Geometric constructions for 3-configurations with non-trivial geometric symmetry

Leah Wrenn Berman

Department of Mathematics & Statistics
University of Alaska Fairbanks
Fairbanks, Alaska, USA

lwberman@alaska.edu

Submitted: May 10, 2012; Accepted: July 25, 2013; Published: Aug 2, 2013
Mathematics Subject Classifications: 51E30, 05B30

Abstract

A geometric 3-configuration is a collection of points and straight lines, typically in the Euclidean plane, in which every point has 3 lines passing through it and every line has 3 points lying on it; that is, it is an (n_3) configuration for some number n of points and lines. We will say that such configuration is *symmetric* if there are non-trivial isometries of the plane that map the configuration to itself. Many symmetric 3-configurations may be easily constructed with computer algebra systems using algebraic techniques: e.g., constructing a number of symmetry classes of points and lines, by various means, and then determining the position of a final class of points or lines by solving some polynomial equation. In contrast, this paper presents a number of ruler-and-compass-type constructions for exactly constructing various types of symmetric 3-configurations, as long as the vertices of an initial regular m-gon are explicitly provided. In addition, it provides methods for constructing chirally symmetric 3-configurations given an underlying unlabelled reduced Levi graph, for extending these constructions to produce dihedrally symmetric 3-configurations, and for constructing 3-configurations corresponding to all 3-orbit and 4-orbit reduced Levi graphs that contain a pair of parallel arcs. Notably, most of the configurations described are movable: that is, they have at least one continuous parameter.

1 Introduction

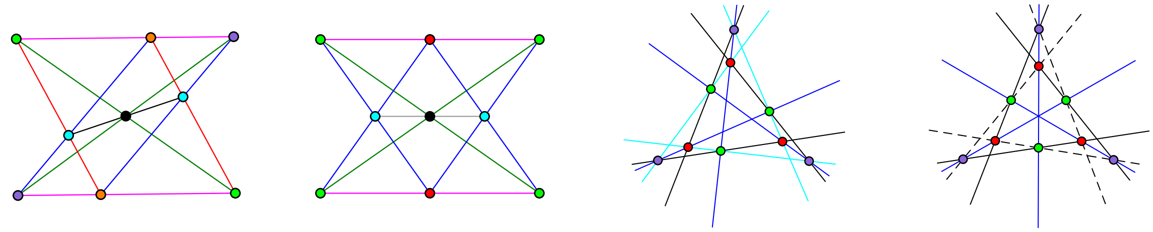
A geometric (q, k)-configuration is a collection of points and straight lines, typically in the Euclidean or projective plane, in which k points lie on every line and q lines pass through every point. If there are p points and n lines, the configuration is called an (p_q, n_k) configuration; if $q = k$ and consequently $p = n$, then we refer to a (n_k) configuration or a k-configuration; the case where $q = k = 3$ is the first non-trivial case.

3-configurations have been studied extensively since the mid-1800s, and there are many results, both existential and enumerative, classifying various types of 3-configurations. Initial investigations focused primarily on classification of combinatorial configurations. We say that a combinatorial configuration is *geometrically realizable* if there exists at least one realization of the combinatorial configuration using points and straight lines. Note that it is possible for there to be two realizations of the same combinatorial configuration that are not even affinely congruent.

We will say that a geometric (n_k) configuration is *symmetric* if it has non-trivial geometric symmetry; that is, under the group of isometries of the plane that map the configuration to itself, the configuration has fewer than n transitivity classes of points and n transitivity classes of lines. This usage of the word "symmetric" follows Branko Grünbaum [7, p. 16] in reserving the word "symmetric" to refer to geometric properties of configurations. In other places in the literature (e.g. [2] and in works connecting the study of configurations to design theory), the word symmetric has been used to refer to (n_k) configurations, which are 'symmetric' in the numbers of points and lines; however, following Grünbaum, we shall call general (n_k) configurations *balanced*, and restrict 'symmetric' to refer to geometric properties of a particular embedding.

Given a geometric configuration with symmetry group S, which consists of all isometries of the plane that map the configuration to itself, we say that the configuration has *chiral* symmetry if S is isomorphic to a cyclic group and *dihedral* symmetry if S is isomorphic to a dihedral group. The action of the symmetry group on the points of the configuration partitions the points into orbits, called the *symmetry classes* of the points, and likewise partitions the lines into symmetry classes of lines; we also can refer to the symmetry classes induced by the rotation subgroup of S, if S is isomorphic to a dihedral group. Typically, symmetric configurations have only a small number of symmetry classes of points and lines, relative to the total number of points and lines of the configuration. Note that the definition of a symmetric configuration given above is slightly broader than that of *polycyclic* configurations [3], which requires that every symmetry class of points and of lines determined by the rotation subgroup of the symmetry group must have the same number of elements. Figure 1 shows several embeddings of the (9_3) Pappus configuration, illustrating chirally and dihedrally symmetric non-polycyclic and polycyclic embeddings. All the configurations constructed in this paper are polycyclic.

