The combinatorics of interval-vector polytopes

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Abstract

An *interval vector* is a $(0, 1)$-vector in $\mathbb{R}^n$ for which all the 1's appear consecutively, and an *interval-vector polytope* is the convex hull of a set of interval vectors in $\mathbb{R}^n$. We study three particular classes of interval vector polytopes which exhibit interesting geometric-combinatorial structures; e.g., one class has volumes equal to the Catalan numbers, whereas another class has face numbers given by the Pascal 3-triangle.

Keywords: Interval vector, lattice polytope, Ehrhart polynomial, root polytope, Catalan number, f-vector

1 Introduction

An *interval vector* is a $(0,1)$-vector $x \in \mathbb{R}^n$ such that, if $x_i = x_k = 1$ for $i < k$, then $x_j = 1$ for every $i \leqslant j \leqslant k$. In [2] Dahl introduced the class of *interval-vector polytopes*, which are formed by taking the convex hull of a set of interval vectors in $\mathbb{R}^n$. Our goal is to derive combinatorial properties of certain interval-vector polytopes.

For $i \leqslant j$, let $\alpha_{i,j} := e_i + e_{i+1} + \cdots + e_j$, where e_i is the i^{th} standard unit vector. The *interval length* of α_{ij} is $j - i + 1$. Let $S \subset \mathbb{N}$. For a fixed n, let $\mathcal{I}_S$ be the set of interval vectors in $\mathbb{R}^n$ with interval length in S. (If S is small, we may leave out the brackets in the set notation; e.g., we will denote $\mathcal{I}_{\{i,j\}}$ by $\mathcal{I}_{i,j}$.) We will denote the set of all non-zero interval vectors in a given dimension as $\mathcal{I}_{[n]}$. Let $\mathcal{P}_n(\mathcal{I}_S)$ be the convex hull of $\mathcal{I}_S \subset \mathbb{R}^n$.

There are three classes of interval vector polytopes that we will consider in this paper. In Section 3 we study the *complete interval vector polytope* $\mathcal{P}_n(\mathcal{I}_{[n]})$, the convex hull of all interval vectors in $\mathbb{R}^n$ except the zero vector. In Section 4 we look at the *fixed interval vector polytope* $\mathcal{P}_n(\mathcal{I}_i)$ given by the convex hull of all interval vectors with interval length i. In Section 5 we introduce the first in a class of *pyramidal interval polytopes*: the *first pyramidal interval vector polytope* $\mathcal{P}_n(\mathcal{I}_{1,n-1})$, the convex hull of all interval vectors in $\mathbb{R}^n$ with interval length 1 or $n-1$. (The reason for the term *pyramidal interval polytope* will also become clear in Section 5.) In Section 6 we generalize this to the i^{th} *pyramidal interval vector polytope* $\mathcal{P}_n(\mathcal{I}_{1,n-i})$. We examine combinatorial characteristics of these polytopes such as the f-vector and volume and discover unexpected relations to well-known numerical sequences.

Let t be a positive integer variable. For a lattice polytope $\mathcal{P}$ (i.e., the vertices of $\mathcal{P}$ all have integer coordinates), the *Ehrhart polynomial* $L_{\mathcal{P}}(t)$ is the counting function yielding the number of lattice points in $t\mathcal{P} := \{tv \mid v \in \mathcal{P}\}$. Ehrhart [5] proved that $L_{\mathcal{P}}(t)$ is indeed a polynomial; see, e.g., [1] for more about Ehrhart polynomials. The Ehrhart polynomial contains useful geometric information about a polytope; in particular, the leading coefficient of the Ehrhart polynomial gives the volume of the polytope.

In [9], Postnikov defines the *complete root polytope* $Q_n \subset \mathbb{R}^n$ as the convex hull of 0 and $e_i - e_j$ for all $i < j$ where e_i is the i^{th} standard unit vector. He showed (among many other things) that the volume of Q_{n+1} is $C_n := \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number. In Section 3 we prove, in a discrete-geometric sense, that Q_{n+1} and the complete interval vector polytope $\mathcal{P}_n(\mathcal{I}_{[n]})$ are interchangeable, that is, the two polytopes have the same Ehrhart polynomial.

Theorem 1. $L_{Q_{n+1}}(t) = L_{\mathcal{P}_n(\mathcal{I}_{[n]})}(t)$.

