New Computational Upper Bounds for Ramsey Numbers $R(3,k)$

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Abstract

Using computational techniques we derive six new upper bounds on the classical two-color Ramsey numbers: $R(3,10) \leq 42$, $R(3,11) \leq 50$, $R(3,13) \leq 68$, $R(3,14) \leq 77$, $R(3,15) \leq 87$, and $R(3,16) \leq 98$. All of them are improvements by one over the previously best known bounds.

Let $e(3,k,n)$ denote the minimum number of edges in any triangle-free graph on n vertices without independent sets of order k. The new upper bounds on $R(3,k)$ are obtained by completing the computation of the exact values of $e(3,k,n)$ for all n with $k \leq 9$ and for all $n \leq 33$ for $k = 10$, and by establishing new lower bounds on $e(3,k,n)$ for most of the open cases for $10 \leq k \leq 15$. The enumeration of all graphs witnessing the values of $e(3,k,n)$ is completed for all cases with $k \leq 9$. We prove that the known critical graph for $R(3,9)$ on 35 vertices is unique up to isomorphism. For the case of $R(3,10)$, first we establish that $R(3,10) = 43$ if and only if $e(3,10,42) = 189$, or equivalently, that if $R(3,10) = 43$ then every critical graph is regular of degree 9. Then, using computations, we disprove the existence of the latter, and thus show that $R(3,10) \leq 42$.

Keywords: Ramsey number; upper bound; computation

1 Definitions and Preliminaries

In this paper all graphs are simple and undirected. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. The set of neighbors of v in G will be written as $N_v(G)$ (or just $N(v)$ if G is fixed). The independence number of G, denoted $\alpha(G)$, is the order of the largest independent set in G, $deg_G(v)$ is the degree of vertex $v \in V(G)$, and $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degree of vertices in G, respectively. For graphs G and H, $G \cong H$ means that they are isomorphic.

For positive integers k and l, the *Ramsey number* $R(k,l)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph K_n with 2 colors, then it contains a monochromatic K_k in the first color or a monochromatic K_l in the second color. If the edges in the first color are interpreted as a graph G and those in the second color as its complement $\overline{G}$, then $R(k,l)$ can be defined equivalently as the smallest n such that every graph on n vertices contains K_k or has independence $\alpha(G) \geq l$. A regularly updated dynamic survey by the second author [17] lists the values and the best known bounds on various types of Ramsey numbers.

Any K_k-free graph G on n vertices with $\alpha(G) < l$ and $e(G) = e$ will be called a $(k,l;n,e)$-*graph*, and by $\mathcal{R}(k,l;n,e)$ we will denote the set of all $(k,l;n,e)$-graphs. We will often omit the parameter e, or both e and n, or give some range to either of these parameters, when referring to special $(k,l;n,e)$-graphs or sets $\mathcal{R}(k,l;n,e)$. For example, a (k,l)-graph is a $(k,l;n,e)$-graph for some n and e, and the set $\mathcal{R}(3,9;35,\leq 139)$ consists of all 35-vertex triangle-free graphs with $\alpha(G) \leq 8$ and at most 139 edges (later we will prove that this set is empty). Any $(k,l;R(k,l)-1)$-graph will be called *critical* for (k,l).

Let $e(k,l,n)$ denote the minimum number of edges in any $(k,l;n)$-graph (or ∞ if no such graph exists). The sum of the degrees of all neighbors of v in G will be denoted by $Z_G(v)$ (or $Z(v)$ if G is fixed), i.e.

$$Z(v) = Z_G(v) = \sum_{\{u,v\} \in E(G)} deg_G(u). \tag{1}$$

In the remainder of this paper we will study only triangle-free graphs. Note that for any $G \in \mathcal{R}(3,k)$ we have $\Delta(G) < k$, since all neighborhoods of vertices in G are independent sets.

Let G be a $(3,k;n,e)$-graph. For any vertex $v \in V(G)$, we will denote by G_v the graph induced in G by the set $V(G) \setminus (N_G(v) \cup \{v\})$. If $d = deg_G(v)$, then clearly G_v is a $(3,k-1;n-d-1,e(G)-Z_G(v))$-graph. Note that this implies that

$$\gamma(v) = \gamma(v,k,G) = e - Z_G(v) - e(3,k-1,n-d-1) \geq 0, \tag{2}$$

where $\gamma(v)$ is the so called *deficiency* of vertex v [8]. Finally, the deficiency of the graph G is defined as

$$\gamma(G) = \sum_{v \in V(G)} \gamma(v,k,G) \geq 0. \tag{3}$$

The condition that $\gamma(G) \geq 0$ will be often sufficient to derive good lower bounds on $e(k,l,n)$, though a stronger condition that all summands $\gamma(v,k,G)$ of (3) are non-negative sometimes implies even better bounds. It is easy to compute $\gamma(G)$ just from the degree sequence of G [8, 10]. If a $(3,k;n,e)$-graph G has n_i vertices of degree i, then

$$\gamma(G) = ne - \sum_i n_i\big(i^2 + e(3,k-1,n-i-1)\big) \geq 0, \tag{4}$$

where $n = \sum_{i=0}^{k-1} n_i$ and $2e = \sum_{i=0}^{k-1} in_i$.

2 Summary of Prior and New Results

In 1995, Kim [12] obtained a breakthrough result by establishing the exact asymptotics of $R(3,k)$ using probabilistic arguments. Recently, the fascinating story of developments and results related to the infinite aspects of $R(3,k)$ was written by Spencer [21].

Theorem 1 ([12]) *$R(3,k) = \Theta(n^2/\log n)$.*

Theorem 1 gives the exact asymptotics of $R(3,k)$, while computing the values for concrete cases remains an open problem for all $k \geq 10$. Still, the progress obtained in the last 50 years in this area is remarkable. Known exact values of $R(3,k)$ for $k \leq 9$, and the best lower and upper bounds for higher k, are listed in [17] together with all the references. We note that much of this progress was obtained with the use of knowledge about $e(3,k,n)$. This direction is also the main focus of our paper: we compute new exact values of $e(3,k,n)$ in several cases and give improved lower bounds for many other, which in turn permits us to prove new upper bounds on $R(3,k)$ for $k = 10, 11, 13, 14, 15$ and 16. Likely, more new upper bounds could be obtained for some $17 \leq k \leq 21$, but we did not perform these computations.

General formulas for $e(3,k,n)$ are known for all $n \leq 13k/4 - 1$ and for $n = 13k/4$ when $k = 0 \bmod 4$.

Theorem 2 ([18, 20]) *For all $n, k \geq 1$, for which $e(3,k+1,n)$ is finite,*

$$e(3,k+1,n) = \begin{cases} 0 & \text{if } n \leq k, \\ n-k & \text{if } k < n \leq 2k, \\ 3n-5k & \text{if } 2k < n \leq 5k/2, \\ 5n-10k & \text{if } 5k/2 < n \leq 3k, \\ 6n-13k & \text{if } 3k < n \leq 13k/4-1. \end{cases} \tag{5}$$

Furthermore, $e(3,k+1,n) = 6n - 13k$ for $k = 4t$ and $n = 13t$, and the inequality $e(3,k+1,n) \geq 6n-13k$ holds for all n and k. All the critical graphs have been characterized whenever the equality in the theorem holds for $n \leq 3k$.

Theorem 2 is a cumulative summary of various contributions [8, 10, 18, 19, 20]. It captures many of the small cases, as presented in Table 3 in Section 4. For example, Theorem 2 gives the exact values of $e(3, 9, n)$ for all $n \leq 26$, of $e(3, 10, n)$ for $n \leq 28$, and of $e(3, 13, n)$ for all $n \leq 39$.

