Ramsey numbers $R(K_3, G)$ for graphs of order 10

Gunnar Brinkmann

Department of Applied Mathematics & Computer Science
Ghent University
Krijgslaan 281-S9,
9000 Ghent, Belgium

gunnar.brinkmann@ugent.be

Jan Goedgebeur

Department of Applied Mathematics & Computer Science
Ghent University
Krijgslaan 281-S9,
9000 Ghent, Belgium

jan.goedgebeur@ugent.be

Jan-Christoph Schlage-Puchta

Department of Mathematics
Ghent University
Krijgslaan 281-S22,
9000 Ghent, Belgium

jcsp@cage.ugent.be

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Abstract

In this article we give the generalized triangle Ramsey numbers $R(K_3, G)$ of 12 005 158 of the 12 005 168 graphs of order 10. There are 10 graphs remaining for which we could not determine the Ramsey number. Most likely these graphs need approaches focusing on each individual graph in order to determine their triangle Ramsey number. The results were obtained by combining new computational and theoretical results. We also describe an optimized algorithm for the generation of all maximal triangle-free graphs and triangle Ramsey graphs. All Ramsey numbers up to 30 were computed by our implementation of this algorithm. We also prove some theoretical results that are applied to determine several triangle Ramsey numbers larger than 30. As not only the number of graphs is increasing very fast, but also the difficulty to determine Ramsey numbers, we consider it very likely that the table of all triangle Ramsey numbers for graphs of order 10 is the last complete table that can possibly be determined for a very long time.

Keywords: Ramsey number; triangle-free graph; generation

1 Introduction

The Ramsey number $R(G, H)$ of two graphs G and H is the smallest integer r such that every assignment of two colours (e.g. red and blue) to the edges of K_r gives G as a red subgraph or H as a blue subgraph. Or equivalently $R(G, H)$ is the smallest integer r such that every graph F with at least r vertices contains G as a subgraph, or its complement F^c contains H as a subgraph. A graph F is a *Ramsey graph* for a pair of graphs (G, H) if F does not contain G as a subgraph and its complement F^c does not contain H as a subgraph.

The existence of $R(G, H)$ follows from Ramsey's theorem [16] from 1930. The classical Ramsey numbers (where both G and H are complete graphs) are known to be extremely difficult to determine. It is even difficult to obtain narrow bounds when H or G have a large order. Therefore only few exact results are known. The last exact result was obtained by McKay and Radziszowski [13] in 1995 when they proved that $R(K_4, K_5) = 25$.

For a good overview of the results and bounds of Ramsey numbers which are currently known, we refer the reader to Radziszowski's dynamic survey [15].

In this article, we focus on triangle Ramsey numbers, that is Ramsey numbers $R(G, H)$ where $G = K_3$. When we speak about Ramsey numbers or Ramsey graphs in the remainder of this article, we always mean triangle Ramsey numbers resp. triangle Ramsey graphs.

Already in 1980 all triangle Ramsey numbers for graphs of order 6 were determined by Faudree, Rousseau and Schelp [8]. In 1993 the Ramsey numbers for connected graphs of order 7 were computed by Jin Xia [19]. Unfortunately some of his results turned out to be incorrect. These were later corrected by Brinkmann [3] who determined all triangle Ramsey numbers for connected graphs of order 7 and 8 by using computer programs. Independently, Schelten and Schiermeyer also determined Ramsey numbers of graphs of order 7 by hand [17, 18].

In [1] all triangle Ramsey numbers for connected graphs of order 9 and all Ramsey numbers $R(K_3, G) \leq 24$ for connected graphs of order 10 are given. For 2001 graphs of order 10 the Ramsey number remained open.

We used the same basic approach for the generation of maximal triangle-free graphs (in short, *mtf graphs*) that was already used in [1], but some observations about the structure of mtf graphs in [1] made it possible to improve the basic algorithm. It was observed that an astonishingly large ratio of small mtf graphs had an automorphism group of size 2 caused by two vertices with identical neighborhoods.

We implemented the optimized algorithm for the generation of mtf graphs and also added improved routines for the restriction to Ramsey graphs. Using this program we independently verified the results from [1] and determined all Ramsey numbers $R(K_3, G)$ up to 30 for connected graphs G of order 10. The improved algorithm is described in section 3. Next to these computational results, we also proved some theoretical results that allowed to determine the Ramsey number of several graphs with Ramsey number larger than 30. Combining these computational and theoretical results, only 10 graphs with 10 vertices are left for which the triangle Ramsey number is unknown. We hope that other researchers will help to complete this list of triangle Ramsey numbers which will then most likely be the last complete list of triangle Ramsey numbers for a very long time.

As human intuition and insight is often based on examples, data about small graphs – like complete lists of Ramsey numbers – can help to discover mathematical theorems, suggest conjectures and give insight into the structure of mathematical problems. An example is given in [5], where a large amount of computational data about *alpha*-labelings gave insight into the structure of *alpha*-labelings of trees so that new theorems could be proven and some unexpexted conjectures were suggested. In order not to be mislead by too small examples, it is important to have as much data as possible to develop a good intuition, as e.g. the following example shows: If $K_n - (m \cdot e)$ denotes the graph obtained by removing m disjoint edges from K_n, then the previously existing lists of triangle Ramsey numbers for graphs of order at most 9 have the property that for fixed n the value $R(K_3, K_n - (m \cdot e))$ is the same for all $2 \leq m \leq n/2$. This may be considered as a hint that it could be true in general, but the list in this article shows that for $n = 10$ this equation does not hold.

2 General results

In this section we prove some general results on Ramsey numbers of the form $R(K_3, G)$, where G is close to a complete graph. Let T_{s+} denote the tree obtained from $K_{1,s}$ by adding an extra vertex and connecting it to a vertex with degree 1 in $K_{1,s}$. We write Δ_s for the graph obtained from $K_{1,s}$ by adding one edge connecting two vertices with degree 1 in $K_{1,s}$, and $D_{s,t}$ for the double star obtained from the disjoint union of $K_{1,s}$ and $K_{1,t}$ by joining the vertices with degrees s and t. We denote the set of vertices of a graph G by $V(G)$ and the set of edges by $E(G)$. We denote the neighbourhood of a vertex $v \in V(G)$ by $N(v)$.

The first result is a slight modification of Theorem 1 from [1]. We give the proof here, as we shall use the same argument repeatedly.