Corollary 2. *The volume of the complete interval vector polytope $\mathcal{P}_n(\mathcal{I}_{[n]})$ equals the n^{th} Catalan number.*

A *unimodular simplex* in $\mathbb{R}^d$ is an n-dimensional lattice simplex Δ whose edge direction at any vertex form a lattice basis for $\mathbb{Z}^d \cap \operatorname{aff}(\Delta)$, where $\operatorname{aff}(\Delta)$ is the affine hull of Δ. In Section 4 we prove:

Theorem 3. *The fixed interval vector polytope $\mathcal{P}_n(\mathcal{I}_i)$ is an $(n-i)$-dimensional unimodular simplex.*

Given an n-dimensional polytope $\mathcal{P}$ with f_k k-dimensional faces, the *f-vector* of $\mathcal{P}$ is written as $f(\mathcal{P}) := (f_{-1}, f_0, f_1, \ldots, f_n)$ where $f_{-1}, f_n := 1$ (see, e.g., [7] for more about f-vectors). In Section 5 we show:

Theorem 4. *For $n \geqslant 3$, the f-vector of the first pyramidal interval vector polytope satisfies $f_k(\mathcal{P}_n(\mathcal{I}_{1,n-1})) = \binom{n-1}{k} + \binom{n+1}{k+1}$.*

The f-vector of $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ is thus the n^{th} row of the *Pascal 3-triangle* (see, e.g., [10, Sequence A028262]), in particular, it is symmetric. We also show that the volume of the 1^{st} pyramidal interval vector polytope is simple:

Theorem 5. *For $n \geqslant 3$, $\mathrm{vol}(\mathcal{P}_n(\mathcal{I}_{1,n-1})) = 2(n-2)$.*

Finally, in Section 6 we lay out future work on i^{th} pyramidal interval vector polytopes.

2 Preliminaries

In this paper, we will be analyzing the properties of certain classes of *convex polytopes* which are formed by taking the convex hull of finitely many points in $\mathbb{R}^n$. The *convex hull* of a set $A = \{v_1, v_2, \ldots, v_m\} \subset \mathbb{R}^n$, denoted $\mathrm{conv}(A)$, is defined as

$$\left\{ \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m \mid \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}_{\geqslant 0} \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\}. \tag{1}$$

The polytope $\mathrm{conv}(A)$ is contained in the *affine hull* $\mathrm{aff}(A)$ of A, defined as in (1) but without the restriction that $\lambda_1, \lambda_2, \ldots, \lambda_m \geqslant 0$. We call a set of points *affinely* (resp. *convexly*) *independent* if each point is not in the affine (resp. convex) hull of the rest. The *vertex set* of a polytope is the minimal convexly independent set of points whose convex hull form the polytope. A polytope is *d-dimensional* if the dimension of its affine hull is d. We denote the dimension of the polytope $\mathcal{P}$ as $\dim(\mathcal{P})$. We call a d-dimensional polytope a *d-simplex* if it has $d+1$ vertices.

A *lattice point* is a point with integral coordinates. A *lattice polytope* is a polytope whose vertices are lattice points. The *normalized volume* of a polytope $\mathcal{P}$, denoted $\mathrm{vol}(\mathcal{P})$, is the volume with respect to a unimodular simplex (recall definition in Section 1). We will refer to the normalized volume of a polytope as its *volume*. Note that the leading coefficient of the Ehrhart polynomial of a lattice polytope $\mathcal{P}$ is $\frac{1}{d!}\,\mathrm{vol}(\mathcal{P})$.

A *hyperplane* is a set of the form

$$H := \{x \in \mathbb{R}^n \mid a_1 x_1 + \cdots + a_n x_n = b\},$$

where not all a_j's are 0. The *half-spaces* defined by this hyperplane are formed by the two weak inequalities corresponding to the equation defining the hyperplane. A *face* of $\mathcal{P}$ is the intersection of a hyperplane and $\mathcal{P}$ such that $\mathcal{P}$ lies completely in one half-space of the hyperplane. This face is a polytope called a *k-face* if its dimension is k. A vertex is a 0-face and an *edge* is a 1-face. Given an n-dimensional polytope $\mathcal{P}$ with

f_k k-dimensional faces, the *f-vector* of $\mathcal{P}$ is written as $f(\mathcal{P}) := (f_{-1}, f_0, \ldots, f_n)$. For example, a triangle $\triangle$ is a 2-dimensional polytope with 3 vertices and 3 edges and thus has f-vector $f(\triangle) = (1, 3, 3, 1)$.