The inequality $e(3, k + 1, n) \geq (40n - 91k)/6$, which is better than $e(3, k + 1, n) \geq 6n - 13k$ for larger parameters, and a number of other improvements and characterizations of graphs realizing specific number of edges, was credited in 2001 by Lesser [13] to an unpublished manuscript by Backelin [1]. As of 2012, the manuscript by Backelin already exceeds 500 pages and it contains numerous additional related results [1, 2], but it still needs more work before it can be published. Therefore, in the remainder of this paper we will not rely on the results included therein, however in several places we will cite the bounds obtained there for reference. In summary, the behavior of $e(3, k + 1, n)$ is clear for $n \leq 13k/4 - 1$, it seems regular but very difficult to deal with for n slightly larger than $13k/4$, and it becomes hopelessly hard for even larger n. In this work we apply computational techniques to establish lower bounds for $e(3, k, n)$ for larger n, for $k \leq 15$. Immediately, our results imply better upper bounds on $R(3, k)$ in several cases, but we hope that they also may contribute to further progress in understanding the general behavior of $e(3, k, n)$.

Full enumeration of the sets $\mathcal{R}(3, \leq 6)$ was established in [18, 16]. The knowledge of the exact values of $e(3, 7, n)$ was completed in [18], those of $e(3, 8, \leq 26)$ in [19], and the last missing value for $\alpha(G) < 8$, namely $e(3, 8, 27) = 85$, was obtained in [4]. The thesis by Lesser [13] contains many lower bounds on $e(3, k, n)$ better than those in [19]. We match or improve them in all cases for $k \leq 10$. For $k \geq 11$ and n slightly exceeding $13k/4 - 1$, the bounds by Lesser (in part credited also to [1]) are better than ours in several cases, however we obtain significantly better ones for larger n.

The general method we use is first to compute, if feasible, the exact value of $e(3, k, n)$ for concrete k and n, or to derive a lower bound using a combination of (2), (3) and (4), and computations. Better lower bounds on $e(3, k - 1, m)$ for $m = n - d - 1$ and various d, lead in general to better lower bounds on $e(3, k, n)$. If we manage to show that $e(3, k, n) = \infty$, i.e. no $(3, k; n)$-graph exists, then we obtain an upper bound $R(3, k) \leq n$. An additional specialized algorithm was needed to establish $R(3, 10) \leq 42$.

Section 3 describes extension algorithms which we used to exhaustively construct all $(3, k; n, e)$-graphs for a number of cases of (n, e), for $k \leq 10$. These results are described in detail in the sequel. This leads to many new lower bounds on $e(3, k, n)$ and full enumerations of $(3, k; n)$-graphs with the number of edges equal to or little larger than $e(3, k, n)$, which are presented in Section 4 (and Appendix 1). These results are then used in Section 5 to prove that there exists a unique critical 35-vertex graph for the Ramsey number $R(3, 9)$. It is known that [5] $40 \leq R(3, 10) \leq 43$ [19]. We establish that $R(3, 10) = 43$ if and only if $e(3, 10, 42) = 189$, or equivalently, that if $R(3, 10) = 43$ then every critical graph in this case is regular of degree 9. Then, in Section 6, using computations we prove that the latter do not exist, and thus obtain $R(3, 10) \leq 42$. Finally, in Section 7, we describe the second stage of our computations, which imply many new lower bounds on $e(3, \geq 11, n)$. This stage uses only degree sequence analysis of potential $(3, k; n, e)$-graphs, which have to satisfy (4). This in turn leads to the new upper bounds on the classical two-color Ramsey numbers marked in bold in Table 1, which presents the values and best bounds on the Ramsey numbers $R(3, k)$ for $k \leq 16$. All the improvements in this work are better by one over the results listed in the latest 2011 revision #13 of the survey [17]. The bound $R(3, 16) \leq 98$ was also obtained by Backelin in 2004, though it was not published [1, 2]. The lower bound $R(3, 11) \geq 47$ was recently obtained by Exoo [6]. The references for all other bounds and values, and the previous upper bounds, are listed in [17].

k	$R(3, k)$	k	$R(3, k)$
3	6	10	40–**42**
4	9	11	47–**50**
5	14	12	52–59
6	18	13	59–**68**
7	23	14	66–**77**
8	28	15	73–**87**
9	36	16	79–**98**

Table 1: Ramsey numbers $R(3, k)$, for $k \leq 16$.

3 Algorithms

Maximum Triangle-Free Method

One method to determine $e(3, k, n)$ is by first generating all *maximal* triangle-free $(3, k; n)$-graphs. A maximal triangle-free graph (in short, an *mtf graph*) is a triangle-free graph such that the insertion of any new edge forms a triangle. It is easy to see that there exists a $(3, k; n)$-graph if and only if there is an mtf $(3, k; n)$-graph. In [4], an algorithm is described that can generate all mtf $(3, k; n)$-graphs efficiently. Using this algorithm, it is much easier to generate all mtf $(3, k; n)$-graphs instead of all $(3, k; n)$-graphs, because the number of the former is in most cases much smaller. For example, there are 477142 $(3, 8; 27)$-graphs, but only 21798 mtf graphs with the same parameters. By recursively removing edges in all possible ways from these mtf $(3, k; n)$-graphs and testing if the resulting graphs G still satisfy $\alpha(G) < k$, the complete set $\mathcal{R}(3, k; n)$ can be obtained.

We applied this method to generate the sets $\mathcal{R}(3, 7; 21), \mathcal{R}(3, 7; 22), \mathcal{R}(3, 8; 26, \leq 77)$ and $\mathcal{R}(3, 8; 27)$ (see Appendix 1 for detailed results). All $(3, 7; 22)$- and $(3, 7; n, e(3, k, n))$-graphs were already known [18], other enumerations are new. This mtf method is infeasible for generating $(3, \geq 9; n)$-graphs for n which were needed in this work. Nevertheless, we used it for verifying the correctness of our other enumerations, and the results agreed in all cases in which more than one method was used (see Appendix 2).

Minimum Degree Extension Method

In their 1992 paper establishing $R(3,8) = 28$, McKay and Zhang [16] proved that the set $\mathcal{R}(3,8;28)$ is empty by generating several sets $\mathcal{R}(3,k;n,e)$ with additional restrictions on the minimum degree $\delta(G)$. Suppose that one wants to generate all $(3,k;n,e)$-graphs. If G is such a graph and one considers its minimum degree vertex v, then we can reconstruct G given all possible graphs G_v. McKay and Zhang described such dependencies, designed an algorithm to reconstruct G, and completed the proof of $R(3,8) = 28$ using this algorithm.

We implemented and used this method by McKay and Zhang [16], and in all cases where more than one algorithm was used it agreed with the other results. However, using this method it was not feasible to generate most classes of graphs with higher parameters needed for our project. For example, we could not generate all $(3,9;28,\leq 69)$-graphs with this method, as the graphs with $\delta(G) = 4$ are obtained from $(3,8;23,\leq 53)$-graphs, but there are already 10691100 $(3,8;23,\leq 52)$-graphs (Table 13 in Appendix 1).

Neighborhood Gluing Extension Method

Our general extension algorithm for an input $(3,k;m)$-graph H produces all $(3,k+1;n,e)$-graphs G, often with some specific restrictions on n and e, such that for some vertex $v \in V(G)$ graph H is isomorphic to G_v. We used the following strategy to determine if the parameters of input graphs to our extender were such that the output was guaranteed to contain all $(3,k+1;n,\leq e)$-graphs.