Lemma 2.1. *Let M be a triangle-free graph on r vertices, such that M^c contains K_{n-1}, and let s be an integer satisfying $1 \leq s < n$ and $(r-n)(s+1) > (n-1)(n-2)$. Then M^c contains $K_n - K_{1,s}$.*

Proof. Suppose otherwise. If there exists a vertex with degree at least n, then M^c contains K_n, since M is triangle-free. Now assume that there exists a vertex v with degree $n-1$. Then the neighbourhood of v consists of an anti-clique of size $n-1$, that is, deleting this vertex we obtain a graph M' with $r-1$ vertices, which contains an anti-clique of size $n-1$, such that each vertex in this anti-clique has degree $\leq n-2$. If there is no vertex with degree $n-1$, we delete an arbitrary vertex not contained in some specified anti-clique of size $n-1$. In each case we obtain an induced subgraph M' of M with $r-1$ vertices, which contains an anti-clique A of size $n-1$, such that every vertex in this anti-clique has degree at most $n-2$. From each vertex in $V \setminus A$ there are at least $s+1$ edges connecting this vertex with an element of A, for otherwise adding this vertex to A we would obtain a supergraph of $K_n - K_{1,s}$ in M^c. Hence, there exists a vertex $v \in A$, which has degree at least $\frac{(r-n)(s+1)}{n-1}$. By assumption this quantity is larger than $n-2$, contradicting the choice of M'. Hence our claim follows. □

Proposition 2.2. *Let r, n, s be integers such that $1 \leq s < n$ and $(r-n)(s+1) > (n-1)(n-2)$. Then for every triangle-free graph M on r vertices, such that M^c contains $K_n - \Delta_{s+1}$, we have that M^c contains $K_n - T_{s+}$.*

Proof. Let M be a counterexample. Since M is triangle-free, at least one of the edges in the triangle missing in $K_n - \Delta_{s+1}$ must be present in the subgraph of M^c containing $K_n - \Delta_{s+1}$. So M^c contains $K_n - T_{s+}$ or $K_n - K_{1,s+1}$. In the former case our claim follows immediatelly, while in the latter we have that M^c contains K_{n-1}, and by Lemma 2.1 we obtain that M^c contains $K_n - K_{1,s}$ and therefore also $K_n - T_{s+}$. □

Proposition 2.3. *Suppose that r, n, s satisfy $r \geq R(K_3, K_{n-1} - e)$, $(r-n+1)s > (n-2)(n-3)$ and $(r-n)(s+1) > (n-1)(n-2)$. Then $r \geq R(K_3, K_n - T_{s+})$.*

Proof. Let M be a triangle-free graph on r vertices. If M^c contains K_{n-1} our claim follows from Lemma 2.1. So we may assume that every vertex in M has degree at most $n-2$ and that M^c does not contain K_{n-1}. From $r \geq R(K_3, K_{n-1} - e)$ it now follows that M^c contains $K_{n-1} - e$ as an induced subgraph. Let A be a set of $n-1$ vertices of M, such that the edge (v, w) is the unique edge between vertices in A. As at most $(n-2)(n-3)$ edges go from $A - \{v, w\}$ to $V \setminus A$ and $(r-(n-1))s > (n-2)(n-3)$, there exists a vertex $x \in V \setminus A$ which is connected to at most $s-1$ elements of $A - \{v, w\}$. If x was connected to both v and w, then M would contain a triangle, hence the induced subgraph on $A \cup \{x\}$ is contained in T_{s+}. We conclude that M^c contains $K_n - T_{s+}$, and our claim follows. □

Proposition 2.4. *Let n be an integer, M be a triangle-free graph, such that M^c contains K_n. Assume further that $|V| \geq 3n+4$. Then M^c contains $K_{n+2} - D_{m,m}$, where $m = \lfloor \frac{n-1}{2} \rfloor$.*

Proof. Assume the statement was false, and that M was a counterexample.

Fix an anti-clique of size n and call it A. We now partition $V - A$ into four sets: L, the set of large vertices, which are connected with more than $n/2$ elements of A; H, the set of medium vertices, which are connected to exactly $n/2$ vertices of A; S, the set of small vertices, which are connected to at least 1, but at most m vertices of A; and X, the set of exceptional vertices, which are not connected to A. Note that medium vertices can only exist for even n.

If there are two different vertices $v, w \in X$, then $A \cup \{v, w\}$ contains at most one edge, and our claim follows. Hence we may assume that $|X| \leq 1$ and therefore $|L \cup H \cup S| \geq 2n+3$. We will prove that the graph induced by $L \cup H \cup S$ is bipartite and therefore contains an anti clique of size $n+2$ contradicting the assumption.

If 2 vertices $v \neq w \in S$ were adjacent, $N(v) \cap A$ and $N(w) \cap A$ would be disjoint as M is triangle-free. But then $A \cup \{v, w\}$ would induce a supergraph of $K_{n+2} - D_{m,m}$ in M^c.

Vertices in L can not be adjacent with vertices in $L \cup H$ as the two endpoints of the edge would have to have a common neighbour in A.

So cycles in the graph induced by $L \cup H \cup S$ contain either only vertices from H or can be split into parts by vertices from S.

If two vertices $v \neq w \in H$ are adjacent, due to M being triangle-free and each vertex having $n/2$ neighbours in A, we have $N(w) \cap A = A \setminus N(v)$.

If $v_1, \ldots, v_k = v_1$ is a cycle containing only vertices from H, then we have $N(v_{i+1}) \cap A = A \setminus N(A)$. So $v_k = v_1$ implies that k must be even.

If a cycle contains elements from S, then each part between two subsequent vertices v, w (which can be the same) from S contains an even number of edges: If v is followed by a vertex from L, then the next vertex is w – so the segment contains two edges. If there was a path $v, x_1, x_2, \ldots, x_k, w$ with $k > 0$ even, then $N(x_1) \cap A = A \setminus N(x_k)$. As M is triangle-free and v, x_1 are adjacent we have $N(v) \subset N(x_k) \cap A$ and analogously $N(w) \subset N(x_1) \cap A$ – so the neighbourhoods of v and w are disjoint in A, so that $A \cup \{v, w\}$ would again induce a supergraph of $K_{n+2} - D_{m,m}$ in M^c. So k must be odd and the segment contains an even number of edges. This implies that each cycle consists of a certain number of segments of even length and is therefore even – proving that the graph induced by $L \cup H \cup S$ is bipartite. $\square$

Proposition 2.5. *Let n, r, s, t be integers, such that $s+t+2 \leq n$, $s \geq t > 0$, $(r-n)(s+1) > (n-1)(n-2)$, and $(r-(n-1))(s+1) > (n+2(s-t)-2)(n-3)$. Then every graph on r vertices which contains $K_{n-1} - e$ contains $K_n - K_{1,s} - K_{1,t}$.*

Proof. Assume our statement is false, and let M be a counterexample. If M^c contains K_{n-1}, then Lemma 2.1 shows that M^c contains $K_n - K_{1,s}$, and we are done. This implies also that all vertices in M have degree at most $n-2$. Let A be a set of $n-1$ vertices, such that among the vertices of A there is a single edge (v, w). Put $X = N(v) - \{w\}$, $Y = N(w) - \{v\}$. As M is triangle-free, the sets X, Y are disjoint anti-cliques and as the degrees of v, W are at most $n-2$, we have $|X|, |Y| \leq n-3$. Each element z of $|X|, |Y|$ has at least two neighbours in A, as otherwise $A \cup \{z\}$ would induce a $K_n - K_{1,2}$ in M^c.