3 Complete Interval Vector Polytopes

In [2] Dahl provides a method for determining the dimension of these polytopes which we will use throughout this paper. We utilized the software packages `polymake` [6] and `LattE` [4, 8] to find most of the patterns described by our results.

Proof of Theorem 1. Each of the vertices of Q_n are vectors with entries that sum to zero, so any linear combination (and specifically any convex combination) of these vertices also has entries who sum to zero. Define $B := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}$; thus $Q_n \subset B$, and B is an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$.

Consider the linear transformation T given by the $n \times n$ lower triangular matrix with entries $t_{i,j} = 1$ if $i \geqslant j$ and $t_{i,j} = 0$ otherwise. Then

$$T(B) \subseteq A := \{x \in \mathbb{R}^n \,|\, x_n = 0\}\,.$$

Since (the matrix representing) T has determinant 1, it is injective when restricting the domain to B. For the same reason, we know that for any $y \in A$, there exists $x \in \mathbb{R}^n$ such that $y = T(x)$. But since $y_n = \sum_{i=1}^n x_i = 0$, then $x \in B$, so that $T|_B : B \to A$ is surjective, and therefore a linear bijection.

Also, the projection $\Pi : A \to \mathbb{R}^{n-1}$ given by

$$\Pi\left((x_1, \ldots, x_{n-1}, 0)\right) = (x_1, \ldots, x_{n-1}),$$

is clearly a linear bijection.

Now we show that the linear bijection $\Pi \circ T|_B : B \to \mathbb{R}^{n-1}$ is a lattice-preserving map, i.e., an isomorphism from $B \cap \mathbb{Z}^n$ to $\mathbb{Z}^{n-1}$ (viewed as additive groups). First we find a lattice basis for B. Consider

$$C := \{e_{i,n} = e_i - e_n \,|\, i < n\}\,.$$

We notice that any integer point of B can be represented as

$$\left(a_1, \ldots, a_{n-1}, -\sum_{i=1}^{n-1} a_i\right) = \sum_{i=1}^{n-1} a_i e_{i,n}$$

and so C is a lattice basis.

Note that $\Pi \circ T(e_{i,n}) = e_i + \cdots + e_{n-1} =: u_i$. Therefore

$$\Pi \circ T(C) = \{u_i \,|\, i \leqslant n-1\} =: U\,.$$

We notice that $e_{n-1} = u_{n-1}$ and $e_i = u_i - u_{i+1}$, so that each of the standard unit vectors e_i of $\mathbb{R}^{n-1}$ is an integral combination of the vectors in U. Since the standard basis is a lattice basis, so is U, thus $\Pi \circ T|_B$ is a lattice-preserving map. Since our bijection is linear and lattice-preserving, all we have left to show is that the vertices of Q_n map to those of $\mathcal{P}_{n-1}(\mathcal{I}_{[n-1]})$. By linearity, $\Pi \circ T(0) = 0$, and given any vertex $\alpha_{i,j}$ of $\mathcal{P}_{n-1}(\mathcal{I}_{[n-1]})$, we know that $\Pi \circ T(e_{i,j+1}) = \alpha_{i,j}$ where $i < j+1 \leqslant n$ so that $\Pi \circ T|_B$ maps vertices to vertices. □

Corollary 2 follows directly from this theorem and [9], since the leading coefficient of the Ehrhart polynomial of $\mathcal{P}_n$ is $\frac{1}{n!}$ times the volume of $\mathcal{P}_n$.

4 Fixed Interval Vector Polytopes

The following construction is due to [2]. We define the set of *elementary vectors* as containing all $e_{i,j} = e_i - e_j$, each unit vector e_i, and the zero vector. Let T be the lower triangular matrix from the proof of Theorem 1. We notice that $T(e_i) = \alpha_{i,n}$ and $T(e_{i,j}) = \alpha_{i,j-1}$. So the image of an elementary vector is an interval vector. Since T is invertible, for any set of interval vectors $\mathcal{I}$, there is a unique set $\mathcal{E}$ of elementary vectors such that $T(\mathcal{E}) = \mathcal{I}$, namely $\mathcal{E} = T^{-1}(\mathcal{I})$.