Given a (q, k)-configuration, and following Grünbaum [7, Section 1.5], we say that the configuration is (h_1, h_2)-astral if it has h_1 symmetry classes of lines and h_2 symmetry classes of points; if $h_1 = h_2 = h$, we call the configuration h-astral. If $h_1 = \lfloor \frac{q+1}{2} \rfloor$ and $h_2 = \lfloor \frac{k+1}{2} \rfloor$, then we say the configuration is *astral* [5, 7]. For example, a 3-configuration is astral if it has 2 symmetry classes of points and lines. Figure 3 shows two symmetric 3-configurations; the configuration shown in Figure 3a is astral, while the configuration shown in Figure 3b is 3-astral.



(a) (b) (c) (d)

Figure 1: Four embeddings of the Pappus Configuration, illustrating various kinds of symmetric embeddings. In each figure, symmetry classes of points and of lines are distinguished by color. (a) A chirally symmetric, but not polycyclic, embedding; note that the symmetry class of blue points contains 2 elements, while the symmetry class of black points contains only one. (b) A dihedrally symmetric, but not polycyclic, embedding; the symmetry class of green points contains four elements, but the symmetry class of red points contains only 2. (c) A polycyclic chiral embedding. (d) A polycyclic dihedral embedding; while the black symmetry class of lines (shown with solid and dashed lines) contains six lines under the dihedral symmetry group, under the rotation subgroup, the symmetry class is partitioned into two sets (solid and dashed) that each contain three lines.

In [7, Section 2.7], Grünbaum presented an exact, ruler-and-compass method of constructing chiral astral 3-configurations (possibly beginning with the vertices of a regular convex m-gon, if such an m-gon is not constructible). His construction method relies on a result discussed very briefly in that text, which is reformulated here as the *Configuration Construction Lemma*. In this paper, we extend this construction method to develop a purely ruler-and-compass technique (given a starting m-gon) to produce a number of different classes of chirally and dihedrally symmetric 3-configurations. Although these configurations could be constructed purely algebraically, using linear algebra, say, to determine numerical constraints to force the last three points of a particular construction sequence to lie on a single line, developing a geometric construction process turns out to be quite useful, experimentally, in finding new construction methods to produce new and interesting examples, such as developing infinite classes of more highly incident configurations that have symmetric 3-configurations as building blocks, where the linear-algebraic techniques might fail, e.g., due to the inability to force more than three points at a time to be collinear. For example, the single known (20_4) configuration, which was found by Grünbaum in 2008 [6], may be analyzed as being formed from two copies (shown in red and black in Figure 2) of an astral 3-configuration (shown in Figure 3a) whose construction is discussed in section 3.1; the construction may be further generalized (with quite a lot of work) to produce an infinite family of $((2^{k-2} \cdot m)_k)$ configurations [1].

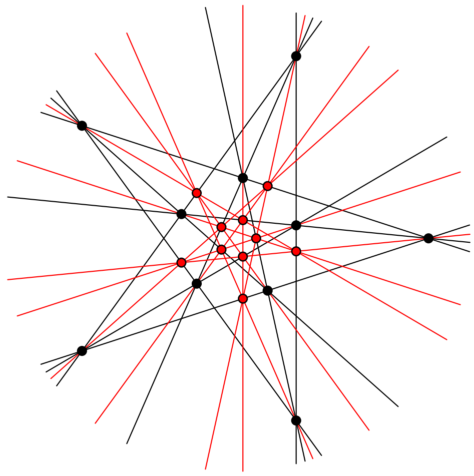
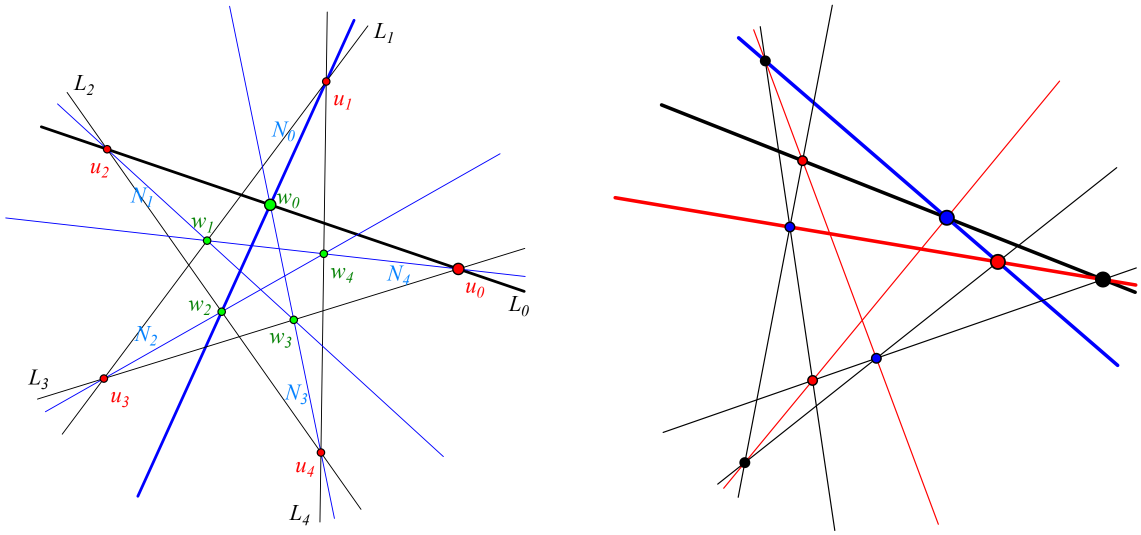