Let $m_i = n - i - 1$, where i ranges over possible degrees in any graph G we look for, $\delta(G) \leq i \leq \Delta(G)$. In the broadest case we have $\delta(G) = \max\{n - R(3,k), 0\}$ and $\Delta(G) = k$, but we also identified a number of special cases where this range was more restricted. Let t_i be an integer such that we have extended all $(3,k;m_i,< e(3,k,m_i)+t_i)$-graphs as potential G_v's of G. Now, if we use $e(3,k,m_i)+t_i$ instead of $e(3,k,m_i)$ in (4) for all relevant values of i, and (4) has no solutions for $(3,k+1;n,\leq e)$-graphs, then we can conclude that all such graphs were already generated. We illustrate this process by an example.

Example. Table 2 lists specific parameters of the general process when used to obtain all $(3,8;25,\leq 65)$-graphs. Every vertex v in any $(3,8;25,\leq 65)$-graph has degree i, for some $2 \leq i \leq 7$. The corresponding graph G_v is of type $(3,7;m_i,e(G_v))$. The values of $e(3,7,m)$ are included in Table 3 of Section 4, and let t_i's be as in Table 2. If we use the values $e(3,7,m_i)+t_i$ instead of $e(3,7,m_i)$ in (4), then there are no solutions for degree sequences of $(3,8;25,\leq 65)$-graphs. Thus, if we run the extender for all possible graphs G_v with the number of edges listed in the last column of Table 2, then we will obtain all $(3,8;25,e)$-graphs for $e \leq 65$.

The set of increments t_i accomplishing this goal is not unique, there are others which work. We just tried to minimize the amount of required computations in a greedy way. Note that the largest increments t_i to $e(3,7,m_i)$ occur for i's which are close to the average degree of G.

$i = deg_G(v)$	$m_i = \|V(G_v)\|$	$e(3,7,m_i)$	t_i	$e(G_v) = e - Z(v)$
2	22	60	1	60
3	21	51	1	51
4	20	44	2	44, 45
5	19	37	3	37, 38, 39
6	18	30	2	30, 31
7	17	25	1	25

Table 2: Obtaining all $(3, 8; 25, \leq 65)$-graphs.

Implementation

In this section we present some details about the extension algorithms implementations for the minimum degree and neighborhood gluing method. Implementation of the algorithm to generate maximal triangle-free Ramsey graphs is described in [4].

Given a $(3, k; n, f)$-graph G' as input and an expansion degree d, a desired maximum number of edges e, and the minimum degree d_m as parameters, our program constructs all $(3, k+1; n+d+1, \leq e)$-graphs G with $\delta(G) \geq d_m$ for which there is a vertex $v \in V(G)$ such that $deg(v) = d$ and $G_v \cong G'$. More specifically, the program adds to G' a vertex v with neighbors $u_1, ..., u_d$ and connects them to independent sets of G' in all possible ways, so that the resulting graph is a $(3, k + 1; n + d + 1, \leq e)$-graph with $\delta(G) \geq d_m$. Note that the neighbors of v have to be connected to independent sets of G', otherwise the expanded graph would contain triangles, and, clearly, $\Delta(G) \leq k$.

The extension program first determines all independent sets of G' of orders t that are possible, namely $d_m - 1 \leq t \leq k - 1$. The program then recursively assigns the d neighbors of v to the eligible independent sets of G', adds the edges joining u_i's to their associated independent sets, and tests if the resulting G is a valid $(3, k + 1; n + d + 1, \leq e)$-graph. If it is, then we output it. This general process is greatly accelerated by the techniques described in the following.

We bound the recursion if a given partial assignment cannot lead to any $(3, k+1; n+d+1, \leq e)$-graphs. Suppose that i independent sets $S_1, \ldots, S_i$ have already been assigned. If $V(G') \setminus (S_1 \cup ... \cup S_i)$ induces an independent set I of order $k+1-i$, then this assignment cannot lead to any output since $I \cup \{u_1, \ldots, u_i\}$ would form an independent set of order $k + 1$ in G. We could test this property for all subsets of S_i's, but we found it to be most efficient to do it only for all pairs. Namely, if $S_1, \ldots, S_i$ is already assigned and we consider the next independent set S, we test if for all j, $1 \leq j \leq i$, $V(G') \setminus (S_j \cup S)$ does not induce any independent set of order $k - 1$. The list of independent sets which can still be assigned is dynamically updated.

For the efficiency of the algorithm it is vital that testing for independence in $V(G') \setminus (S_1 \cup ... \cup S_i)$ is fast, and hence we precompute the independence numbers of all induced subgraphs of G'. This precomputation also needs to be done very efficiently. We represent a set of vertices $S \subset V(G')$ by a bitvector. The array `indep_number[S]` of 2^n elements stores the independence number of the graph induced by S in G'. It is very important that `indep_number[]` fits into the memory. On the computers on which we performed the expansions this was still feasible up to $n = 31$. We investigated various approaches to precompute `indep_number[S]`, and Algorithm 1 below was by far be the most efficient one. If the superset S' of S already has `indep_number[S']` $\geq j$, then we can break the recursion of making the supersets. Usually one can break very quickly. For small extension degrees $d \leq 3$, it is more efficient not to precompute these independence numbers, but instead to compute them as needed.

Algorithm 1 Precomputing independence number

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for $i = 0$ **upto** $2^n - 1$ **do**
  set `indep_number[i]` $= 0$
end for
for $j = k - 1$ **downto** $k + 1 - d$ **do**
  for all independent sets S of order j in G' **do**
    Recursively make all supersets S' of S, and
    if `indep_number[S']` $= 0$ **then** set `indep_number[S']` $= j$
    else break making supersets of S
  end for
end for

If a neighbor u_i of v has been assigned to an independent set S, we also update the degrees of the vertices in G'. If u_i is being connected to S, the degree of every vertex of S increases by one. If the degree of a vertex w of G' becomes k, then other neighbors of v cannot be assigned to independent sets which contain w. We call such vertices which are no longer eligible *forbidden vertices*, and all of them are stored in a dynamically updated bitvector. We also dynamically update the list of independent sets which can still be assigned to u_i's. Independent sets which contain forbidden vertices are removed from the list of eligible independent sets. We perform bitvector operations whenever suitable. If no eligible independent sets are left, we can bound the recursion. Note that we cannot break the recursion when the number of eligible independent sets is smaller than the number of neighbors of v that still have to be considered, since they can be assigned to the same independent set. If i neighbors of v are already assigned and the forbidden vertices form an independent of set order $k+1-(d-i)$, then the recursion can also be bounded, though this criterion in general is weak.

We assign the neighbors u_i of v to independent sets in ascending order, i.e. if u_i is assigned to S_i, then $|S_i| \leq |S_{i+1}|$ for all $1 \leq i < d$. Doing this rather than in descending order allows us to eliminate many candidate independent sets early in the recursion. If $|S_i|$ is small, then it is very likely that $V(G') \setminus S_i$ induces a large independent set. Hence, it is also very likely that S_i cannot be assigned to a new u_i or that assigning S_i eliminates many eligible independent sets.

Assigning sets in ascending order also gives us an easy lower bound for the number of edges in any potential output graph which can be obtained from the current graph and assignment. If the sets $S_1, \ldots, S_i$ have already been assigned to neighbors of v and the current minimal order of eligible independent sets is t, then any expanded graph will have at least $f = e(G') + d + |S_1| + ... + |S_i| + t(d - i)$ edges. If $f > e$, then we can bound the recursion as well.

