NOTES ON THE NEW GRAPH POLYNOMIAL

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Hi Dirk, here are some notes on your new polynomial.

1. The 3-regular case

Let $G = (V, E)$ be a 3-regular graph where external edges are permitted and are viewed as unmatched half-edges. Associate a variable to each half-edge. One way to index the variables is by a vertex and an incident edge (external or internal); in this indexing the variables will be denoted $a_{v,e}$ where v is a vertex and e is an edge.

We need the following definitions

- For a vertex $v \in V$ let $n(v)$ be the set of edges incident to v (internal or external).
- For a vertex $v \in V$ let $D_v = \sum_{j \in n(v)} a_{v,j}$.
- Let $\mathcal{C}$ be the set of all cycles of G (cycles, not circuits).
- For C a cycle and v a vertex in V, since G is 3-regular, there is a unique edge of G incident to v and not in C, let $o(C, v)$ be this edge.
- For $i \geq 0$ let

$$C^i_{3g} = \sum_{\substack{C_1, C_2, \ldots C_i \in \mathcal{C} \\ C_j \text{pairwise disjoint}}} \left(\left(\prod_{j=1}^{i} \prod_{v \in C_j} a_{v,o(C_j,v)} \right) \prod_{v \notin C_1 \cup C_2 \cup \cdots \cup C_i} D_v \right)$$

- Let

$$C_{3g} = \sum_{j \geq 0} (-1)^j C^j_{3g}$$

This is a polynomial because $C^i_{3g} = 0$ for $i > |\mathcal{C}|$ (and in fact it will usually be 0 quite a lot sooner – only in the case of a disjoint union of cycles and trees will $C^{|\mathcal{C}|}_{3g}$ be nonzero).

The first nice thing is that these things count something, and the alternating sum cancels away entirely, as follows.

Theorem 1. *Let $\mathcal{T}$ be the set of sets T of half edges of G with the property that*

- *every vertex of G is incident to exactly one half edge of T*
- *$G \smallsetminus T$ has no cycles*

Then

$$C_{3g} = \sum_{T \in \mathcal{T}} \prod_{h \in T} a_h$$

Proof. First notice that every monomial in each C^i_{3g} includes exactly one variable for each vertex of G. This is because if a vertex v is in one of the cycles, then it is in exactly one of the cycles (as the cycles are disjoint), and so appears once as a $a_{v,o(C,v)}$, and if v is not in one of the cycles, then v contributes D_v.

Consider a set T of half edges of G with the property that every vertex of G is incident to exactly one half edge of T. Note that $G \setminus T$ is 2-regular, and so consists of a disjoint union of cycles and lines (the lines are possible because of the external edges). Let k be the number of cycles of $G \setminus T$.

Now we wish to count how many times $\prod_{h \in T} a_h$ appears in C_{3g}^j. $\prod_{h \in T} a_h$ appears once for every set of cycles $C_1, C_2, \ldots C_j$ with the property that $a_{v,o(C_i,v)} \in T$ for all $v \in C_1, C_2 \ldots C_j$, that is whenever $C_1 \cup C_2 \cup \cdots \cup C_j \subseteq G \setminus T$. There are $\binom{k}{j}$ ways this can occur.

Thus the number of times $\prod_{h \in T} a_h$ appears in C_{3g} is

$$\sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} = \begin{cases} (1-1)^k = 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$$

The result follows. $\square$

We can determine the number of components of $G \setminus T$ for $T \in \mathcal{T}$.

Proposition 2. *Let G be connected and otherwise as above. Take $T \in \mathcal{T}$. Let t be the number of components of $G \setminus T$. Let d be the number of edges e of G for which both halves of e are in T. Let E_{ext} be the set of external edges of G. Then*

$$t + d + |E_{ext} \cap T| = \frac{|V| + |E_{ext}|}{2}$$

Proof. The proof is just a case of being careful with Euler's formula.

Let v be the number of vertices, e_{int} the number of internal edges, e_{ext} the number of external edges, and ℓ the number of independent cycles. Then we have the following facts

- $v - e_{\text{int}} + \ell = 1$ by Euler's formula
- $2e_{\text{int}} + e_{\text{ext}} = 3v$ as the graph is 3-regular

Remove the half edges in T from G one by one. Each removal either decreases the number of independent cycles by one, creates a new component, completes the removal of an internal edge of G, or removes an external edge of G. Thus

$$\ell + t + d + |E_{\text{ext}} \cap T| = |T| = v + 1$$

where the final 1 is for the single initial component of G. Combining these facts together we get

$$\begin{aligned} t + d + |E_{\text{ext}} \cap T| &= v - \ell + 1 \\ &= 2v - e_{\text{int}} \\ &= 2v + \frac{e_{\text{ext}} - 3v}{2} \\ &= \frac{e_{\text{ext}} - v}{2} \end{aligned}$$

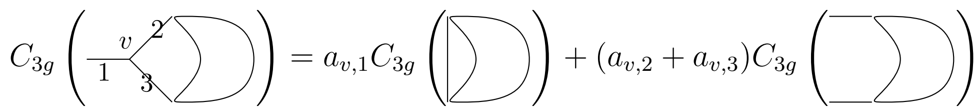
$\square$

The second nice thing is how these objects decompose upon removing a vertex. Note that when we remove a vertex in a graph then we remove the half edges incident to the vertex, but we leave the other half of any affected edges. These remaining halves are now external edges.