Figure 2: A (20_4) configuration, formed from two nested copies of the symmetric 3-configuration shown in Figure 3a.

2 Double-arcs and reduced Levi graphs

The configurations whose construction is described in this paper typically have the property that there are at least two symmetry classes of points, say u and w, and two symmetry classes of lines, say L and N, with the property that two points labelled u lie on a line labelled L, and two points from w lie on a line labelled N; we say that such a configuration has the *double-arc* property. An example of a small configuration with this property, an astral (10_3) configuration, is shown in Figure 3a; the configuration in Figure 4 also has this property. Note that there are symmetric geometric configurations which do not have this property. For example, the version of the (9_3) Pappus configuration shown in Figure 3b has the property that each point contains one line from each of three symmetry classes of points, and similarly, each line contains one point from each of three symmetry classes of lines.

Given a configuration, it is possible to construct an associated bipartite graph called a Levi graph, which has a black node for each vertex of the configuration and a white node for each line of the configuration, and two nodes are connected with an arc in the Levi graph if and only if the corresponding vertex and line are incident in the configuration. In their article *Polycyclic Configurations* [3], Marko Boben and Tomaž Pisanski presented a way of analyzing the (rotational) symmetry of configurations via a graph-theoretic object called a *voltage graph* (for a general discussion of voltage graphs, see [4]). In their presentation, the voltage graph is a bipartite quotient, with respect to a cyclic group, of the Levi graph of a combinatorial configuration, and they asked whether it is possible to determine a geometric rotational realization of the configuration.



(a) A (10_3) configuation with the "double-arc" property.

(b) A chiral representation of the Pappus configuration, which has no double-arcs

Figure 3: (a) A (10_3) configuration with two symmetry classes of points and lines, with two points from one symmetry class and one point from a second symmetry class on lines labelled L and M, and two lines from each of two symmetry classes of lines passing through points labelled u and w. (b) A (9_3) configuration with three symmetry classes of points and lines, with one point from each of three symmetry classes of points on each line, and one line from each of three symmetry classes of lines through each point. In both figures, symmetry classes are indicated by color.

In this article, we modify their presentation of a voltage graph to emphasize the connection between the Levi graph and the voltage graph, to help develop new infinite families of configurations: that is, we consider construction techniques to produce infinite families of configurations for which the underlying unlabelled reduced Levi graph is the same. We use the labeling conventions from the notion of a *reduced Levi graph* introduced in Grünbaum's monograph [7, Section 1.6] (note that his object is different from the "reduced Levi graph" discussed by Artzy; see [7, p. 150, exercise 6]) and the use of a directed multigraph from the definition of a voltage graph [3] to produce a modified reduced Levi graph (henceforth, simply *reduced Levi graph*) which encapsulates significant structural properties of classes of symmetric configurations.

In what follows, we use the terms *nodes* and *arcs* to refer to graph-theoretic objects, and we reserve the words *points* and *lines* for elements of a configuration.

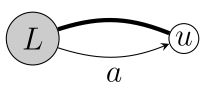
Beginning with a particular polycyclic geometric configuration with $\mathbb{Z}_m$ cyclic symmetry (which may or may not be the full symmetry group of the configuration), the construction and interpretation of the reduced Levi graph is as follows.

Suppose the configuration has h_1 symmetry classes of lines and h_2 symmetry classes of points under the rotational symmetries. We denote the classes of lines as $L_0, L_1, \dots, L_{h_1-1}$ and the classes of points as $v_0, v_1, \dots, v_{h_2-1}$, and we assume each symmetry class contains m elements. Within a symmetry class, say within class v_0, the elements of the symmetry class are labelled cyclically, so that point class v_0 contains points $(v_0)_0, (v_0)_1, \dots, (v_0)_{m-1}$ which form the vertices of a regular convex m-gon. (Note that if the number of symmetry classes is small, the symmetry classes may be labelled so as to avoid the double-indexing; for example, in Example 1 below, the symmetry classes of vertices are labelled as u, v, w and the symmetry classes of lines are labelled L, M, N.)