The pseudocode of the recursive extension is listed below as Algorithm 2. It is assumed that `indep_number[]` (see Algorithm 1) and the list of eligible independent sets are already computed. The parameters for `Construct()` are the order of the sets which are currently being assigned and the number of neighbors of v which were already assigned to independent sets. The recursion is bounded if any of the bounding criteria described above can be applied.

Algorithm 2 `Construct`(current_order, num_assigned)

```
if num_assigned = d then
   expand graph G' to G
   if G is a (3, k+1; n+d+1, ≤ e)-graph then
      output G
   end if
else
   for every eligible set S of order current_order do
      assign S to u_{num_assigned+1}
      update the set of eligible independent sets
      Construct(current_order, num_assigned + 1)
   end for
   if current_order < k − 1 then
      Construct(current_order + 1, num_assigned)
   end if
end if
```

Our extension program does not perform any isomorphism rejection. We canonically label the output graphs with *nauty* [14, 15] and remove the isomorphic copies. This is not a bottleneck as there are usually only a few $(3, k+1; n+d+1, \leq e)$-graphs which are constructed by our program. The results obtained by our extension algorithms are described in Sections 4 and 6. In the appendices we describe how the correctness of our implementation was tested.

Degree Sequence Feasibility

Suppose we know the values or lower bounds on $e(3, k, m)$ for some fixed k and we wish to know all feasible degree sequences of $(3, k+1; n, e)$-graphs. We construct the system of integer constraints consisting of $n = \sum_{i=0}^{k} n_i$, $2e = \sum_{i=0}^{k} in_i$, and (4). If it has no solutions then we conclude that $e(3, k+1, n) > e$. Otherwise, we obtain solutions for n_i's which include all desired degree sequences. This algorithm is similar in functionality to the package FRANK developed by Lesser [13].

4 Progress on Computing Small $e(3, k, n)$

vertices							k							
n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1													
4	2	1												
5	5	2	1											
6	∞	3	2	1										
7		6	3	2	1									
8		10	4	3	2	1								
9		∞	7	4	3	2	1							
10			10	5	4	3	2	1						
11			15	8	5	4	3	2	1					
12			20	11	6	5	4	3	2	1				
13			26	15	9	6	5	4	3	2	1			
14			∞	20	12	7	6	5	4	3	2	1		
15				25	15	10	7	6	5	4	3	2	1	
16				**32**	20	13	8	7	6	5	4	3	2	1
17				**40**	25	16	11	8	7	6	5	4	3	2
18				∞	30	20	14	9	8	7	6	5	4	3
19					**37**	25	17	12	9	8	7	6	5	4
20					**44**	30	20	15	10	9	8	7	6	5
21					**51**	35	25	18	13	10	9	8	7	6
22					**60**	**42**	30	21	16	11	10	9	8	7
23					∞	**49**	35	25	19	14	11	10	9	8
24						**56**	40	30	22	17	12	11	10	9
25						**65**	46	35	25	20	15	12	11	10
26						**73**	52	40	30	23	18	13	12	11
27						**85**	**61**	45	35	26	21	16	13	12
28						∞	**68**	51	40	30	24	19	14	13
29							**77**	**58**	45	35	27	22	17	14
30							**86**	**66**	50	40	30	25	20	15
31							**95**	**73**	56	45	35	28	23	18

Table 3: Exact values of $e(3, k, n)$, for $3 \le k \le 16$, $3 \le n \le 31$.

Table 3 presents the exact values of $e(3, k, n)$ for small cases, where clear regularities are well described by Theorem 2. Empty entries in the upper-right triangle of the table are 0's, while those in the lower-left triangle are equal to ∞. The columns correspond to fixed values of k. Almost all entries are given by Theorem 2. We list them for a better perspective and completeness. The entries beyond the range of Theorem 2 are marked in bold, and they were obtained as follows: $e(3, 6, 16)$ and $e(3, 6, 17)$ in [8], all cases for $k = 7$

in [8, 10, 18], all cases for $k = 8$ and $22 \leq n \leq 26$ in [19], $e(3, 8, 27) = 85$ was computed in [4], and those for $k \geq 9$ are obtained here. The smallest n for which there is an open case is 32, namely that of $e(3, 11, 32)$. Tables 4 and 5 below and 7–11 in Section 7 present the details of what we found about these harder parts of each column k, for $9 \leq k \leq 16$.

The exact counts of $(3, k; n, e)$-graphs for $k = 7, 8, 9, 10$ which were obtained by the algorithms described in Section 3 are listed in Tables 12, 13, 14, 15, respectively, in Appendix 1. All $(3, \leq 9; n, e(3, k, n))$-graphs which were constructed by our programs can be obtained from the *House of Graphs* [3] by searching for the keywords "minimal ramsey graph".

Exact values of $e(3, 9, n)$

The values of $e(3, 9, \leq 26)$ are determined by Theorem 2. The values of $e(3, 9, n)$ for $27 \leq n \leq 34$ were obtained by computations, mostly by the gluing extender algorithm described in Section 3, and they are presented in Table 4. All of these values improve over previously reported lower bounds [19, 13]. The equality $e(3, 9, 35) = 140$ will be established by Theorem 3 in Section 5.

n	$e(3,9,n)$	comments
27	61	
28	68	
29	77	
30	86	
31	95	
32	104	not enough for $R(3, 10) \leq 42$
33	118	just enough for Theorem 4
34	129	122 required for $R(3, 10) \leq 43$
35	140	Theorem 3
36	∞	hence $R(3, 9) \leq 36$, old bound

Table 4: Exact values of $e(3, 9, n)$, for $n \geq 27$

Values and lower bounds on $e(3, 10, n)$

The values of $e(3, 10, \leq 28)$ are determined by Theorem 2. The values for $29 \leq n \leq 33$ were obtained by the gluing extender algorithm described in Section 3. The lower bounds on $e(3, 10, \geq 34)$ are included in the second column of Table 5. They are based on solving integer constraints (3) and (4), using the exact values of $e(3, 9, n)$ listed in Table 4. Our bounds on $e(3, 10, n)$ improve over previously reported lower bounds [19, 13] for all $n \geq 30$.

By Theorem 4 (see Section 5) we know that any $(3, 10; 42)$-graph must be 9-regular with 189 edges, and thus its all graphs G_v are necessarily of the type $(3, 9; 32, 108)$. There exists a very large number of the latter graphs. Their generation, extensions to possible $(3, 10; 42, 189)$-graphs, and implied nonexistence of any $(3, 10; 42)$-graphs will be described in Section 6.

n	$e(3,10,n) \geq$	comments
29	58	exact, the same as in [13]
30	66	exact
31	73	exact
32	81	exact
33	90	exact
34	99	
35	107	
36	117	
37	128	
38	139	146 required for $R(3,11) \leq 49$
39	151	as required for $R(3,11) \leq 50$, Theorem 7
40	161	
41	172	184 maximum
42	∞	hence $R(3,10) \leq 42$, new bound, Theorem 6
43	∞	hence $R(3,10) \leq 43$, old bound

Table 5: Values and lower bounds on $e(3,10,n)$, for $n \geq 29$.

5 Better Lower Bounds for $e(3,9,35)$ and $e(3,10,42)$

Sometimes we can improve on the lower bounds on $e(3,k,n)$ implied by (3) and (4) by a more detailed analysis of feasible degree sequences. Such improvements typically can be done in cases for which (4) gives a small number of possible degree sequences, none of which is of a regular graph, furthermore with only one heavily dominating degree. We have such a situation in the proofs of the two following theorems.

