}_6$	$X^3\mathcal{E}_3 + 2X^2\mathcal{E}_2^2 + X\mathcal{E}_5 + \mathcal{E}_2\mathcal{E}_4$ $+\mathcal{E}_2 \circ \mathcal{E}_3 + \mathcal{E}_3 \circ \mathcal{E}_2$ $-X^4\mathcal{E}_2 - X^2\mathcal{E}_4 - 2X\mathcal{E}_2\mathcal{E}_3$ $-\mathcal{E}_2 \circ (X\mathcal{E}_2) - \mathcal{E}_2(\mathcal{E}_2 \circ \mathcal{E}_2) - \mathcal{E}_6$

we notice that in the orders from $n = 1$ through $n = 4$, there is no repeated terms in $\mathcal{T}^c_n$ and $\mathcal{S}^c_n$. When $n = 5$, there is one, namely, $X\mathcal{E}_2 \circ \mathcal{E}_2$. When $n = 6$, there are two repeated terms $X\mathcal{E}_2\mathcal{E}_3$ and $X^2\mathcal{E}_2 \circ \mathcal{E}_2$. When $n = 7$, there are 11 repeated terms in $\mathcal{T}^c_7$ and $\mathcal{S}^c_7$, and the situation becomes complicated.

It is desirable to obtain a combinatorial interpretation of further reduced form of the combinatorial logarithm Ω.

5 List of Species

$\mathcal{C}\,(\mathcal{C}^c)$	species of (connected) cographs.
$\mathcal{B}\,(\mathcal{B}^c)$	species of (connected) bi-point-determining graphs.
$\mathcal{E}$	species of edgeless graphs.
$\mathcal{G}\,(\mathcal{G}^c)$	species of (connected) simple graphs.
$\mathcal{H}$	species of phylogenetic trees.
$\mathcal{K}$	species of complete graphs.
$\Omega = (1 + X)^c$	the combinatorial logarithm.
$\mathcal{P}\,(\mathcal{P}^c)$	species of (connected) point-determining graphs.
$\mathcal{Q}\,(\mathcal{Q}^c)$	species of (connected) co-point-determining graphs.
$\mathcal{R}$	species of special pure 2-trees.
$\mathcal{S}\,(\mathcal{S}^c)$	species of (connected) point-determining cographs.
$\mathcal{T}\,(\mathcal{T}^c)$	species of (connected) co-point-determining cographs.

Achknowledgements

Thanks to Professor Ira Gessel for his support.

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