We define the corresponding reduced Levi graph to be an edge-labelled bipartite multigraph (we are allowing doubled edges), with h_1 large nodes, each corresponding to a symmetry class of lines, and h_2 small nodes, each corresponding to a symmetry class of points.

We define incidence between nodes according to the incidence between the points and lines in the configuration. In particular, we construct a directed arc from node L_k to node v_j with label a precisely when line $(L_k)_0$ passes through vertex $(v_j)_a$, or alternately, when the line labelled $(L_k)_{-a}$ intersects the node $(v_j)_0$. Note that this labeling convention is opposite to the labeling scheme described for voltage graphs in [3]. In many cases, we will need the label of 0, meaning that vertex $(v_j)_0$ lies on line $(L_j)_0$; to lessen the clutter of the diagram we will **suppress the 0 label and draw such lines thick, unlabeled, and without arrows.**

If two lines P and Q intersect, we sometimes will refer to the intersection point as $P \wedge Q$, and if two points v and w are connected by a line, then we may call the line $v \vee w$.

A double-arc 

in a reduced Levi graph connecting line class L and vertex class u with labels 0 and a corresponds to the situation where line L_i is incident with vertex u_i and u_{i+a}; that is, $L_i = u_i \vee u_{i+a}$ (or conversely, $u_i = L_i \wedge L_{i-a}$) and the lines L_i are said to be *of span a*. (More generally, a double-arc is any pair of parallel arcs in a reduced Levi graph, but it is convenient to work with double-arcs in which one label is 0.)

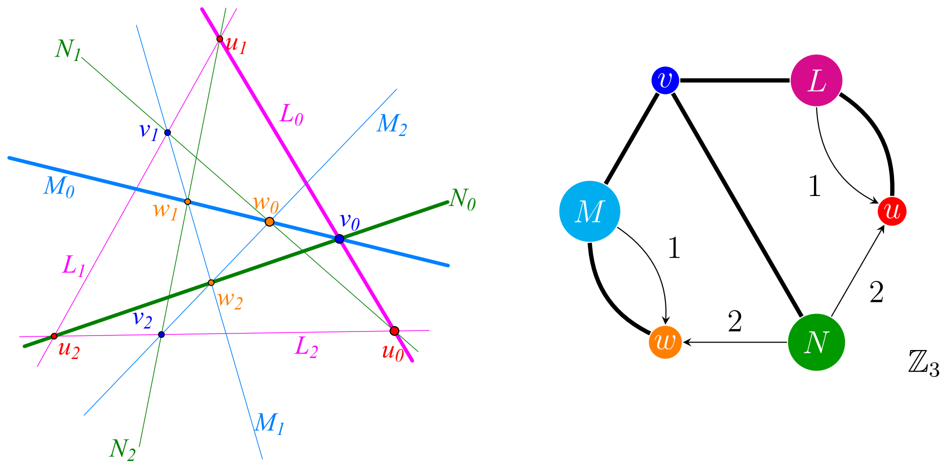
Example 1. *Constructing a reduced Levi graph.*

We illustrate the construction of a reduced Levi graph using the (9_3) configuration shown in Figure 4a. It has 3-fold rotational symmetry, three classes of lines, called L, M and N, and three classes of points u, v, w; in the figures the 0-th element of each symmetry class is shown larger. Incidences between the points and lines are shown in Table 1; in the table, all indices are taken modulo 3.

A useful fact is that given a (labelled) reduced Levi graph, by “rotating labels” about consecutive nodes appropriately, it is possible to zero-out labels along any given spanning tree of the graph. To see this, consider the underlying multigraph for the reduced Levi graph shown in Figure 4b. In Figure 5, we have chosen and highlighted a particularly desirable spanning tree for the graph, namely, a Hamiltonian path that begins and ends at a double-arc.

Table 1: Incidences between points and lines of the (9_3) configuration shown in Figure 2.

Point	incident to			Line	incident to		
u_i	L_i	L_{i+2}	N_{i+1}	L_i	u_i	u_{i+1}	v_i
v_i	L_i	M_i	N_i	M_i	v_i	w_i	w_{i+1}
w_i	M_i	M_{i+2}	N_{i+1}	N_i	u_{i+2}	v_i	w_{i+2}



(a) A (9_3) configuration

(b) The corresponding reduced Levi graph

Figure 4: A 3-configuration and its corresponding reduced Levi graph. Note that the thick lines indicate that the labelling on the arc is 0, and nodes corresponding to lines are shown larger. For example, since point v_i lies on line L_i, the arc between node v and node L is drawn thick, but is unlabelled. On the other hand, since point w_{i+2} lies on line N_i, the arc between nodes w and N is labelled with a 2.