Thus for any interval vector polytope $\mathcal{P}_n(\mathcal{I}_S) \subset \mathbb{R}^n$, we can construct the corresponding *flow-dimension graph* $G_{\mathcal{I}_S} = (V, E)$ as follows. Let $\mathcal{E}_S = T^{-1}(\mathcal{I}_S)$. Let the vertex set $V = [n]$. Specify a subset $V_1 = \{j \in V \mid e_j \in \mathcal{E}_S\}$, and define the directed edge set $E = \{(i,j) \mid e_{i,j} \in \mathcal{E}_S\}$. Let k_0 denote the number of connected components $\mathcal{C}$ of the graph G (ignoring direction) so that $\mathcal{C} \cap V_1$ is empty.

Recall that the fixed interval vector polytope $\mathcal{P}_n(\mathcal{I}_i)$ is the convex hull of all interval vectors in $\mathbb{R}^n$ with interval length i. For example, the fixed interval vector polytope with $n = 5$, $i = 3$ is

$$\mathcal{P}_5(\mathcal{I}_3) = \text{conv}\left((1,1,1,0,0)\,,\,(0,1,1,1,0)\,,\,(0,0,1,1,1)\right)$$

and its flow-dimension graph is depicted in Figure 1.

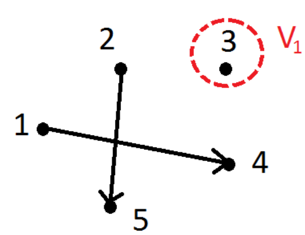

Figure 1: The flow-dimension graph of $\mathcal{P}_5(\mathcal{I}_3)$.

Theorem 6 (Dahl [2])**.** *If* $0 \in \text{aff}(\mathcal{I}_S)$*, then the dimension of* $\mathcal{P}_n(\mathcal{I}_S)$ *is* $n - k_0$*. Else, if* $0 \notin \text{aff}(\mathcal{I}_S)$ *then the dimension of* $\mathcal{P}_n(\mathcal{I}_S)$ *is* $n - k_0 - 1$*.*

For a fixed i,

$$T^{-1}(\mathcal{I}_i) = \mathcal{E}_i = \{e_{k,k+i} \mid k \leqslant n-i\} \cup \{e_{n-i+1}\}.$$

The corresponding flow-dimension graph is $G_{\mathcal{P}_n(\mathcal{I}_i)} = (V, E)$ where $V = \{1, \ldots, n\}$ and $E = \{(k, k+i) \,|\, k \in [n-i]\}$. Then $V_1 = \{n-i+1\}$ corresponds to $e_{n-i+1} \in \mathcal{E}_i$.

Two nodes a, b in a graph $G = (V, E)$ are said to be *connected* if there exists a *path* from a to b, that is there exist $q_0, \ldots, q_s \in V$ such that $(a, q_0), (q_0, q_1), \ldots, (q_s, b) \in E$.

Lemma 7. *Let a, b be nodes in the flow-dimension graph $G_{\mathcal{P}_n(\mathcal{I}_i)}$. Then a and b are connected if and only if $a \equiv b \mod i$.*

Proof. The edges in $G_{\mathcal{P}_n(\mathcal{I}_i)}$ are of the form $(k, k+i)$, and therefore the nodes of a path in $G_{\mathcal{P}_n(\mathcal{I}_i)}$ are all in the same congruence class modulo i. □

Proposition 8. *$\mathcal{P}_n(\mathcal{I}_i)$ is an $(n-i)$-dimensional simplex.*

Proof. For a given dimension and interval length, an interval vector is uniquely determined by the location of the first 1, hence we can determine the number of vertices of $\mathcal{P}_n(\mathcal{I}_i)$ by counting all possible placements of the first 1 in an interval of i 1's. Since the string must have length i, the number of spaces before the first 1 must not exceed $n-i$ and so there are $n-i+1$ possible locations for the first 1 in the interval to be placed. Thus, $\mathcal{P}_n(\mathcal{I}_i)$ has $n-i+1$ vertices.