Proposition 3. *Let v be a vertex of G which is incident to an external edge of G. Label this external edge by 1 and the other edges incident to v by 2 and 3. Let H be G with v removed and the two remaining half edges of 2 and 3 joined to make a new internal edge. Then*

$$C_{3g}(G) = a_{v,1}C_{3g}(H) + (a_{v,2} + a_{v,3})C_{3g}(G \setminus v)$$

In pictures



Note that $G \setminus v$ does not need to be connected.

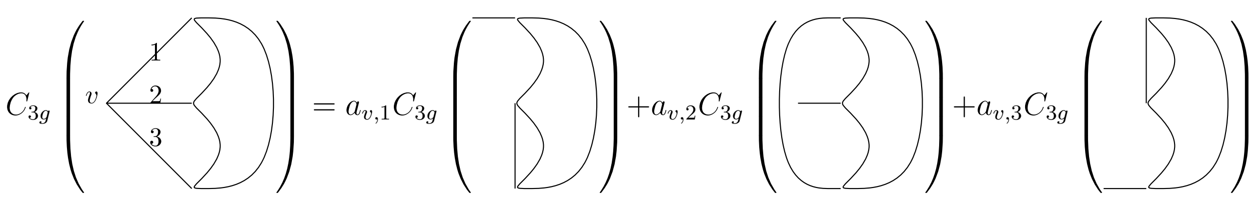
Proof. Collect together those terms of $C_{3g}(G)$ where the $v, 1$ half edge is removed. These are exactly the ways of removing a half edge from each remaining vertex so as to result in no cycles of $G \setminus v$ and so that joining the remaining ends of 1 and 2 (if they have not been removed) does not cause a cycle. These are exactly the terms of $C_{3g}(H)$.

Collect together those terms of $C_{3g}(G)$ where the $v, 2$ half edge is removed. The connection between 1 and 2 is now broken, so these are exactly the ways of removing a half edge from each remaining vertex so as to result in no cycles of $G \setminus v$ with no additional restrictions. These are exactly the terms of $C_{3g}(G \setminus v)$. By symmetry the same is true for the $v, 3$ half edge. $\square$

Proposition 4. *Let v be a vertex of G which is not incident to an external edge of G. Label the edges incident to v by 1, 2, and 3. Let H_i for $i = 1, 2, 3$ be G with v removed and the two remaining half edges which are not from edge i joined to make a new internal edge. Then*

$$C_{3g}(G) = a_{v,1}C_{3g}(H_1) + a_{v,2}C_{3g}(H_2) + a_{v,3}C_{3g}(H_3)$$

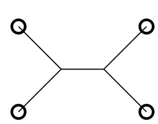
In pictures



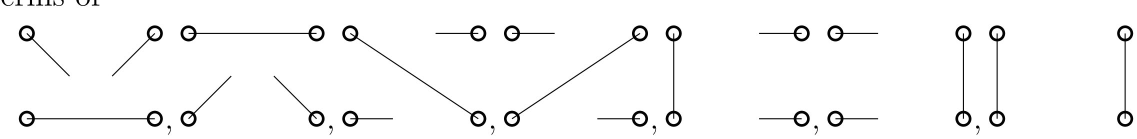
The proof follows the same idea as the previous one, so I'll leave it out for now.

I would argue that for these objects, the two previous propositions are a more natural decomposition than contraction-deletion, but you may disagree. If we want contraction-deletion, then we need to allow higher valences.

(One thing you could do while staying 3 regular is apply the previous proposition twice to adjacent vertices. This results in writing



in terms of



where the rest of the graph connects at the four circled vertices.)

2. Higher valences

You proposed the following definition for higher valences. Suppose G is no longer necessarily 3-regular. Define

- For C a cycle and v a vertex in V, let $r(C,v) = \sum_{e\in n(v), e\notin C} a_{v,e}$
- For $i \geq 0$ let

$$C_d^i = \sum_{\substack{C_1,C_2,\dots C_i \in \mathcal{C} \\ C_j \text{pairwise disjoint}}} \left(\left(\prod_{j=1}^{i} \prod_{v\in C_j} r(C,v) \right) \prod_{v\notin C_1\cup C_2\cup\cdots\cup C_i} D_v \right)$$

- Let

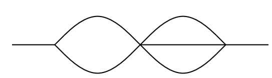
$$C_d = \sum_{j\geq 0} (-1)^j C_d^j$$

Note that if G is 3-regular then $C_d(G) = C_{3g}(G)$.