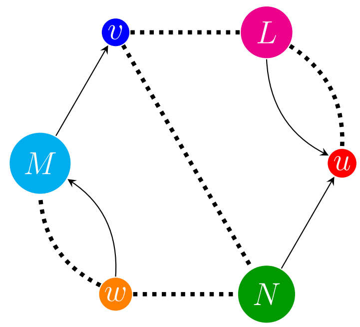


Figure 5: A nice spanning tree, indicated on an unlabeled underlying reduced Levi graph.

How can we change the labelling on the original configuration so that in the reduced Levi graph, we have a spanning tree (in this case, a spanning path) of zero-labelled edges? We already have the path $u - L - v - N$ all labelled with 0, so we need to continue the path by making the arc from N to w be labelled 0. To do this, we add -2 to every arc incident with w, with addition done modulo 3 since the symmetry group of the configuration is $\mathbb{Z}_3$. This is equivalent to subtracting 2 from every index of the vertex labels associated with w in the labelling of the configuration (that is, rotating the labels assigned to the elements of the symmetry class w by two steps backwards). Next, we want to make one of the arcs between w and M have label 0, so we add 2 to every label incident with M (both on the node, and for the labels of the lines in the configuration). We now have achieved the desired 0-labelled spanning path; the sequence of operations is shown in Figure 6.

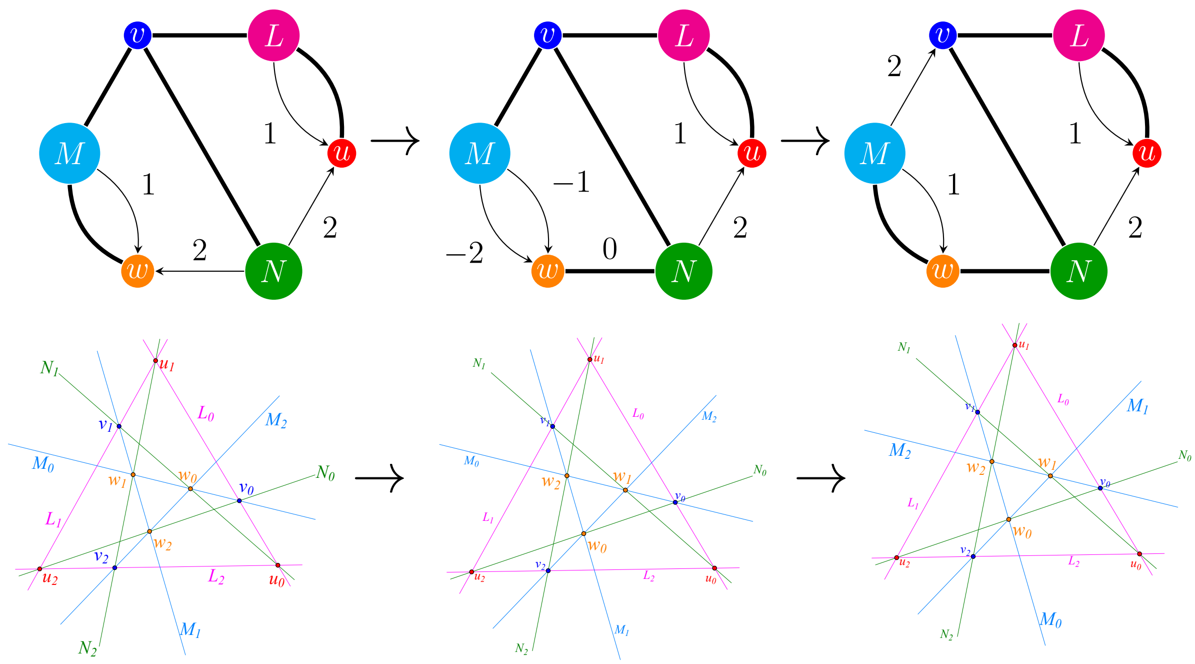


Figure 6: Achieving a desired spanning tree. Initial graph: the original reduced Levi graph. The middle graph is obtained by "rotating about w" by addiing -2 to each label for arcs incident with w in the original graph. The right-hand graph is obtained from the middle graph by "rotating about M" by adding 2 to each label for arcs incident with M.