Theorem 3 *There exists a unique* $(3,9;35)$*-graph, and* $e(3,9,35) = 140$*.*

Proof. Any $(3,9;35)$-graph G has $\Delta(G) \leq 8$, hence we have $e(G) \leq 140$. Suppose $G \in \mathcal{R}(3,9;35,140-s)$ for some $s \geq 0$. Since $R(3,8) = 28$, the degrees of vertices in G are 7 or 8, and let there be n_7 and n_8 of them, respectively. We have $n_7 + n_8 = 35$, $n_7 = 2s$. In this case there are five solutions to (4) with $0 \leq s \leq 4$. In particular, this shows that $e(3,9,35) \geq 136$. If $n_7 > 0$ (equivalently $s > 0$), then consider graph H induced in G by n_7 vertices of degree 7. Observe that $\delta(H) \leq s$, since H is triangle-free on $2s$ vertices. Let v be a vertex in $V(G)$ of degree 7 connected to at most s other vertices of degree 7. Thus we have $Z_G(v) \geq 7s + 8(7-s) = 56 - s$, and $e(G_v) \leq (140-s) - (56-s) = 84$. However G_v is a $(3,8;27)$-graph which contradicts the fact that $e(3,8,27) = 85$.

The computations extending all $(3,8;26,76)$-graphs, using the neighborhood gluing extension method described in Section 3, established that there exists a unique (up to isomorphism) 8-regular $(3,9;35)$-graph. We note that it is a cyclic graph on 35 vertices with circular distances {1,7,11,16}, found by Kalbfleisch [11] in 1966. Clearly, any $(3,9;35,140)$-graph must be 8-regular, and thus the theorem follows. $\square$

Theorem 4 $R(3,10)=43$ *if and only if* $e(3,10,42)=189$.

Proof. It is known that $R(3,10) \leq 43$ [19], i.e. there are no $(3,10;43)$-graphs. We will prove the theorem by showing that any $(3,10;42)$-graph must be regular of degree 9. The essence of the reasoning is very similar to that for $e(3,9,35)=140$ in the previous theorem, except that this time it is little more complicated.

Suppose $G \in \mathcal{R}(3,10;42,189-s)$ for some $s \geq 0$. The computations described in Section 3 established that G cannot have the unique $(3,9;35)$-graph as one of its G_v's. Hence, $7 \leq \deg_G(v) \leq 9$ for all vertices $v \in V(G)$. The solutions n_i to (4) which contain all possible degree sequences for G with this restriction are presented in Table 6.

n_7	n_8	n_9	$e(G)$	$\gamma(G)$	s
0	8	34	185	24	4
1	6	35	185	25	4
2	4	36	185	26	4
3	2	37	185	27	4
4	0	38	185	28	4
0	6	36	186	60	3
1	4	37	186	61	3
2	2	38	186	62	3
3	0	39	186	63	3
0	4	38	187	96	2
1	2	39	187	97	2
2	0	40	187	98	2
0	2	40	188	132	1
1	0	41	188	133	1
0	0	42	189	168	0

Table 6: Solutions to (4) for $(3,10;42,189-s)$-graphs.

Note that for all $0 \leq s \leq 4$ we have $0 \leq n_7 \leq s$, $n_8 + 2n_7 = 2s$, $n_9 = 42 - n_8 - n_7$, and $e(G) = 189 - s$. Since $e(3,9,34) = 129$, using (2) we see that $Z(v) \leq 60 - s$ for every vertex v of degree 7. Similarly, since $e(3,9,33) = 118$, $Z(v) \leq 71 - s$ for every vertex v of degree 8. If $s = 0$, then we are done, otherwise consider graph H induced in G by $2s - n_7$ vertices of degree 7 or 8. Observe that $\delta(H) \leq s - n_7/2$, since H is triangle-free.

Case 1: $n_7 = 0$**.** Let v be a vertex in $V(G)$ of degree 8 connected to at most s other vertices of degree 8. This gives $Z_G(v) \geq 8s + 9(8-s) = 72 - s$, which is a contradiction.

Case 2: $n_8 = 0$**.** Let v be a vertex in $V(G)$ of degree 7 connected to at most $s/2$ other vertices of degree 7 (in this case $|V(H)| = s$). This gives $Z_G(v) \geq 7s/2 + 9(7 - s/2) = 63 - s$, which is a contradiction.

Case 3: $n_7 = 1$**.** If v is the only vertex of degree 7, then $n_8 = 2s - 2$ and we easily have $Z_G(v) \geq 8n_8 + 9(7 - n_8) = 65 - 2s > 60 - s$, which again is a contradiction.

Case 4: $n_7 = 2$**.** Both vertices of degree 7 must have $Z_G(v) \geq 7 + 8n_8 + 9(7 - n_8 - 1) = 61 - (2s - 2n_7) = 65 - 2s$, which is a contradiction.

Case 5: $n_7 > 2$**.** The only remaining degree sequence not covered by previous cases is $n_7 = 3$ and $n_8 = 2$, for $s = 4$ and $e = 185$. There is a vertex v of degree 7 connected to at most one other of degree 7, and thus $Z_G(v) \geq 7 + 2 \cdot 8 + 4 \cdot 9 > 60 - s$, a contradiction. □

6 $R(3, 10) \leq 42$

Theorem 4 implies that any $(3, 10; 42)$-graph G must be regular of degree 9 with 189 edges. Removing any vertex v with its neighborhood from G yields a $(3, 9; 32, 108)$-graph G_v. Hence, our first task is to obtain all $(3, 9; 32, 108)$-graphs.

We used the neighborhood extension method to generate $(3, 9; 32, 108)$-graphs H with a vertex v for which H_v is one of the following types: $(3, 8; 27)$, $(3, 8; 26, \leq 77)$, $(3, 8; 25, \leq 68)$, $(3, 8; 24, \leq 59)$ or $(3, 8; 23, 49)$. These extensions yielded the set of 2104151 $(3, 9; 32, 108)$-graphs $\mathcal{X}$. Using notation of the example in Section 3, now with $4 \leq i \leq 8$, $m_i = 31 - i$, and $t_i = 10, 5, 4, 4, 1$, respectively, the only remaining degree sequence passing (4) for a $(3, 9; 32, 108)$-graph is $n_6 = 8, n_7 = 24$.

Potentially, the complete set of $(3, 9; 32, 108)$-graphs could be obtained by performing additional extensions of degree 6 to $(3, 8; 25, 69)$-graphs or extensions of degree 7 to $(3, 8; 24, 60)$-graphs. However, there are already 12581543 $(3, 8; 25, \leq 68)$-graphs and 3421512 $(3, 8; 24, \leq 59)$-graphs (see Table 13 in Appendix 1), and there are many more with one additional edge. Hence, further refinement of the construction method of the $(3, 9; 32, 108)$-graphs not in $\mathcal{X}$ was needed. It is described in the following Lemma 5, which permitted a fast computation and the completion of the task.

Lemma 5 *Every* $(3, 9; 32, 108)$*-graph* $H \notin \mathcal{X}$ *has* $n_6 = 8, n_7 = 24$*, and furthermore in such* H *every vertex of degree* 6 *has exactly* 3 *neighbors of degree* 7 *and every vertex of degree* 7 *has exactly* 1 *neighbor of degree* 6*.*

Proof. As stated after the definition of $\mathcal{X}$ above, (4) implies the specified degree sequence of $H \notin \mathcal{X}$. Suppose that H has a vertex v of degree 6 with at least 4 neighbors of degree 7. One can easily see that $Z_H(v) \geq 40$ and thus $e(H_v) \leq 68$. All such graphs, however, were included in the set of inputs producing $\mathcal{X}$, so we have a contradiction. Similarly, suppose that H has a vertex v of degree 7 with no neighbors of degree 6. Then $Z_H(v) = 49$ and $e(H_v) = 59$, but all such graphs were used as inputs producing $\mathcal{X}$, hence again we have a contradiction. Now, by the pigeonhole principle, there are exactly 24 edges connecting vertices of distinct degrees, and we can easily conclude that every vertex of degree 6 must have exactly 3 neighbors of degree 7 and every vertex of degree 7 exactly 1 neighbor of degree 6. □

We adapted the extension algorithm from Section 3 to generate this very restricted set of $(3,9;32,108)$-graphs by performing extensions of all 64233886 $(3,8;24,60)$-graphs (Table 13 in Appendix 1). The result is that there are no $(3,9;32,108)$-graphs not in $\mathcal{X}$.