By Lemma 7 we know there are i connected components in the flow-dimension graph $G_{\mathcal{P}_n(\mathcal{I}_i)}$ and since V_1 has only one element, $k_0 = i-1$. Thus by Theorem 6 the dimension of $\mathcal{P}_n(\mathcal{I}_i)$ is $n-i$. Therefore $\mathcal{P}_n(\mathcal{I}_i)$ is an $(n-i)$-dimensional simplex. □

Proof of Theorem 3. It remains to show that $\mathcal{P}_n(\mathcal{I}_i)$ is unimodular. Consider the affine space where the sum over every i^{th} coordinate is 1,

$$A = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \sum_{j \equiv k \bmod i} x_j = 1, \quad \text{for all } \; k \in [i] \right\}.$$

Since the vertices of $\mathcal{P}_n(\mathcal{I}_i)$ have interval length i, they are in A; thus $\mathcal{P}_n(\mathcal{I}_i) \subset A$. We want to show that the following vectors in $\mathcal{P}_n(\mathcal{I}_i)$ form a lattice basis for A:

$$\begin{aligned} w_1 &= \alpha_{1,i} - \alpha_{n-i+1,n} \\ w_2 &= \alpha_{2,i+1} - \alpha_{n-i+1,n} \\ &\;\;\vdots \\ w_{n-i} &= \alpha_{n-i,n-1} - \alpha_{n-i+1,n}. \end{aligned}$$

We will do this by showing that any integer point $p \in A$ can be expressed as an integral linear combination of the proposed lattice basis, that is, there exist integer coefficients $Y_1, \ldots, Y_{n-i}$ so that $p = Y_1 w_1 + \cdots + Y_{n-i} w_{n-i} + \alpha_{n-i+1,n}$.

We first notice that p can be expressed as

$$\left(p_1, p_2, \ldots, p_{n-i}, \sum_{\substack{j \leqslant n-i \\ j \equiv t-i+1 \bmod i}} (-p_j) + 1, \sum_{\substack{j \leqslant n-i \\ j \equiv t-i+2 \bmod i}} (-p_j) + 1, \ldots, \sum_{\substack{j \leqslant n-i \\ j \equiv t \bmod i}} (-p_j) + 1 \right)$$

by solving for the last term in each of the equations defining A. Let

$$Y_t = \begin{cases} p_1 & \text{if} \quad t = 1, \\ p_t - p_{t-1} & \text{if} \quad 1 < t \leqslant i, \\ p_t - Y_{t-i} & \text{if} \quad i < t \leqslant n - i. \end{cases}$$

Then each Y_t is an integer. We claim that

$$Y_1 w_1 + \cdots + Y_{n-i} w_{n-i} + \alpha_{n-i+1,n} = p.$$

Clearly the first coordinate is p_1 since w_1 is the only vector with an element in the first coordinate. Next consider the t^{th} coordinate of this linear combination for $1 < t \leqslant i$, by summing the coefficients of all the vectors who have a 1 in the t^{th} position:

$$Y_t + Y_{t-1} + Y_{t-2} + \cdots + Y_1 = p_t - p_{t-1} + p_{t-1} - p_{t-2} + \cdots + p_2 - p_1 + p_1 = p_t$$

We next consider the t^{th} coordinate of the combination for $i < t \leqslant n - i$ by summing the coefficients of the vectors who have a 1 in the t^{th} position.

$$Y_t + Y_{t-1} + \cdots + Y_{t-i+1} = (p_t - Y_{t-1} - \cdots - Y_{t-i+1}) + Y_{t-1} + \cdots + Y_{t-i+1} = p_t$$

Finally, we consider the t^{th} coordinate of the combination for $n - i < t \leqslant n$, noticing that each coordinate from w_1 to w_t has a -1 in the $(t-i)^{\text{th}}$ position, and $\alpha_{n-i+1,n}$ has a 1 in this position. This gives

$$-(Y_1 + Y_2 + \cdots + Y_{t-i}) + 1.$$

Applying the two relations we have defined between coordinates, and calling $\langle t \rangle$ the least residue of $t \bmod i$, we see that

$$\begin{aligned} -(Y_1 + Y_2 + \cdots + Y_{t-i}) + 1 &= -(Y_1 + Y_2 + \cdots + Y_{t-2i} + p_{t-i}) + 1 \\ &= -(Y_1 + Y_2 + \cdots + Y_{t-3i} + p_{t-2i} + p_{t-i}) + 1 \\ &= -\left(Y_1 + Y_2 + \cdots + Y_{\langle t \rangle} + \sum_{\substack{i < j \leqslant n-i \\ j \equiv t \bmod i}} p_j \right) + 1 \\ &= -\left(\sum_{\substack{j \leqslant n-i \\ j \equiv t \bmod i}} p_j \right) + 1. \end{aligned}$$