A useful feature of the reduced Levi graph is that since every arc in the graph corresponds to a point-line incidence in the configuration, the degree of the vertex-nodes and the line-nodes of the reduced Levi graph is equal to the type of configuration the graph corresponds to. For instance, in the example shown in Figure 4, every node in the graph is of degree 3, and the configuration is a 3-configuration. In general, a (q, k)-configuration corresponds to a reduced Levi graph with q-valent line-nodes and k-valent vertex-nodes. A (q, k)-configuration where $q = k$ is said to be *balanced*; if $q \neq k$ the configuration is *unbalanced*. (In the literature (e.g., [2]), balanced combinatorial configurations have often been called symmetric, but that terminology clashes with our desire to use *symmetric* to refer to a geometric property of a configuration. In this language—as in so much else—we follow Grünbaum; see [7, p. 16] for his discussion of the issue.) An unbalanced (3,4)-configuration and its corresponding reduced Levi graph is shown in Figure 7. Note that under the cyclic symmetry group $\mathbb{Z}_7$, it has three symmetry classes of lines and four symmetry classes of points (indicated by color); each of the vertex-nodes is 3-valent, while each of the line-nodes is 4-valent. (On the other hand, if the configuration is viewed as having dihedral symmetry, it has two symmetry classes of points and two symmetry classes of lines.)

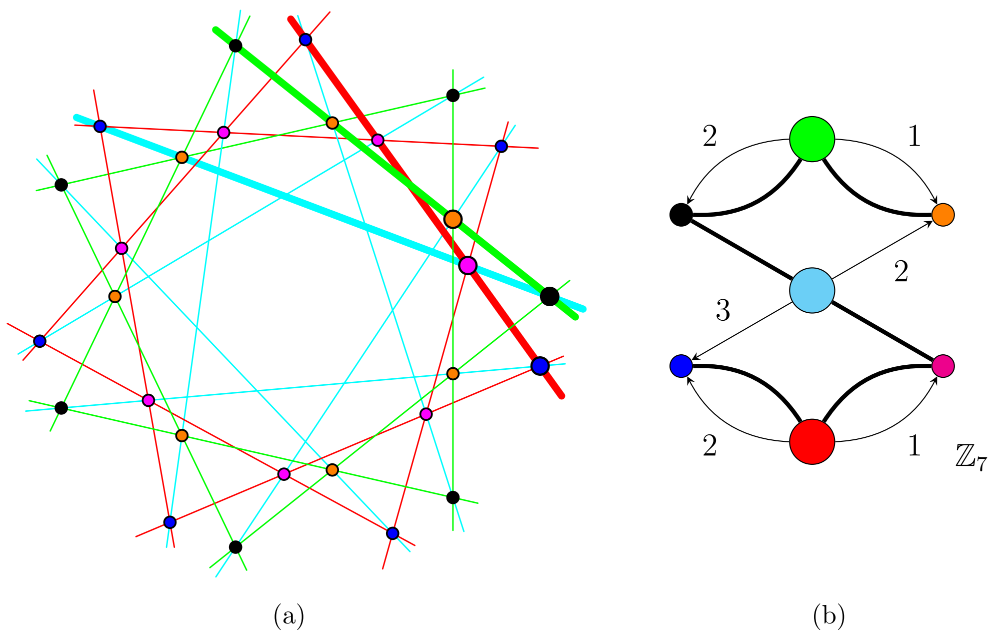


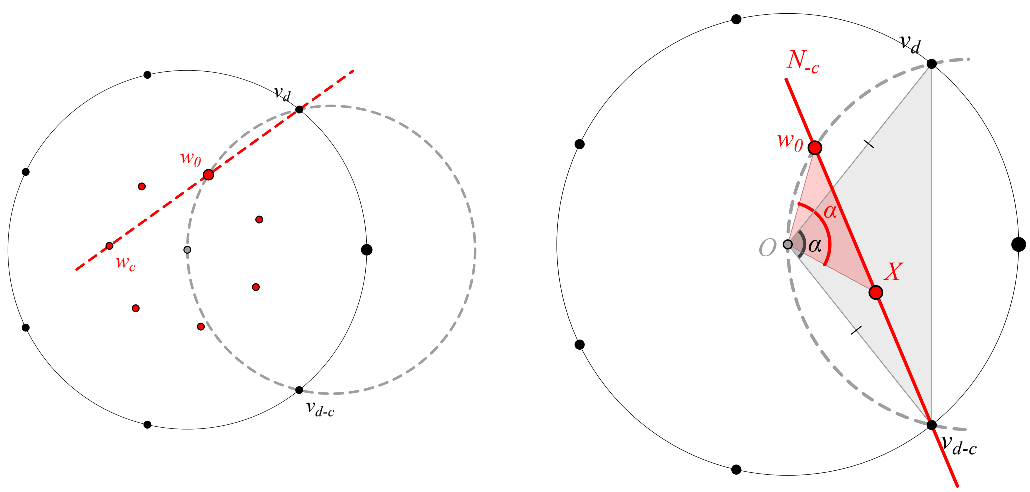
Figure 7: (a) An unbalanced $(3, 4)$-configuration; (b) the corresponding reduced Levi graph. In the reduced Levi graph, the large nodes correspond to the lines and the smaller nodes to the vertices of the configuration; symmetry classes are coded by color. In the configuration, the 0-th element of each symmetry class is shown larger, and subsequent elements are labelled cyclically, counterclockwise.