Theorem 6 $R(3,10) \leq 42$.

Proof. For contradiction, suppose that G is a $(3,10;42)$-graph. By Theorem 4 it must be a 9-regular $(3,10;42,189)$-graph whose all G_v's are $(3,9;32,108)$-graphs. By Lemma 5 and the computations described above there are exactly 2104151 such graphs. A specialized extension algorithm (a modification of the gluing extender) was run for all of them in an attempt to obtain a 9-regular $(3,10;42,189)$-graph. The neighbors of v have to be connected to independent sets of order 8 in G_v. For every pair of (possibly equal) independent sets $\{S_i, S_j\}$ of order 8, we test if they can be assigned to two neighbors of v by checking if $V(G_v) \setminus (S_i \cup S_j)$ induces an independent set of order 8 in G_v, and if so we can bound the recursion. We used for this task a precomputed table storing the results of such tests for all pairs of independent sets of order 8. The concept of eligible candidates (Section 3) was also used, and the condition $\Delta(G) = 9$ turned out to be particularly strong in pruning the recursion. No 9-regular $(3,10;42,189)$-graphs were produced, and thus $R(3,10) \leq 42$. $\square$

Theorem 6 improves over the bound $R(3,10) \leq 43$ obtained in 1988 [19]. The correctness tests of our implementations and the computational effort required for various parts of the computations are described in Appendix 2.

Geoffrey Exoo [6] found almost 300000 $(3,10;39)$-graphs, we extended this set to more than $4 \cdot 10^7$ graphs, and very likely there are more of them. The known $(3,10;39)$-graphs have the number of edges ranging from 161 to 175, hence we have $151 \leq e(3,9,39) \leq 161$. We expect that the actual value is much closer, if not equal, to 161. Despite many attempts by Exoo, us, and others, no $(3,10;40)$-graphs were constructed. The computations needed for the upper bound in Theorem 6 were barely feasible. Consequently, we anticipate that any further improvement to either of the bounds in $40 \leq R(3,10) \leq 42$ will be very difficult.

7 Lower Bounds for $e(3, k, n)$ and Upper Bounds for $R(3, k)$, for $k \geq 11$

We establish five further new upper bounds on the Ramsey numbers $R(3, k)$ as listed in Theorem 7. All of the new bounds improve the results listed in the 2011 revision of the survey [17] by 1. Note that we don't improve the upper bound on $R(3, 12)$.

Theorem 7 *The following upper bounds hold:*
$R(3, 11) \leq 50$*,* $R(3, 13) \leq 68$*,* $R(3, 14) \leq 77$*,* $R(3, 15) \leq 87$*, and* $R(3, 16) \leq 98$*.*

Proof. Each of the new upper bounds $R(3, k) \leq n$ can be obtained by showing that $e(3, k, n) = \infty$. The details of the intermediate stages of computations for all k are presented in the tables and comments of the remaining part of this section. For $k = 16$ no data is shown except some comments in Table 11, in particular the data in this table implies $e(3, 16, 98) = \infty$ by (4). □

In the Tables 7, 8 and 9, for $k = 11, 12$ and 13, respectively, we list several cases in the comments column, where the lower bounds on $e(3, k, n)$ listed in [13] (some of them credited to [1]) are better than our results. This is the case for n slightly larger than $13k/4 - 1$, mostly due to the theorems claimed in the unpublished manuscript by Backelin [1, 2]. Our lower bounds on $e(3, k, n)$, and implied upper bounds on $R(3, k)$, do not rely on these results. We have checked that assuming the results from [1, 2, 13] would not imply, using the methods of this paper, any further improvements on the upper bounds on $R(3, k)$ for $k \leq 16$, but they may for $k \geq 17$. Hence, if the results in [1, 13] are published, then using them jointly with our results may lead to better upper bounds on $R(3, k)$, at least for some $k \geq 17$.

Lower bounds for $e(3,11,n)$

The exact values of $e(3,11,\leq 31)$ are determined by Theorem 2. The bounds for $n = 32, 33$ marked with a 't' are from Theorem 2. The lower bounds on $e(3,11,\geq 32)$ are included in the second column of Table 7. They are based on solving integer constraints (4), using known values and lower bounds on $e(3,10,n)$ listed in Table 5 in Section 4. They are better than those in [13] for all $36 \leq n \leq 50$.

n	$e(3,11,n) \geq$	comments
32	62t	63 in [13], credit to [1]
33	68t	69 in [13], credit to [1]
34	75	76 in [13], credit to [1]
35	83	84 in [13], credit to [1]
36	92	
37	100	
38	109	
39	117	unique solution, 6-regular
40	128	
41	138	
42	149	
43	159	
44	170	
45	182	
46	195	199 required for $R(3,12) \leq 58$
47	209	
48	222	unique solution: $n_9 = 36, n_{10} = 12$,
		215 required for $R(3,12) \leq 59$, old bound
49	237	245 maximum
50	∞	hence $R(3,11) \leq 50$, new bound, Theorem 7
51	∞	hence $R(3,11) \leq 51$, old bound

Table 7: Lower bounds on $e(3,11,n)$, for $n \geq 32$.

The maximum number of edges in any $(3,11;49)$-graph is that of a 10-regular graph, so a proof of $e(3,11,49) > 245$ would imply $R(3,11) \leq 49$. Observe that any graph G_v of any 10-regular $(3,11;50)$-graph must be a $(3,10;39,150)$-graph. Thus, our improvement of the upper bound on $R(3,11)$ from 51 to 50 is mainly due to the new lower bound $e(3,10,39) \geq 151$ (together with not-too-much-off adjacent bounds).

Lower bounds for $e(3,12,n)$

The exact values of $e(3,12,\leq 34)$ are determined by Theorem 2. The bounds for $35 \leq n \leq 37$ marked with a 't' are from Theorem 2. The lower bounds on $e(3,12,\geq 35)$ are included in the second column of Table 8. They are based on solving integer constraints (4), using known values and lower bounds on $e(3,11,n)$ given in Table 7. They are better than those in [13] for all $43 \leq n \leq 58$.

An improvement of the upper bound on $R(3,12)$ obtained by Lesser [13] from 60 to 59 is now immediate (it formed a significant part of her thesis), but a further improvement from 59 to 58 would require an increase of the lower bound on $e(3,12,58)$ by 4.

n	$e(3,12,n) \geq$	comments
35	67t	68 in [13], credit to [1]
36	73t	74 in [13], credit to [1]
37	79t	81 in [13], credit to [1]
38	86	88 in [13], credit to [1]
39	93	95 in [13], credit to [1]
40	100	102 in [13]
41	109	111 in [13]
42	119	the same as in [13]
43	128	
44	138	
45	148	
46	158	
47	167	
48	179	
49	191	
50	203	
51	216	
52	229	
53	241	
54	255	259 required for $R(3,13) \leq 67$
55	269	265 required for $R(3,13) \leq 68$, Theorem 7
56	283	
57	299	
58	316	319 maximum
59	∞	hence $R(3,12) \leq 59$, old bound

Table 8: Lower bounds on $e(3,12,n)$, for $n \geq 35$.