This is still a sum for which every monomial has one factor for each vertex, so this is still a sum over sets of half edges with the property that there is one half edge for each vertex. The cancellation behaves rather differently, in particular negative terms may appear. For example the banana with 4 edges labelled 1,2,3,4 and vertices a and b gives the polynomial

$$-2(a_{a,1}a_{b,1} + a_{a,2}a_{b,2} + a_{a,3}a_{b,3} + a_{a,4}a_{b,4})$$

which matches half edges – quite a different behaviour. Furthermore one gets mixed signs with the following graph



Contraction-deletion is also in bad shape for this definition. Consider an edge 1 with vertices v and w. There are three things which may occur in a given term

- half edges $v, 1$ and $w, 1$ are both removed
- exactly one of $v, 1$ and $w, 1$ are removed
- neither are removed

In the middle case we could contract the edge, each vertex still has one edge removed, but the connectivity has changed. Since I don't have a characterization of which terms appear, this might be ok but it is suspicious. . In the third case we can cut the edge (truly removing it, not leaving external edges), however we have broken a cycle. The first term cutting is the only possibility; we have to leave in the halves as external edges to keep one cut at each vertex, but then this requires other terms for that same graph where other choices were cut.

Another possibility is the following

- For a vertex $v \in V$ let

$$E_v = \sum_{\substack{j,k\in n(v) \\ j\neq k}} \prod_{\substack{i\neq j \\ i\neq k}} a_{v,i}.$$

- For C a cycle and v a vertex in V, let

$$p(C,v) = \prod_{\substack{i\in n(v) \\ i\notin C}} a_{v,i}$$

- For $i \geq 0$ let

$$C_k^i = \sum_{\substack{C_1, C_2, \ldots C_i \in \mathcal{C} \\ C_j \text{pairwise disjoint}}} \left(\left(\prod_{j=1}^{i} \prod_{v \in C_j} p(C,v) \right) \prod_{v \notin C_1 \cup C_2 \cup \cdots \cup C_i} E_v \right)$$

- Let

$$C_k = \sum_{j \geq 0} (-1)^j C_k^j$$

For this definition we are summing over sets T of half edges for which all but 2 of the half edges adjacent to any vertex v of G are in T. As a consequence $G \setminus T$ is again a disjoint union of cycles and lines, so the cancellation argument goes through unchanged. There is also something resembling contraction-deletion (the next proposition). But of course if physics demands C_d, then C_k is of no relevance regardless of its convenience.

Proposition 5. *Let* 1 *be an edge of* G *with vertices* a *and* b*. Let* $e_{a,1}$ *and* $e_{b,1}$ *be the two half edges making up edge* 1*. Then*

$$C_k(G) = C_k(G/1) + a_{a,1} C_k(G \setminus e_{a,1}) + a_{b,1} C_k(G \setminus e_{b,1}) - a_{a,1} a_{b,1} C_k(G \setminus e_{a,1} e_{b,1})$$

Proof. Consider a set T of half edges of G for which all but 2 of the half edges adjacent to any vertex v of G are in T and $G \setminus T$ has no cycles.

If neither $e_{a,1}$ nor $e_{b,1}$ are in T, then edge 1 can be contracted without changing the cycle structure of $G \setminus T$ and the new vertex still has exactly two edges in $T \setminus 1$.

If $e_{a,1}$ is not in T but $e_{b,1}$ is, then edge 1 can be cut without changing the cycle structure of $G \setminus T$. To preserve all but two edges in T at every vertex, we must not remove $e_{a,1}$ from G. Thus this term appears in $a_{b,1} C_k(G \setminus e_{b,1})$.

However, $a_{b,1} C_k(G \setminus e_{b,1})$ also contains the terms where $e_{a,1}$ happens to be in the set of half edges. That is, it contains the terms of $C_k(G)$ where both $e_{a,1}$ and $e_{b,1}$ are in T. Such terms are exactly the terms of $a_{a,1} a_{b,1} C_k(G \setminus e_{a,1} e_{b,1})$.

The same argument applies with a and b swapped. So we get that all terms of $C_k(G)$ appear in $C_k(G/1) + a_{a,1} C_k(G \setminus e_{a,1}) + a_{b,1} C_k(G \setminus e_{b,1})$ with those in $a_{a,1} a_{b,1} C_k(G \setminus e_{a,1} e_{b,1})$ appearing twice and the others appearing once. Subtracting off the over counting gives the result. □

3. Still to do

It's still not clear to me whether or not there is a determinantal form for C_{3g}. Hopefully the above observations are useful even though they are not everything.