Suppose that X contains elements x, x', such that x is connected with at most t elements of $A - \{v, w\}$, and x' is connected with at most s elements of $A - \{v, w\}$. Then all edges in $A \cup \{x, x'\} - \{v\}$ are between $\{x, x'\}$ and elements of $A - \{v, w\}$, and we obtain $K_n - K_{1,s} - K_{1,t}$ in M^c.

Hence either each element in X is connected with at least $t+1$ elements in $A - \{v, w\}$, or all but at most one element of X is connected with at least $s+1$ elements of $A - \{v, w\}$ and the remaining element is connected with at least one element of $A - \{v, w\}$. The same argument applies for Y. An element x of $V - (A \cup X \cup Y)$ is not connected to v or w, hence if this element is connected with $p \leq s$ elements of $A - \{v, w\}$, then $A \cup \{v\}$ forms a $K_n - K_{1,p} - e$ in the complement that has $K_n - K_{1,s} - K_{1,t}$ as a subgraph. We conclude that each vertex in $V - (A \cup X \cup Y)$ has at least $s+1$ neighbours in $A - \{v, w\}$.

Counting the edges between $A - \{v, w\}$ and $V - A$ we get a lower bound of

$$\min\big((t+1)|X|,(s+1)(|X|-1)+1\big)+\min\big((t+1)|Y|,(s+1)(|Y|-1)+1\big) \\ +(s+1)(r-(n-1)-|X|-|Y|)$$

As a function of $|X|$ and $|Y|$ this expression is non-increasing, hence this quantity has its minimum for $|X|=|Y|=n-3$, which gives

$$2\min\big((t+1)(n-3),(s+1)(n-4)+1\big)+(s+1)(r-3n+7).$$

On the other hand each vertex in $A-\{v,w\}$ has degree at most $n-2$, giving an upper bound of $(n-2)(n-3)$. Now for $s=t$ we obtain $(s+1)(r-n-1)+2 \leq (n-2)(n-3)$, which contradicts our assumption $(r-n)(s+1)>(n-1)(n-2)$. If $t<s$, we obtain $(s+1)(r-n+1)-2(s-t)(n-3) \leq (n-2)(n-3)$, which contradicts our assumption $(r-(n-1))(s+1)>(n+2(s-t)-2)(n-3)$. Hence, in both cases our claim follows. □

Applying these results to the case of graphs on 10 vertices, we obtain the following:

Corollary 2.6. *1. For $9 \geq s \geq 2$ we have $R(K_3, K_{10}-K_{1,s})=36$;*

2. For $8 \geq s \geq 3$ we have $R(K_3,K_{10}-T_{s+})=R(K_3,K_{10}-\Delta_{s+1})=R(K_3,K_{10}-K_{1,s}-e)=31$;

3. We have $R(K_3,K_{10}-D_{3,3})=28$.

Proof. The upper bounds follow from the propositions, while the lower bounds are implied by $R(K_3,K_9)=36$, $R(K_3,K_9-e)=31$, and $R(K_3,K_8)=28$, respectively. □

3 The algorithm

A *maximal* triangle-free graph (in short, an *mtf graph*) is a triangle-free graph so that the insertion of each new edge introduces a triangle. For $|V|>2$ this is equivalent to being triangle-free and having diameter 2.

As adding edges to a triangle-free graph removes edges from its complement, it is easy to see that there is a triangle Ramsey graph of order r for some graph G if and only if there is an mtf graph of order r that is a Ramsey graph for G (in short, an *mtf Ramsey graph*).

In order to prove that $R(K_3,G)=r$, we have to show that:

- There are no mtf Ramsey graphs for G with r vertices (which implies $R(K_3,G) \leq r$).
- There is an mtf Ramsey graph for G with $r-1$ vertices (which implies $R(K_3,G)>r-1$).

Even though only a very small portion of the triangle-free graphs are also maximal (e.g. 0.002% for 13 vertices and 0.000044% for 16 vertices), the number of mtf graphs still grows very fast (see Table 1). Thus it is not possible for large r to generate all mtf graphs with r vertices and test if they are Ramsey graphs for a given G. Therefore it is necessary to include the restriction to Ramsey graphs already in the generation process.

In section 3.1 we describe an algorithm for the generation of all non-isomorphic mtf graphs. This algorithm follows the same lines as the algorithm in [2] but uses some structural information obtained from [2] to speed up the generation. In section 3.2 we describe how we extended this algorithm to generate only mtf Ramsey graphs for a given graph G. In section 3.3 we describe how we used the generator for Ramsey graphs to determine the Ramsey numbers $R(K_3, G)$. The main difference to the approach described in [1] is that the approach used here is optimized for small lists of graphs with larger Ramsey numbers instead of large lists with comparatively small Ramsey numbers like the approach in [1].

3.1 Generation of maximal triangle-free graphs

Mtf graphs with $n+1$ vertices are generated from mtf graphs with n vertices using the same construction method as in [2] but different isomorphism rejection routines. To describe the construction, we first introduce the concept of *good dominating sets*.

Definition 1. $S \subseteq V(G)$ is a dominating set of G if $S \cup \{N(s) \mid s \in S\} = V(G)$.

A dominating set S of an mtf graph G is *good* if after removing all edges with both endpoints in S for every $s \in S$ and $v \in V(G) \setminus S$, the distance from s to v is at most two.

The basic construction operation removes all edges between vertices of a good dominating set S and connects all vertices of S to a new vertex v. It is easy to see that this is a recursive structure for the class of all mtf graphs [2].

In [2] it was observed that a surprisingly large number of mtf graphs had automorphism groups of size 2. This was caused by two vertices with identical neighbourhoods. We exploit this observation to improve the efficiency of the isomorphism rejection routines. To this end we distinguish between 3 types of good dominating sets.

type 0: A set $S = N(v)$ for some $v \in V$. Note that in an mtf graph for each vertex v the set $N(v)$ is a good dominating set without internal edges.

type 1: A good dominating set S without internal edges, but $S \neq N(v)\, \forall v \in V$.

type 2: A good dominating set S with internal edges.

We call construction operations also *expansions* and the inverse operations *reductions* and will also talk about reductions or expansions of type 0, 1 and 2 if the good dominating sets involved are of this type. If G' is obtained from G by an expansion, we call G' the child of G and G the parent of G'.