3 The Configuration Construction Lemma

The configuration construction lemma is very easy to state and prove, but it provides a broadly useful method for a purely geometric construction for a number of symmetric configurations. In particular, it lets us precisely construct geometric 3-configurations whose reduced Levi graphs contain two non-adjacent pairs of double-arcs.

A variant of this lemma was presented by Grünbaum in [7, Section 2.7] in the context of constructing a single, highly restricted class of 3-configurations, the astral 3-configurations with cyclic symmetry.

Lemma 1 (Configuration Construction Lemma)**.** *Let $v_0, v_1, \ldots, v_{m-1}$ form the vertices of a regular m-gon with center $\mathcal{O}$. Let $\mathcal{C}$ be the circle passing through the points $\mathcal{O}, v_d, v_{d-c}$, and let $\angle v_d \mathcal{O} v_{d-c} = \frac{2\pi c}{m}$. Let w_0 be any point on $\mathcal{C}$, and define w_i to be the rotation of w_0 by $\frac{2\pi i}{m}$ about $\mathcal{O}$. Then w_0, w_c, v_d are all collinear.*



(a) Lemma statement (b) proving the lemma

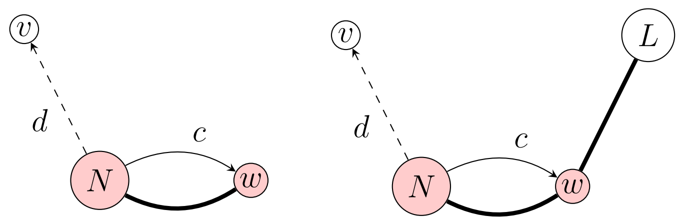
Figure 8: Illustrating the configuration construction lemma.

Proof. Construct a point X on the line $\overline{v_{d-c}w_0}$ so that the oriented angle $\angle w_0 \mathcal{O} X = -\alpha$. (See Figure 8b.) Note that both $\angle \mathcal{O} v_d v_{d-c}$ and $\angle \mathcal{O} w_0 v_{d-c}$ are subtended by chord $\mathcal{O} v_{d-c}$ of circle $\mathcal{C}$, so they are equal. Moreover, since X lies on $\overline{v_{d-c}w_0}$, $\angle \mathcal{O} w_0 X = \angle \mathcal{O} v_d v_{d-c}$. Therefore, triangles $\triangle w_0 \mathcal{O} X$ and $\triangle v_d \mathcal{O} v_{d-c}$, which have two pairs of equal angles, are similar, so because $\triangle v_d \mathcal{O} v_{d-c}$ is isosceles by construction, it follows that $\triangle w_0 \mathcal{O} X$ is also isosceles. That is, X and w_0 lie on the same circle centered at $\mathcal{O}$. Because $\angle w_0 \mathcal{O} X = -\frac{2\pi c}{m}$, it follows that $X = w_{-c}$, by the definition of the w_i.

If we define N_{-c} to be the line passing through w_0 and v_{d-c}, we have just shown that the point $X = w_{-c}$ also lies on N_{-c}. Defining N_{i-c} to be the rotation of N_{-c} through $\frac{2\pi i}{m}$, it follows that N_{i-c} contains the points w_{0+i} and v_{d-c+i}, and w_{-c+i}. That is, the line $N_0 = N_{c-c}$ contains the points w_c, v_d, and w_0, as was to be shown. $\square$

If we consider the effect of the configuration construction lemma on corresponding nodes and arcs of a partial reduced Levi graph under the action of the group $\mathbb{Z}_m$, then we arrive at the following "gadget" in the reduced Levi graph, shown in the left-hand side of

Figure 9, where the dashing between N and v indicates that the collinearity is forced. If in addition w_0 is the intersection of $\mathcal{C}$ with some other line L_0 which is an element of a symmetry class L — which is the case in all of the subsequent constructions — then we have in the reduced Levi graph the gadget shown in the right-hand side of Figure 9.



(a) w_0 is on $\mathcal{C}$... (b) ...and also on L_0

Figure 9: "Gadgets" in reduced Levi graphs found as a consequence of using the construction in the configuration construction lemma. (a) The point w_0 lies on the circle $\mathcal{C}$ passing through $\mathcal{O}, v_d, v_{d-c}$, and N_0 is a line of span c through w_0; (b) In addition, w_0 also lies on some line L_0.

3.1 Chiral astral 3-configurations

We first apply the Configuration Construction Lemma to construct a chiral astral 3-configuration, which has two symmetry classes of points and lines, and whose reduced Levi graph is shown in Figure 10; since there are three discrete parameters in the reduced Levi graph, this type of configuration has been represented by the symbol $m\#(a, b; d)$ (for example, in Section 2.7 of [7] as $m\#(b, c; d)$). The smallest example of such a configuration is shown in Figure 3a, with symmetry group $\mathbb{Z}_5$. The method presented here for constructing chiral astral 3-configurations is a variant of the method described in [7, Section 2.7] but is significantly more straightforward.