Lower bounds for $e(3, 13, n)$

The exact values of $e(3, 13, \leq 39)$ are determined by Theorem 2. The bound for $n = 40$ is from Theorem 2. The lower bounds on $e(3, 13, \geq 40)$ are included in the second column of Table 9. They are based on solving integer constraints (4), using lower bounds on $e(3, 12, n)$ listed in Table 8. They are better than those in [13] for all $51 \leq n \leq 68$.

n	$e(3, 13, n) \geq$	comments
40	84t	86 in [13]
41	91	93 in [13], credit to [1]
42	97	100 in [13], credit to [1]
43	104	107 in [13], credit to [1]
44	112	114 in [13]
45	120	122 in [13]
46	128	130 in [13]
47	136	139 in [13], credit to [1]
48	146	148 in [13]
49	157	158 in [13]
50	167	the same as in [13]
51	177	
52	189	
53	200	
54	212	
55	223	
56	234	
57	247	
58	260	
59	275	
60	289	
61	303	
62	319	326 required for $R(3, 14) \leq 76$
63	334	
64	350	345 required for $R(3, 14) \leq 77$, Theorem 7
65	365	
66	381	
67	398	402 maximum
68	∞	hence $R(3, 13) \leq 68$, new bound
69	∞	hence $R(3, 13) \leq 69$, old bound

Table 9: Lower bounds on $e(3, 13, n)$, for $n \geq 40$.

Lower bounds for $e(3, 14, n)$

The exact values of $e(3, 14, \leq 41)$ are determined by Theorem 2. Only lower bounds on $e(3, 14, \geq 66)$ are included in the second column of Table 10, since these are relevant for our further analysis of $R(3, 15)$ and $R(3, 16)$. They are based on solving integer constraints (4), using lower bounds on $e(3, 13, n)$ listed in Table 9. They are better than those in [13] for all $66 \leq n \leq 77$.

n	$e(3, 14, n) \geq$	comments
66	321	
67	334	
68	350	
69	365	
70	381	
71	398	407 required for $R(3, 15) \leq 86$
72	415	414 required for $R(3, 15) \leq 87$, Theorem 7
73	432	
74	449	
75	468	
76	486	494 maximum
77	∞	hence $R(3, 14) \leq 77$, new bound
78	∞	hence $R(3, 14) \leq 78$, old bound

Table 10: Lower bounds on $e(3, 14, n)$, for $n \geq 66$.

Lower bounds for $e(3, 15, n)$

The exact values of $e(3, 15, \leq 44)$ are determined by Theorem 2. Only lower bounds on $e(3, 15, \geq 81)$ are included in the second column of Table 11, since these are relevant for further analysis of $R(3, 16)$. They are based on solving integer constraints (4), using lower bounds on $e(3, 14, n)$ listed in Table 10. They are better than those in [13] for all $81 \leq n \leq 87$.

n	$e(3, 15, n) \geq$	comments
81	497	
82	515	518 required for $R(3, 16) \leq 97$
		511 required for $R(3, 16) \leq 98$, Theorem 7
83	533	
84	552	
85	572	
86	592	602 maximum
87	∞	hence $R(3, 15) \leq 87$, new bound
88	∞	hence $R(3, 15) \leq 88$, old bound

Table 11: Lower bounds on $e(3, 15, n)$, for $n \geq 81$.

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Appendix 1: Graph Counts

Tables 12–15 below contain all known exact counts of $(3, k; n, e)$-graphs for specified n, for $k = 7, 8, 9$ and 10, respectively. All graph counts were obtained by the algorithms described in Section 3. Empty entries indicate 0. In all cases, the maximum number of edges is bounded by $\Delta(G)n/2 \leq (k-1)n/2$. All $(3, \leq 9; n, \leq e(3, k, n) + 1)$-graphs which were constructed by our programs can be obtained from the *House of Graphs* [3] by searching for the keywords "minimal ramsey graph" or from [7].

edges	number of vertices n						
e	16	17	18	19	20	21	22
20	2						
21	15						
22	201						
23	2965						
24	43331						
25	498927	2					
26	4054993	30					
27	?	642					
28	?	13334					
29	?	234279					
30	?	2883293	1				
31	?	?	15				
32	?	?	382				
33	?	?	8652				
34	?	?	160573				
35	?	?	2216896				
36	?	?	?				
37	?	?	?	11			
38	?	?	?	417			
39	?	?	?	10447			
40	?	?	?	172534			
41	?	?	?	1990118			
42-43	?	?	?	?			
44	?	?	?	?	15		
45	?	?	?	?	479		
46	?	?	?	?	10119		
47	?	?	?	?	132965		
48	?	?	?	?	1090842		
49-50		?	?	?	?		
51		?	?	?	?	4	
52			?	?	?	70	
53			?	?	?	717	
54			?	?	?	5167	
55				?	?	27289	
56				?	?	97249	
57				?	?	219623	
58					?	307464	
59					?	267374	
60					?	142741	1
61						43923	6
62						6484	30
63						331	60
64							59
65							25
66							10

Table 12: Number of $(3, 7; n, e)$-graphs, for $n \geq 16$.

edges	number of vertices n								
e	19	20	21	22	23	24	25	26	27
25	2								
26	37								
27	763								
28	16939								
29	?								
30	?	3							
31	?	60							
32	?	1980							
33	?	58649							
34	?	1594047							
35	?	?	1						
36	?	?	20						
37	?	?	950						
38	?	?	35797						
39	?	?	1079565						
40-41	?	?	?						
42	?	?	?	21					
43	?	?	?	1521					
44	?	?	?	72353					
45	?	?	?	2331462					
46-48	?	?	?	?					
49	?	?	?	?	102				
50	?	?	?	?	8241				
51	?	?	?	?	356041				
52	?	?	?	?	10326716				
53-55	?	?	?	?	?				
56	?	?	?	?	?	51			
57	?	?	?	?	?	3419			
58	?	?	?	?	?	129347			
59	?	?	?	?	?	3288695			
60	?	?	?	?	?	64233886			
61-64	?	?	?	?	?	?			
65	?	?	?	?	?	?	396		
66	?	?	?	?	?	?	21493		
67		?	?	?	?	?	613285		
68		?	?	?	?	?	11946369		
69-72		?	?	?	?	?	?		
73			?	?	?	?	?	62	
74				?	?	?	?	1625	
75				?	?	?	?	23409	
76				?	?	?	?	216151	
77				?	?	?	?	1526296	
78-84					?	?	?	?	
85							?	?	4
86							?	?	92
87							?	?	1374
88								?	11915
89								?	52807
90								?	122419
91								?	151308
92									99332
93									33145
94									4746

Table 13: Number of $(3, 8; n, e)$-graphs, for $n \geq 19$.