We use the canonical construction path method [12] to make sure that only pairwise non-isomorphic mtf graphs are generated. Two reductions of mtf graphs G and G' (which may be identical) are called equivalent if there is an isomorphism from G to G' mapping the vertices that are removed onto each other and inducing an isomorphism of the reduced graphs. In order to use this method, we first have to define which of the various possible reductions of an mtf graph G to a smaller mtf graph is the *canonical reduction* of G. This canonical reduction must be uniquely determined up to equivalence. We call the graph obtained by applying the canonical reduction to G the *canonical parent* of G and an expansion that is the inverse of a canonical reduction a *canonical expansion*.

Furthermore, we also define an equivalence relation on the set of possible expansions of a graph G. Note that the expansions are uniquely determined by the good dominating set S to which they are applied. Therefore we define two expansions of G to be equivalent if and only if there is an automorphism of G mapping the two good dominating sets onto each other.

The two rules of the canonical construction path method are:

(a) Only accept a graph if it was constructed by a canonical expansion.

(b) For every graph G to which construction operations are applied, perform exactly one expansion from each equivalence class of expansions of G.

If we start with K_1 and recursively apply these rules to each graph until the output size is reached, exactly one graph of each isomorphism class of mtf graphs is generated. We refer the reader to [2] for a proof. The coarse structure of the algorithm is given as pseudocode in Algorithm 1.

Algorithm 1 Construct(mtf graph G)

```
if G has the desired number of vertices then
  output G
else
  find expansions
  compute classes of equivalent expansions
  for each equivalence class do
    choose one expansion X
    perform expansion X
    if expansion is canonical then
      Construct(expanded mtf graph)
    end if
    perform reduction X^{-1}
  end for
end if
```

For deciding whether or not a reduction is canonical, we use a two step strategy. First we decide which vertex should be removed by the canonical reduction. In case the graph is not an mtf graph after the removal of this vertex, we determine the canonical way to insert edges. A 5-tuple $t(v) = (x_0(v), \ldots, x_4(v))$ represents the vertex v involved in the reduction in such a way that two vertices have the same 5-tuple if and only if they are in the same orbit of the automorphism group. The canonical reduction will be a reduction using the vertex with lexicographically smallest 5-tuple.

The first entry $x_0(v)$ is the type of the neighbourhood of v in the reduced graph. The most expensive part in computing the canonical reduction is the computation of how edges have to be inserted between the former neighbours of the removed vertex. If the graph has vertices with identical neighbourhood, a reduction with $x_0 = 0$ is always possible and other reductions do not have to be considered in order to find the one with minimal 5-tuple. In case there are exactly two vertices with identical neighbourhood, the canonical reduction is even found after this step: no matter how the remaining entries of the 5-tuple are defined, removing one of these two vertices is the canonical reduction as they are the only ones with minimal value for x_0. Furthermore there is an automorphism exchanging the vertices and fixing the rest, so the two reductions are equivalent and both are canonical.

The way the remaining values are chosen is the result of a lot of performance tests comparing different choices. The value of $x_1(v)$ is the degree $-deg(v)$ of the vertex v that is to be removed in case $x_0(v) \in \{0, 1\}$ and $deg(v)$ in case $x_0(v) = 2$. Furthermore $x_2(v) = -\sum_{w \in N(v)} deg(w)$ and $x_3(v)$ can be described as $-\sum_{w \in N(v)} |V|^{deg(w)}$. In the program x_3 is in fact implemented as a sorted string of degrees, but it results in the same ordering.

We call a vertex v *eligible* for position j if it is among the vertices for which $(x_0(v), ..., x_{j-1}(v))$ is minimal among all possible reductions. For for step (a) of the canonical construction path method we do not have to find the canonical reduction, but only have to determine whether the last expansion producing vertex w is canonical. Therefore each x_i is only computed if the vertex w is still eligible for position i and only for vertices which are eligible for position i. If $x_0(w) \in \{0, 1\}$ and w is the only vertex eligible for position i, we know that the expansion was canonical. If $x_0(w) = 2$, we still have to determine whether the edges that have been removed are equivalent to the edges that would be inserted for a canonical reduction.

If there are also other vertices eligible for position 4, we canonically label the graph G using the program *nauty* [11] and define $x_4(v)$ to be the negative of the largest label in the canonical labelling of G of a vertex which is in the same orbit of the automorphism group of G as v. The discriminating power of $x_0, ..., x_3$ is usually enough to decide whether or not a reduction is canonical. For example for generating all mtf graphs with $n = 20$ vertices, the more expensive computation of x_4 is only required in 7.8% of the cases. This fraction is decreasing with the number of vertices to e.g. 6.2% for $n = 22$. After computing x_4 the vertex in the canonical reduction is uniquely defined up to isomorphism.

In case $x_0 = 2$ the canonical reduction is not completely determined by the vertex v which is removed by the canonical reduction as there can be multiple ways to insert the edges in the former neighbourhood of v. In this case we use the same method as in [2], which is essentially a canonical choice of a set of edges that can be inserted which gives priority to sets of small size. This part hardly has any impact on the time consumption of the program. For generating mtf graphs with $n = 18$ vertices, only about 2.5% of the time is spent on the routines dealing with this part and already for $n = 20$ this decreases to 1.5%. Therefore we decided not to develop any improvements for this part and refer the reader to [2] for details.

The priority of the operations expressed in the 5-tuple allows look-aheads for deciding whether or not an expansion can be canonical before actually performing it. This is also an advantage when constructing good dominating sets for expansion as it often allows to reduce the number of sets that have to be constructed.

A vertex which has the same neighbourhood as another vertex is called a *double vertex*. An mtf graph with double vertices can be reduced by a reduction of type 0. If two vertices have the same neighbourhood in an mtf graph, each good dominating set without internal edges either contains both vertices or none, so after an operation of type 0 or 1 the vertices still have identical neighbourhoods allowing a reduction of type 0. So if a graph G contains a reduction of type 0, we do not have to apply expansions of type 1. Furthermore we only have to apply expansions of type 0 to neighbourhoods of vertices v of G for which $deg(v)$ is at least as large as the degree of the canonical double vertex in G, otherwise the new vertex will not have the maximal value of x_1. If G did not contain any double vertices, we have to apply operations of type 0 to the neighbourhoods of all vertices.

After a canonical operation of type 2 no reductions of type 0 are possible, so we only have to apply operations of type 2 that make sure that afterwards no vertices with identical neighbourhoods exist. Therefore the good dominating sets to which an operation of type 2 is applied must contain at least one vertex from the neighbourhood of each double vertex. Since if no vertex of the common neighbourhood of a pair of double vertices is included, both vertices must be contained in the dominating set themselves. But then they would still have identical neighbourhoods after the operation. Each good dominating set must also contain a vertex from each set of vertices with identical neighbourhood. In the program we use this in its strongest form only if there is just one common neighbourhood, else we use a weaker form. This is not a problem for the efficiency as there is usually only one common neighbourhood.