Thus $p = Y_1 w_1 + Y_2 w_2 + \cdots + Y_{n-i} w_{n-i} + \alpha_{n-i+1,n}$ and so $w_1, \ldots, w_{n-i}$ form a lattice basis of A. Thus the vertices of $\mathcal{P}_n(\mathcal{I}_i)$ form a lattice basis, and so $\mathcal{P}_n(\mathcal{I}_i)$ is a unimodular simplex. $\square$

5 The first pyramidal interval vector polytope

Recall that $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ is the convex hull of all vectors in $\mathbb{R}^n$ with interval length 1 or $n-1$. For example,

$$\mathcal{P}_4(\mathcal{I}_{1,3}) = \operatorname{conv}\big((1,0,0,0)\,,\,(0,1,0,0)\,,\,(0,0,1,0)\,,\,(0,0,0,1)\,,\,(1,1,1,0)\,,\,(0,1,1,1)\big)\,,$$

whose flow-dimension graph is depicted in Figure 2.

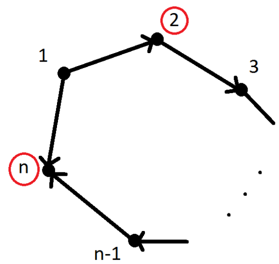

Figure 2: $G_{\mathcal{P}_n(\mathcal{I}_{1,n-1})}$.

Proposition 9. *The dimension of $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ is n.*

Proof. The affine hull of $e_1,\ldots,e_n$ is the $(n-1)$-dimensional set

$$\{x \in \mathbb{R}^n \,|\, x_1 + \cdots + x_n = 1\}\,.$$

Since $\alpha_{1,n-1}$ is not in this set, $\dim(\mathcal{P}_n(\mathcal{I}_{1,n-1})) = n$. $\square$

Recall that the f-vector of a polytope tells us the number of faces the polytope has of each dimension. Our next task is to compute the f-vector of $\mathcal{P}_n(\mathcal{I}_{1,n-1})$.

Lemma 10. *Let $n \geqslant 3$. Then $\mathcal{B} := \operatorname{conv}(e_1, e_n, \alpha_{1,n-1}, \alpha_{2,n})$ is a 2-dimensional face of $\mathcal{P}_n(\mathcal{I}_{1,n-1})$.*

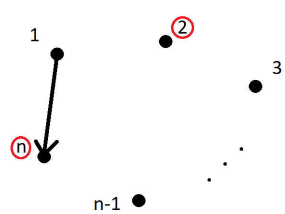

Figure 3: $G_{\mathcal{A}}$.

Proof. We first consider $\mathcal{A} = \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$. The corresponding elementary vectors of the vertex set are $\{e_{1,n}, e_2, e_n\}$. So we build the flow-dimension graph as seen in Figure 2, $G_{\mathcal{A}} = (V, E)$ where $V = [n]$, $E = \{(1, n)\}$ corresponding to $e_{1,n}$. The subset $V_1 = \{2, n\}$ (circled in Figure 2) corresponds to e_2 and e_n. This graph has $n-1$ connected components, two of which contain elements of V_1 so that $k_0 = n - 3$.

If we let $\lambda_1 e_n + \lambda_2 \alpha_{1,n-1} + \lambda_3 \alpha_{2,n} = \mathbf{0}$, we first notice that $\lambda_2 = 0$ since $\alpha_{1,n-1}$ is the only vector with a nonzero first coordinate. But this implies that $\lambda_1 = \lambda_3 = 0$. Since the coefficients cannot sum to 1, we conclude that $\mathbf{0} \notin \text{aff}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$. So now by Theorem 6,

$$\dim(\text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})) = n - k_0 - 1 = n - (n-3) - 1 = 2.$$

Finally $e_1 = (1)\alpha_{1,n-1} + (-1)\alpha_{2,n} + (1)e_n$ is in the affine hull of $\mathcal{A}$ and thus does not add a dimension. We conclude that $\dim(\mathcal{B}) = 2$. $\square$

Corollary 11. *Let $\mathcal{I} := \{e_1, e_2, \ldots, e_n, \alpha_{1,n-1}, \alpha_{2,n}\}$. For $2 \leqslant i \leqslant n-1$ each e_i adds a dimension to $\mathcal{P}_n(\mathcal{I}_{1,n-1})$, that is, $e_i \notin \text{aff}(\mathcal{I} \setminus \{e_i\})$.*

Proof. This follows from Proposition 9 and Lemma 10. Since $\mathcal{B}$ has dimension 2 and $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ has dimension n, then the $n-2$ remaining vertices must add the remaining $n-2$ dimensions. $\square$

Lemma 12. *Let $\mathcal{B}$ as in Lemma 10. Then $\mathcal{B}$ has f-vector* $(1, 4, 4, 1)$.