edges	number of vertices n											
e	24	25	26	27	28	29	30	31	32	33	34	35
40	2											
41	32											
42	2089											
43	115588											
44-45	?											
46	?	1										
47	?	39										
48	?	4113										
49	?	306415										
50-51	?	?										
52	?	?	1									
53	?	?	1									
54	?	?	444									
55	?	?	58484									
56-60	?	?	?									
61	?	?	?	700								
62	?	?	?	95164								
63	?	?	?	6498191								
64-67	?	?	?	?								
68	?	?	?	?	126							
69	?	?	?	?	17223							
70	?	?	?	?	1202362							
71-76	?	?	?	?	?							
77	?	?	?	?	?	1342						
78	?	?	?	?	?	156686						
79-85	?	?	?	?	?	?						
86	?	?	?	?	?	?	1800					
87	?	?	?	?	?	?	147335					
88-94	?	?	?	?	?	?	?					
95	?	?	?	?	?	?	?	560				
96	?	?	?	?	?	?	?	35154				
97-103		?	?	?	?	?	?	?				
104			?	?	?	?	?	?	39			
105				?	?	?	?	?	952			
106				?	?	?	?	?	18598			
107				?	?	?	?	?	234681			
108				?	?	?	?	?	2104151			
109-117					?	?	?	?	?			
118							?	?	?	5		
119							?	?	?	69		
120							?	?	?	≥ 1223		
121								?	?	≥ 13081		
122								?	?	≥ 90235		
123								?	?	≥ 401731		
124								?	?	≥ 1188400		
125									?	≥ 2366474		
126									?	≥ 3198596		
127									?	≥ 2915795		
128									?	≥ 1758241		
129										≥ 673600	1	
130										≥ 153676	4	
131										≥ 18502	≥ 15	
132										≥ 922	≥ 40	
133											≥ 54	
134											≥ 43	
135											≥ 20	
136											≥ 7	
137-139												
140												1

Table 14: Number of $(3, 9; n, e)$-graphs, for $n \geq 24$.

edges	number of vertices n					
e	29	30	31	32	33	34
58	5					
59	1364					
60–65	?					
66	?	5084				
67	?	1048442				
68–72	?	?				
73	?	?	2657			
74	?	?	580667			
75–80	?	?	?			
81	?	?	?	6592		
82–89	?	?	?	?		
90	?	?	?	?	57099	
91–98	?	?	?	?	?	
99	?	?	?	?	?	≥ 1
≥ 100	?	?	?	?	?	?

Table 15: Number of $(3, 10; n, e)$-graphs, for $29 \leq n \leq 34$.

We showed that $e(3, 10, 34) \geq 99$ (see Section 4), a $(3, 10; 34, 99)$-graph was constructed by Backelin [2], and thus $e(3, 10, 34) = 99$.

Appendix 2: Testing Implementations

Correctness

Since most results obtained in this paper rely on computations, it is very important that the correctness of our programs has been thoroughly verified. Below we list the main tests and agreements with results produced by more than one computation.

- For every $(3,k)$-graph which was output by our programs, we verified that it does not contain an independent set of order k by using an independent program.
- For every $(3,k;n,e(3,k,n))$-graph which was generated by our programs, we verified that dropping any edge creates an independent set of order k.
- For various $(3,k;n,\leq e)$-graphs we added up to f edges in all possible ways to obtain $(3,k;n,\leq e+f)$-graphs. For the cases where we already had the complete set of $(3,k;n,\leq e+f)$-graphs we verified that no new $(3,k;n,\leq e+f)$-graphs were obtained. We used this, amongst other cases, to verify that no new $(3,9;24,\leq 43)$, $(3,9;28,\leq 70)$, $(3,9;30,\leq 87)$ or $(3,10;30,\leq 67)$-graphs were obtained.
- For various $(3,k;n,\leq e+f)$-graphs we dropped one edge in all possible ways and verified that no new $(3,k;n,\leq e+f-1)$-graphs were obtained. We used this technique, amongst other cases, to verify that no new $(3,9;24,\leq 42)$, $(3,9;28,\leq 69)$, $(3,9;33,\leq 119)$, $(3,9;34,\leq 130)$, $(3,10;30,66)$ or $(3,10;32,81)$-graphs were obtained.
- For various sets of $(3,k+1;n,\leq e)$-graphs we took each member G and constructed from it all G_v's. We then verified that this did not yield any new $(3,k;n-deg(v)-1,\leq e-Z(v))$-graphs for the cases where we have all such graphs. We performed this test, amongst other cases, on the sets of $(3,9;28,\leq 70)$- and $(3,10;31,\leq 74)$-graphs.
- Various sets of graphs can be obtained by both the minimum degree extension method and the neighborhood gluing extension method. We performed both extension methods for various cases (e.g. to obtain the sets of $(3,9;24,\leq 43)$ and $(3,9;25,\leq 48)$-graphs). In each of these cases the results obtained by both methods were in complete agreement.
- The sets of $(3,7;21,\leq 55)$, $(3,7;22)$, $(3,8;26,\leq 76)$ and $(3,8;27,\leq 88)$-graphs were obtained by both the maximal triangle-free method [4] and the neighborhood gluing extension method. The results were in complete agreement. As these programs are entirely independent and the output sets are large, we think that this provides strong evidence of their correctness.
- The counts of $(3,7;16,20)$, $(3,7;17,25)$, $(3,7;18,30)$, $(3,7;19,37)$, $(3,7;20,44)$, $(3,7;21,51)$, and $(3,7;22,e)$ for all $60\leq e\leq 66$, are confirmed by [18].

- The counts of $(3, 7; 18, 31)$, $(3, 7; 19, 38)$, $(3, 7; 20, 45)$ and $(3, 7; 21, \leq 53)$-graphs are confirmed by [19].

- The counts of $(3, 8; 19, 25)$, $(3, 8; 20, 30)$, $(3, 8; 21, 35)$ and $(3, 9; 24, 40)$-graphs are confirmed by [20].

- The counts of $(3, 7; 16, 21)$, $(3, 7; 17, 26)$, $(3, 8; 22, 42)$ and $(3, 9; 25, 47)$-graphs are confirmed by [2].

Additional implementation correctness tests of specialized algorithms described in Section 6 were as follows:

- The specialized program of Section 6 was used to extend $(3, 8; 26, 76)$- to $(3, 9; 35, 140)$-graphs and it produced the unique $(3, 9; 35, 140)$-graph.

- We relaxed the conditions to generate all $(3, 9; 32, 108)$-graphs from Lemma 5 by dropping the requirement that each vertex of degree 6 has 3 neighbors of degree 7, and enforcing just one vertex of degree 7 with exactly one neighbor of degree 6. This yielded 21602 graphs. We verified that each of these graphs was indeed already included in the set $\mathcal{X}$, and that $\mathcal{X}$ does not contain any additional such graphs.

Since our results are in complete agreement with previous results and since all our consistency tests passed, we believe that this is strong evidence for the correctness of our implementations.

Computation Time

The implementations of extension algorithms described in Sections 3 and 6 are written in C. Most computations were performed on a cluster with Intel Xeon L5520 CPU's at 2.27 GHz, on which a computational effort of one CPU year can be usually completed in about 8 elapsed hours. The overall computational effort of this project is estimated to be about 50 CPU years, which includes the time used by a variety of programs. The most cpu-intensive tasks are listed in the following.

The first phase of obtaining $(3, 9; 32, 108)$-graphs required about 5.5 CPU years. The bottlenecks of this phase were the computations required for extending all $(3, 8; 24, \leq 59)$-graphs (which required approximately 3.5 CPU years), and extending the $(3, 8; 25, \leq 68)$-graphs (which took more than 2 CPU years). The second phase of obtaining the special $(3, 9; 32, 108)$-graphs with $n_6 = 8$, $n_7 = 24$ as in Lemma 5 took about 5.8 CPU years. The specialized program of Section 6 extended all $(3, 9; 32, 108)$-graphs to 9-regular $(3, 10; 42, 189)$-graphs quite fast, in about only 0.25 CPU years. Performing computations to generate all $(3, 10; 39, \leq 150)$-graphs (there are none of these), which were needed for the bound $R(3, 11) \leq 50$, took about 4.8 CPU years.

The CPU time needed to complete the computations of Section 7 was negligible, however their variety caused that they were performed during the span of several weeks.