If a graph has at least 3 vertices which have the same neighbourhood, every graph obtained by applying an expansion of type 2 to G has a reduction of type 0. Therefore we do not have to apply expansions of type 2 to this kind of graphs.

Due to the choice of x_1 the degree of the vertex to be removed is minimal for canonical reductions of type 2. If we apply an operation of type 2 to a good dominating set S, the new vertex v will have degree $|S|$. If the minimum degree of a graph is m, we only have to apply operations of type 2 to good dominating sets of size at most m (or size $m + 1$ if the good dominating set contains all vertices of minimum degree).

Recall that we also have to compute the equivalence classes of expansions of a graph in order to comply with rule (b) of the canonical construction path method. We use *nauty* to compute the automorphism group of the graph and then compute the orbits of good dominating sets using the generators of the group. In case we know that only an operation of type 0 can be canonical we actually compute the orbits of vertices representing the good dominating sets formed by their neighbourhoods.

In some cases we do not have to call *nauty* to compute the automorphism group. For example if G has a trivial automorphism group and we apply an operation of type 0 by inserting a vertex v' with the same neighbourhood as v, the expanded graph G' will have an automorphism group of size 2 generated by the automorphism exchanging v and v' and fixing all other vertices.

Testing and results

We used our program to generate all mtf graphs up to 23 vertices. The number of graphs generated were in complete agreement with the numbers obtained by running the program from Brandt et al. [2] (which is called *MTF*). The graph counts, running times and a comparison with *MTF* are given in Table 1. Our program is called *triangleramsey*. Both programs were compiled by gcc and the timings were performed on an Intel Xeon L5520 CPU at 2.27 GHz. The timings for $|V(G)| \geq 20$ include a small overhead due to parallelisation.

Table 2 gives an overview how many graphs are constructed by canonical operations of the different types. This table shows that operations of type 2 are by far the least common canonical operations.

$\lvert V(G)\rvert$	number of graphs	MTF (s)	triangleramsey (s)	speedup
17	164 796	4.0	0.8	5.00
18	1 337 848	30.5	6.2	4.92
19	13 734 745	315	67	4.70
20	178 587 364	4 390	972	4.52
21	2 911 304 940	75 331	17 109	4.40
22	58 919 069 858	1 590 073	373 417	4.26
23	1 474 647 067 521	40 895 299	10 431 362	3.92

Table 1: Counts and generation times for mtf graphs.

3.2 Generation of Ramsey graphs

The construction operations for mtf graphs never add edges between vertices of the parent. So if G is contained in the complement of an mtf graph M, G will also be contained in the complement of all descendants of M. Thus if M is not a Ramsey graph for G, its descendants also won't be Ramsey graphs. So we can prune the generation process.

The same pruning was already used in [1], but as the graphs with 10 vertices whose Ramsey number could not be determined in [1] are all very dense, we mainly optimized our algorithm for this kind of graphs and will describe these optimisations here.

number of vertices	number of mtf graphs	num. generated by an operation of type 0	num. generated by an operation of type 1	num. generated by an operation of type 2
4	2	2	0	0
5	3	2	0	1
6	4	4	0	0
7	6	6	0	0
8	10	9	0	1
9	16	15	0	1
10	31	29	1	1
11	61	57	3	1
12	147	139	4	4
13	392	368	15	9
14	1 274	1 183	75	16
15	5 036	4 595	391	50
16	25 617	22 889	2 420	308
17	164 796	142 718	19 577	2 501
18	1 337 848	1 105 394	213 743	18 711
19	13 734 745	10 674 672	2 855 176	204 897
20	178 587 364	129 333 325	46 244 514	3 009 525

Table 2: The number of mtf graphs which were generated by operations of each type.

For a graph G and an mtf graph M the following criteria are equivalent:

(i) G is subgraph of M^c

(ii) M contains a spanning subgraph of G^c as an induced subgraph

If G is dense, G^c has relatively few edges and therefore it is easier to test (ii) instead of (i) in this case.

By just applying this simple algorithm, even with the faster generator we were not able to go much further than the results in [1]. Therefore we designed and applied several optimisations specifically for dense graphs. These optimisations are crucial for the efficiency of the algorithm.

The bottleneck of the algorithm is the procedure which tests if the generated mtf graphs contain a spanning subgraph of G^c as induced subgraph. This procedure basically constructs all possible sets with $|V(G)|$ vertices and an upper bound of $|E(G^c)|$ on the number of edges and tests for each set if the graph induced by this set is a subgraph of G^c. Various bounding criteria are used to avoid the construction of sets which cannot be a subgraph of G^c.

If the algorithm as described so far is applied and the order of the mtf graphs is sufficiently large, by far most of the mtf graphs that are generated are rejected as they turn out to be no Ramsey graphs for the testgraph G. For example for $G = K_{10} - P_5$ and $|V(M)| = 28$ (without other optimisations) approximately 99% of the mtf graphs which were generated are no Ramsey graphs (and are thus rejected). So most of the tests for making spanning induced subgraphs give a positive result – that is: there is an induced subgraph of M that is a subgraph of G^c. We take this into account by first using some heuristics to try to find a set of vertices which is a spanning subgraph of G^c quickly. If such a set is found, we can abort the search.

More specifically: when an mtf graph is rejected because it is not a Ramsey graph for G, we store the set of vertices which induces a spanning subgraph of G^c. For each order n, we store up to 100 sets of vertices which caused an mtf graph with n vertices to be rejected. When a graph with n vertices is generated, we first investigate if one of those 100 sets of vertices induces a spanning subgraph of G^c. Only if this is not the case, we continue the search. Experimental results showed that storing 100 sets seemed to be a good compromise between cost to test if a set induces a spanning subgraph of G^c and the chance to have success. Without other optimisations this makes the program e.g. 5 times faster for $G = K_{10} - P_5$ and $|V(M)| = 26$.

The second step in trying to prove that M is not a Ramsey graph is a greedy heuristic. We construct various sets of $|V(G)|$ vertices which have as few neighbours with each other as possible. These sets are good candidates to induce a subgraph of G^c. Only if none of these sets induces a subgraph of G^c, we have to continue to investigate the graph. This gives an additional speedup of approximately 10%.

These heuristics allow to find a set of vertices which induces a spanning subgraph of G^c quickly in about 98% of the cases. If these heuristics did not yield such a set of vertices, we start a complete search. In about 70% of the cases the graphs passing the heuristical search are actually Ramsey graphs for G. The coarse pseudocode of the procedure which tests if an mtf graph M is a Ramsey graph for G is given in Algorithm 3.2.