Proof. Since $\mathcal{B}$ has dimension 2, $f_1 = f_0$. We know that $\{e_n, \alpha_{1,n-1}, \alpha_{2,n}\}$ are three vertices of $\mathcal{B}$. If $e_1 \in \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$ then

$$e_1 = \lambda_1 e_n + \lambda_2 \alpha_{1,n-1} + \lambda_3 \alpha_{2,n} \tag{2}$$

where the coefficients sum to 1. Since $\alpha_{1,n-1}$ is the only vector with a nonzero coordinate in the first position, $\lambda_2 = 1$. This in turn implies that $\lambda_1 = \lambda_3 = 0$, contradicting (2). So $e_1 \notin \text{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n})$ and therefore forms a fourth vertex. $\square$

We can tie all this together with the following theorem. First we define a *d-pyramid* P as the convex hull of a $(d-1)$-dimensional polytope K (the *basis* of P) and a point $A \notin \text{aff}(K)$) (the *apex* of P).

Theorem 13 (see, e.g., [7])**.** *If P is a d-pyramid with basis K then*

$$\begin{aligned}
f_0(P) &= f_0(K) + 1 \\
f_k(P) &= f_k(K) + f_{k-1}(K) \quad \textit{for } 1 \leqslant k \leqslant d-2 \\
f_{d-1}(P) &= 1 + f_{d-2}(K)\,.
\end{aligned}$$

We notice that the rows of Pascal's 3-triangle act in the same manner and we claim the face numbers for $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ can be derived from Pascal's 3-triangle.

Proof of Theorem 4. Recall that $\mathcal{I} = \{e_1, e_2, \ldots, e_n, \alpha_{1,n-1}, \alpha_{2,n}\}$ is the vertex set for $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ with $n \geqslant 3$, and let $\mathcal{R}_k := \operatorname{conv}(\mathcal{I} \setminus \{e_k, e_{k+1}, \ldots, e_{n-1}\})$ for $1 \leqslant k < n$. Then it is clear that $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ is the convex hull of the union of the $(n-1)$-dimensional polytope $\mathcal{R}_{n-1}$ and $e_{n-1} \notin \operatorname{aff}(\mathcal{R}_{n-1})$ (by Corollary 11), and thus is a pyramid and its face numbers can be computed as in Theorem 13 from the face numbers of $\mathcal{R}_{n-1}$.

Notice next that $\mathcal{R}_{n-1}$ is the convex hull of the $(n-2)$-dimensional polytope $\mathcal{R}_{n-2}$ and $e_{n-2} \notin \operatorname{aff}(\mathcal{R}_{n-2})$ (again by Corollary 11), so we can compute the face numbers of $\mathcal{R}_{n-1}$ from those of $\mathcal{R}_{n-2}$ as in Theorem 13.

We can continue this process until we get that $\mathcal{R}_3$ is the convex hull of $\mathcal{R}_2$ and $e_2 \notin \operatorname{aff}(\mathcal{R}_2)$. But we notice that $\mathcal{R}_2 = \mathcal{B}$, so by Lemma 12, $f_0(\mathcal{R}_2) = f_1(\mathcal{R}_2) = 4$. From here we can build the f-vector of $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ recursively, using Theorem 13. $\square$

Our next goal is to compute the volume of $\mathcal{P}_n(\mathcal{I}_{1,n-1})$. A simple induction proof gives:

Lemma 14. *The determinant of the $n \times n$-matrix*

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ & & \ddots & & \\ 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

is $(-1)^{n-1}(n-1)$.