Algorithm 2 Is_Ramsey_graph(mtf graph M, testgraph G)

```
for each stored set S with n = |V(M)| do
  if S induces a spanning subgraph of G^c in M then
    return M is not a Ramsey graph for G
  end if
end for
construct sets of |V(G)| vertices in a greedy way
if set found which induces a spanning subgraph of G^c in M then
  store set
  return M is not a Ramsey graph for G
end if
construct all possible sets of |V(G)| vertices
if set found which induces a spanning subgraph of G^c in M then
  store set
  return M is not a Ramsey graph for G
else
  return M is a Ramsey graph for G
end if
```

The construction of all possible sets of $|V(G)|$ vertices can also be improved. Recall that our algorithm constructs Ramsey graphs from Ramsey graphs. Therefore if an mtf graph M was constructed by operations of type 0 or 1 (i.e. no edges were removed), we only have to investigate sets of vertices which contain the new vertex which was added by the construction. The subgraphs induced by the other sets did not change and are already proven not to induce a spanning subgraph of G^c. Moreover if M was constructed by an operation of type 0, we only have to investigate sets of vertices which contain the new vertex and all other vertices which have the same neighbourhood as the new vertex. Since if a set does not contain a vertex v which has the same neighbourhood as the new vertex, we can swap v and the new vertex.

Similarly, if M was constructed by an operation of type 2 and one edge e was removed (say $e = \{v_1, v_2\}$), we only have to investigate sets of vertices which contain the new vertex or which contain both v_1 and v_2. Similar optimisations can also be used when more edges are removed, but this does not speed up the program as in most cases such operations turn out to be not canonical. So then the graph is already rejected before it is tested whether or not this graph is a Ramsey graph.

We also avoid constructing mtf graphs that are no Ramsey graphs for G. This is of course even better than efficiently rejecting graphs after they are constructed. More specifically, each time a new mtf Ramsey graph M for a graph G was constructed, we search and store *approximating* sets of vertices. We call a set of vertices *approximating* if it induces a spanning subgraph of G^c_δ, where G^c_δ is a graph obtained by removing a vertex of minimum degree from G^c. For all graphs G with 10 vertices whose Ramsey number could not be determined in [1], G^c has minimum degree 0.

If for a graph M' which is constructed from M there is an approximating set S of

F for which no vertex $s \in S$ is a neighbour of the new vertex v, the graph induced by $S \cup \{v\}$ in F' is a spanning subgraph of G^c. So graphs constructed from M can only be Ramsey graphs if the good dominating set of M contains at least one vertex from each approximating set in F. On average this optimisation avoids the construction of more than 90% of the children.

Since searching for all approximating sets is expensive, we search for them during the search for sets of vertices which induce a spanning subgraph of G^c: when a set of $|V(G^c_\delta)|$ vertices was formed, we test if it is an approximating set and store it if it is the case.

3.3 Computing the Ramsey numbers

To determine the Ramsey numbers with our algorithm, we again use the same basic strategy as Brandt et al. used in [1]:

Assume we have a list of all graphs G with Ramsey number $R(K_3, G) \geq r$. We want to split this list into those with $R(K_3, G) = r$ and those with $R(K_3, G) > r$. We have a (possibly empty) list of MAXGRAPHs. These are graphs which have Ramsey number r. We also have a (possibly empty) list of RAMSEYGRAPHs, which are triangle-free graphs with r vertices which are (or might be) Ramsey graphs for some of the remaining graphs.

The procedure to test whether the remaining graphs have Ramsey number at most r or at least $r + 1$ works as follows:

for $k = \binom{n}{2}$ **downto** $n - 1$ **do**
  for every connected graph G with k edges in the list **do**
    if G is not contained in any MAXGRAPH **then**
      if G is contained in the complement of every RAMSEYGRAPH **then**
        if *triangleramsey* applied to G finds a Ramsey graph of order r **then**
          add this Ramsey graph to the list of RAMSEYGRAPHs
          $R(K_3, G) > r$
        else
          add G to the list of MAXGRAPHs
          $R(K_3, G) \leq r$
        end if
      else
        $R(K_3, G) > r$
      end if
    else
      $R(K_3, G) \leq r$
    end if
  end for
end for

For large orders of r (i.e. $r \geq 26$), the bottleneck of the procedure is computing individual Ramsey graphs by *triangleramsey*. Here we used some additional optimisations: if r is close to $R(K_3, G)$, there are usually only very few Ramsey graphs of order r for G. Therefore for certain graphs G where we expected r to be close to $R(K_3, G)$, we used *triangleramsey* to compute all Ramsey graphs of order r for G, instead of aborting the program as soon as one Ramsey graph was found. *Triangleramsey* constructs Ramsey graphs from smaller Ramsey graphs, so in order to construct all Ramsey graphs for G of order $r+1$, we can start the program from the Ramsey graphs of order r. This avoids redoing the largest part of the work. Of course this approach only works if there are not too many Ramsey graphs of order r to be stored. We used this strategy amongst others to generate all Ramsey graphs of order 28 for $K_{10}-P_5$ and $K_{10}-2P_3$ (where P_x stands for the path with x vertices). Computing all Ramsey graphs with 28 vertices for $K_{10}-2P_3$ for example, required almost 4 CPU years. This yielded 7 Ramsey graphs and constructing the Ramsey graphs with 29 vertices from these 7 graphs took less than 2 seconds.

Let H be a subgraph of G for which we know that $R(K_3, H) \geq r$. If we have the list of all Ramsey graphs of order r for G and none of these Ramsey graphs is a Ramsey graph for H, we know that $R(K_3, H) = r$. This allowed us amongst others to determine that several subgraphs of $K_{10} - P_5$ and $K_{10} - 2P_3$ have Ramsey number 28.

3.4 Testing and results

By using the algorithm described in section 3.3 we were able to compute all Ramsey numbers $R(K_3, G) = r$ for connected graphs of order 10 for which $r \leq 30$ and to determine which Ramsey graphs have Ramsey number larger than 30. Since *triangleramsey* is more than 20 times faster than *MTF* for generating triangle Ramsey graphs of large order r, we could only compute Ramsey numbers up to $r = 26$ with the original version of *MTF*. In order to be able to compare our results for larger r, we added several of the optimisations which were described in section 3.2 to *MTF*. With this improved version of *MTF* we were also able to determine all Ramsey numbers up to $r = 30$ and to determine which graphs have Ramsey number larger than 30. All results were in complete agreement.

In the cases where we generated all Ramsey graphs of order r for a given testgraph, the results obtained by *triangleramsey* and *MTF* were also in complete agreement.

For each Ramsey graph which was generated, we also used an independent program to confirm that the Ramsey graph does not contain G in its complement.

There are 34 graphs G for which $R(K_3, G) > 30$. In section 2, we proved that $R(K_3, K_{10} - T_{3+}) = R(K_3, K_{10} - K_{1,3} - e) = 31$. So the graphs with $R(K_3, G) > 30$ which are a subgraph of $K_{10} - T_{3+}$ or $K_{10} - K_{1,3} - e$ also have Ramsey number 31. In that section, we also proved that $R(K_3, K_{10} - K_{1,s}) = 36$ (for $2 \leq s \leq 9$). This leaves us with 10 graphs with $R(K_3, G) > 30$ for which we were unable to determine their exact Ramsey number. Also note that if G^c contains a triangle and H^c is the only graph which can be obtained by removing an edge from that triangle of G^c, then $R(K_3, G) = R(K_3, H)$. Thus among the 10 remaining graphs, $R(K_3, K_{10} - K_3 - e) = R(K_3, K_{10} - P_3 - e)$ and $R(K_3, K_{10} - K_4) = R(K_3, K_{10} - K_4^-)$ (where K_4^- stands for K_4 with 1 edge removed).

Table 4 contains the number of connected graphs G of order 10 which have $R(K_3, G) = r$. The triangle Ramsey numbers of connected graphs of order 10 are given in section 5.

Previously only the Ramsey numbers for disconnected graphs of order at most 8 were known (see [3]). We independently verified these results for order 8 and also determined all Ramsey numbers for disconnected graphs of order 9 and 10. These results are listed in Table 5. The Ramsey numbers smaller than 28 were obtained computationally. We also independently confirmed the computational results by using *MTF*. The other Ramsey numbers were obtained by some simple reasoning. More specifically, if a disconnected graph is the union of 2 connected graphs G_1 and G_2 and $R(K_3, G_1) - |V(G_1)| \geq R(K_3, G_2)$, then $R(K_3, G_1 \cup G_2) = R(K_3, G_1)$.

Unfortunately we did not succeed to compute new values of the functions $f(), g()$ and $h()$ given in Table 2 of [1]. Nevertheless we could confirm all the values given in Table 2 of [1] with the new program.

Classical Ramsey numbers

In 1992 McKay and Zhang [14] proved that $R(K_3, K_8) = 28$, but the complete set of Ramsey graphs with 27 vertices for K_8 was not yet known. Until now 430 215 such graphs were known (most of these were generated by McKay).

We used *triangleramsey* to compute all maximal triangle-free Ramsey graphs with 27 vertices for K_8. This yielded 21 798 mtf graphs. We also independently generated these Ramsey graphs with *MTF* and obtained the same results. We then recursively removed edges in all possible ways from these mtf Ramsey graphs to obtain the complete set of Ramsey graphs for $R(K_3, K_8)$ with 27 vertices. This yielded 477 142 Ramsey graphs. As a test we verified that all of the 430 215 previously known Ramsey graphs are indeed included in our list. Goedgebeur and Radziszowski [9] generated all Ramsey graphs for $R(K_3, K_8)$ with 27 vertices and at most 88 edges using an independent program and obtained the same results. The list can be downloaded from [10]. Table 3 contains the counts of these graphs according to their number of edges.

We did not construct the lists of all Ramsey graphs with less than 27 vertices for $R(K_3, K_8)$ as there are too much of these graphs to store.

4 Closing remarks

Since all computational results were independently obtained by both *MTF* and *triangleramsey*, the chance of wrong results caused by errors in the implementation is extremely small.

We believe that specialized algorithms and/or new theoretical results will be required to determine the triangle Ramsey number of the remaining 10 graphs and hope that this challenge to complete the possibly last complete list of triangle Ramsey numbers for a very long time will be taken up by the mathematical community..

Besides the already mentioned property that $n = 10$ is the first case where the Ramsey numbers of $R(K_3, K_n - (m \cdot e))$ are not the same for all $2 \leq m \leq n/2$, the most striking observation in the list is possibly that while for $7 \leq n \leq 9$ the graph $K_n - P_5$ has a smaller Ramsey number than $K_n - 2P_3$, for $n = 10$ they have the same Ramsey number (i.e. 29).

The latest version of *triangleramsey* can be downloaded from [6]. The list of the Ramsey graphs used in this research can be obtained from *House of Graphs* [4] by searching

Number of edges	Number of Ramsey graphs
85	4
86	92
87	1 374
88	11 915
89	52 807
90	122 419
91	151 308
92	99 332
93	33 145
94	4 746

Table 3: Counts of all 477 142 Ramsey graphs with 27 vertices for $R(K_3, K_8)$ according to their number of edges.

for the keywords "ramsey * order 10" and the Ramsey numbers can be obtained from [7].

Acknowledgements

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	$\|H\| = 3$	$\|H\| = 4$	$\|H\| = 5$	$\|H\| = 6$	$\|H\| = 7$	$\|H\| = 8$	$\|H\| = 9$	$\|H\| = 10$
$r = 5$	1							
$r = 6$	1							
$r = 7$		5						
$r = 8$								
$r = 9$		1	18					
$r = 10$								
$r = 11$			2	98				
$r = 12$				6				
$r = 13$				2	772			
$r = 14$			1	4	40			
$r = 15$						9 024		
$r = 16$					13	1 440		
$r = 17$				1	19	498	242 773	
$r = 18$				1	7	119	16 024	
$r = 19$							311	10 101 711
$r = 20$								504
$r = 21$					1	28	1 809	1 602 240
$r = 22$							22	3 155
$r = 23$					1	6	98	6 960
$r = 24$								
$r = 25$						1	26	1 384
$r = 26$							5	316
$r = 27$							3	92
$r = 28$						1	7	142
$r = 29$								30
$r = 30$								3
$r = 31$							1	≥ 16
$r = 36$							1	≥ 8

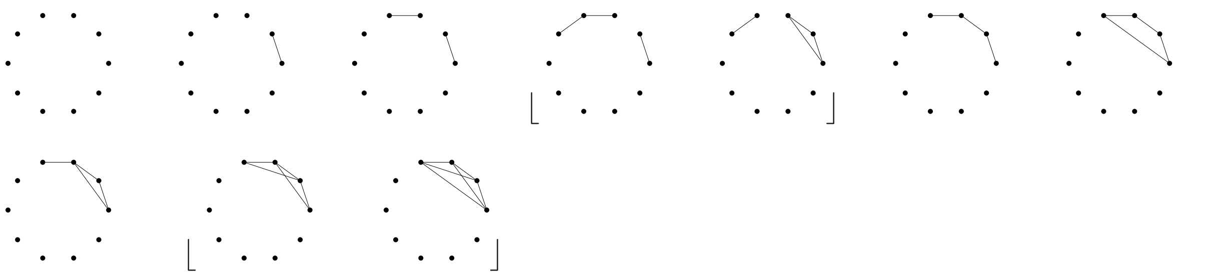
Table 4: Numbers of connected graphs H with Ramsey number $R(K_3, H) = r$. Note that the 10 graphs with $R(K_3, H) \geq 31$, but whose Ramsey number we were unable to determine are not included in the table.