Proof of Theorem 5. In order to calculate the volume of $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ we will first triangulate the 2-dimensional base of the pyramid $\mathcal{B}$ from Lemma 10: namely, $\mathcal{B}$ is the union of

$$\triangle_1 = \operatorname{conv}(e_1, e_n, \alpha_{1,n-1}) \qquad \text{and} \qquad \triangle_2 = \operatorname{conv}(e_n, \alpha_{1,n-1}, \alpha_{2,n}).$$

By Corollary 11, each $e_2, \ldots, e_{n-1}$ adds a dimension so that the convex hull of these points and $\triangle_1$ is an n-dimensional simplex. The same can be said of $\triangle_2$. Call these simplices S_1 and S_2 respectively; thus S_1 and S_2 triangulate $\mathcal{P}_n(\mathcal{I}_{1,n-1})$, and the sum of their volumes is equal to the volume of $\mathcal{P}_n(\mathcal{I}_{1,n-1})$. In order to calculate the volume of S_1 and S_2, we will use the Cayley Menger determinant [3]. Consider S_1, whose volume is the determinant of the matrix

$$\begin{bmatrix} e_1 - \alpha_{1,n-1} & e_2 - \alpha_{1,n-1} & \cdots & e_n - \alpha_{1,n-1} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 & \cdots & -1 & -1 \\ -1 & 0 & -1 & \cdots & -1 & -1 \\ -1 & -1 & 0 & -1 & \cdots & -1 \\ & & & \ddots & & \\ -1 & -1 & \cdots & -1 & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Cofactor expansion on the last row will leave us with the determinant, up to a sign, of the $(n-1) \times (n-1)$ matrix

$$\begin{bmatrix} 0 & -1 & -1 & \cdots & -1 \\ -1 & 0 & -1 & \cdots & -1 \\ & & \ddots & & \\ -1 & \cdots & -1 & 0 & -1 \\ -1 & -1 & \cdots & -1 & 0 \end{bmatrix}, \tag{3}$$

which, when ignoring sign, by Lemma 14 is $n-2$. Therefore the volume of S_1 is $n-2$.

A similar computation gives the volume of S_2 as $n-2$, and so the volume of $\mathcal{P}_n(\mathcal{I}_{1,n-1})$ is $2(n-2)$, as desired. □

6 The i^{th} pyramidal interval vector polytope

Recall that the i^{th} pyramidal interval vector polytope is $\mathcal{P}_n(\mathcal{I}_{1,n-i})$, the convex hull of all interval vectors in $\mathbb{R}^n$ with interval length 1 or $n-i$.

Example 15. For $n=6$ and $i=2$,

$$\begin{aligned} \mathcal{P}_6(\mathcal{I}_{1,4}) = \operatorname{conv}\big(&(1,0,0,0,0,0)\,,\ (0,1,0,0,0,0)\,,\ (0,0,1,0,0,0)\,,\ (0,0,0,1,0,0) \\ &(0,0,0,0,1,0)\,,\ (0,0,0,0,0,1)\,,\ (1,1,1,1,0,0)\,,\ (0,1,1,1,1,0) \\ &(0,0,1,1,1,1)\big)\,. \end{aligned}$$

The following proposition collects certain properties of $\mathcal{P}_n(\mathcal{I}_{1,n-i})$. We omit its proof, since it is similar to the proofs in Section 5.

Proposition 16. *The dimension of $\mathcal{P}_n(\mathcal{I}_{1,n-i})$ is n. Furthermore, $\mathcal{P}_n(\mathcal{I}_{1,n-i})$ can be constructed by taking iterative pyramids (with the sequence of top vertices $e_{i+1}, e_{i+2}, \ldots, e_{n-i}$) over the $2i$-dimensional base*

$$\operatorname{conv}\left(\{e_1, e_2, \ldots, e_i, e_{n-i+1}, \ldots, e_n, \alpha_{1,n-i}, \alpha_{2,n-i-1} \ldots, \alpha_{i+1,n}\}\right).$$

Using `polymake` to generate f-vectors for varying n, we observed that the f-vectors of $\mathcal{P}_n(\mathcal{I}_{1,n-i})$ correspond to the sum of multiple shifted Pascal triangles; this is again due to its pyramid property. We also offer the following:

Conjecture 17. The volume of $\mathcal{P}_n(\mathcal{I}_{1,n-i})$ equals $2^i(n-(i+1))$.

We conjecture something more concrete: namely, that $\mathcal{P}_n(\mathcal{I}_{1,n-i})$ can be triangulated into 2^i simplices, and pyramiding over each of these simplices each yields a volume of $n-(i+1)$.

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