	$\lvert H\rvert = 3$	$\lvert H\rvert = 4$	$\lvert H\rvert = 5$	$\lvert H\rvert = 6$	$\lvert H\rvert = 7$	$\lvert H\rvert = 8$	$\lvert H\rvert = 9$	$\lvert H\rvert = 10$
$r = 3$	2							
$r = 4$		2						
$r = 5$		2	4					
$r = 6$		1	3	7				
$r = 7$			5	11	18			
$r = 8$				3	5	23		
$r = 9$			1	20	50	60	83	
$r = 10$						36	68	151
$r = 11$				2	102	225	427	596
$r = 12$					6	12	144	168
$r = 13$					2	776	1 552	3 734
$r = 14$				1	6	52	107	447
$r = 15$							9 024	18 048
$r = 16$						13	1 466	2 933
$r = 17$					1	21	540	243 856
$r = 18$					1	9	137	16 301
$r = 19$								311
$r = 20$								
$r = 21$						1	30	1 869
$r = 22$								22
$r = 23$						1	8	114
$r = 24$								
$r = 25$							1	28
$r = 26$								5
$r = 27$								3
$r = 28$							1	9
$r = 31$								1
$r = 36$								1

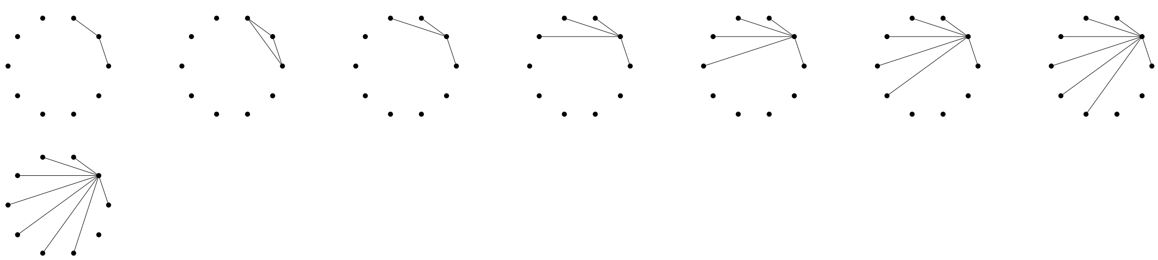
Table 5: Numbers of disconnected graphs H with Ramsey number $R(K_3, H) = r$.

5 The triangle Ramsey number for connected graphs of order 10

The following 10 graphs H^c have $R(K_3, H) > 30$, but we were unable to determine their Ramsey number. Graphs which must have the same Ramsey number are grouped by $\lfloor$ and $\rfloor$.

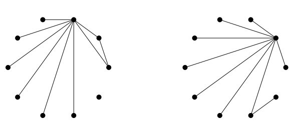


If H^c is one of the graphs:

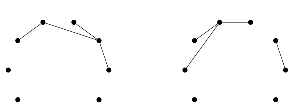


Then $R(K_3, H) = 36$. These are possibly not **all** graphs with $R(K_3, H) = 36$.

If H^c is contained in one of the graphs:

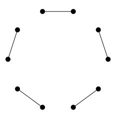


and contains one of the graphs:

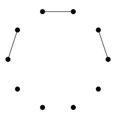


Then $R(K_3, H) = 31$. These are possibly not **all** graphs with $R(K_3, H) = 31$.

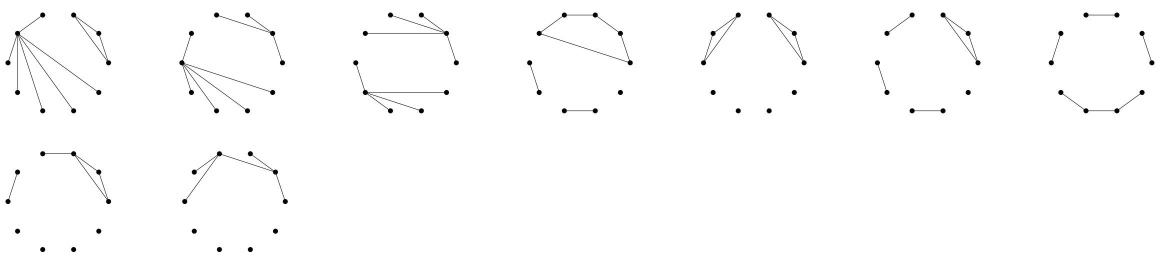
$R(K_3, H) = 30$ if and only if H^c is contained in:



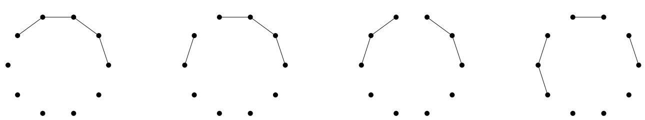
and contains:



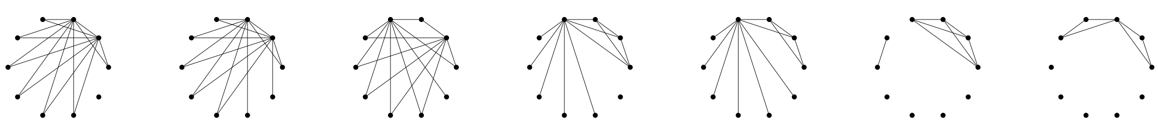
$R(K_3, H) = 29$ if and only if H^c is contained in one of the graphs:



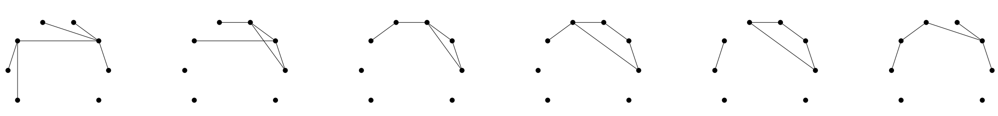
and contains one of the graphs:



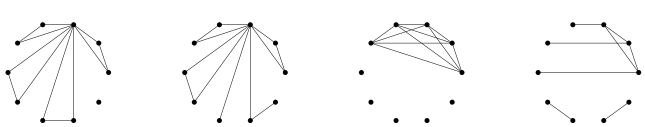
$R(K_3, H) = 28$ if and only if H^c is contained in one of the graphs:



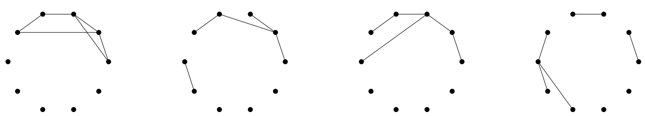
and contains one of the graphs:



$R(K_3, H) = 27$ if and only if H^c is contained in one of the graphs:



and contains one of the graphs:



The graphs with Ramsey number $R(K_3, H) < 27$ can be obtained from [7].

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