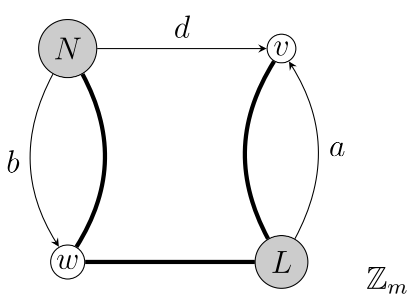


Figure 10: The reduced Levi graph for a cyclic astral 3-configuration

The existence of two pairs of double-arcs in the reduced Levi graph shown in Figure 10 mean that we need a class of lines L of span a with respect to some points v and a class of lines N of span b with respect to some points w. The 0-labelled line between L

and w indicates that point w_0 must lie somewhere on line L_0. Finally, the $N \xrightarrow{d} v$ arc indicates that line N_0 must also contain v_d (as well as w_0 and w_b, since N_0 is span b with respect to the w_i). A simple application of the configuration construction lemma allows us to construct such a configuration. First, we construct the points v_i as the vertices of a regular convex m-gon, and next we construct the lines L_i as the lines of span a with respect to the v_i. We then define w_0 to be an intersection of line L_0 with the circle $\mathcal{C}$ passing through $\mathcal{O}$, v_d and v_{b-d}; the notation d' in the symbol $m\#(a,b;d')$ indicates we should choose the leftmost intersection, while $m\#(a,b;d)$ indicates we should choose the rightmost intersection. Finally, the configuration construction lemma lets us conclude that the points w_0, w_b, v_d are collinear, so we label the line through those three points as N_0, and define N_i to pass through w_i, w_{i+b}, v_{i+d}. Note that with the exception, possibly, of constructing the original vertices of a regular m-gon, the entire construction is achieved using only a straightedge and compass.

As an example, consider the construction of $7\#(3,2;1')$, shown in Figure 11.

Example 2. *Constructing an astral 3-configuration* $7\#(3,2;1')$

1. Construct the points $v_i = \left(\cos\left(\frac{2\pi i}{7}\right), \sin\left(\frac{2\pi i}{7}\right)\right)$.
2. Construct lines $L_i = v_i \wedge v_{i+3}$.
3. Construct the circle $\mathcal{C}$ passing through $\mathcal{O}$, v_1 and $v_{1-2} = v_{-1}$.
4. Let w_0 be an intersection of line L_0 with $\mathcal{C}$. In this case, we choose the leftmost intersection, because of the $1'$ (instead of 1) in the symbol description.
5. Construct the lines M_i connecting w_i and w_{i+2} and note that by the configuration construction lemma, M_i also passes through v_{i+1}.

In general, the circumcircle $\mathcal{C}$ intersects line L_0 in two places. In particular, suppose the line L_0 is parametrized as $(1-t)v_0 + t(v_a)$ and $\mathcal{C}$ intersects L_0 at two points with parameter values $t = \gamma$ and $t = \gamma'$, where $\gamma \leqslant \gamma'$. By writing $m\#(a,b;d)$ we mean that we choose the intersection point with $t = \gamma$, that is, the rightmost intersection, and $m\#(a,b;d')$ means that we choose the intersection with $t = \gamma'$, or the leftmost intersection. Figure 12 shows the two configurations $9\#(4,1;3)$ and $9\#(4,1;3')$.

4 Construction of infinite classes of symmetric 3-configurations

Other infinite classes of symmetric 3-configurations may be constructed geometrically using the configuration construction lemma.

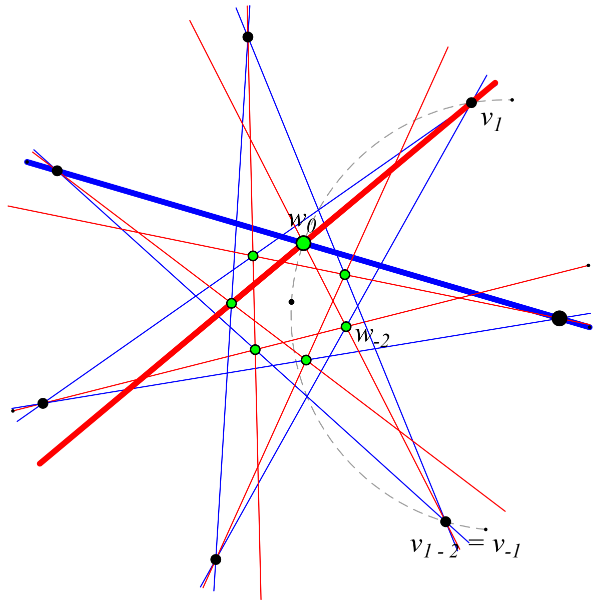
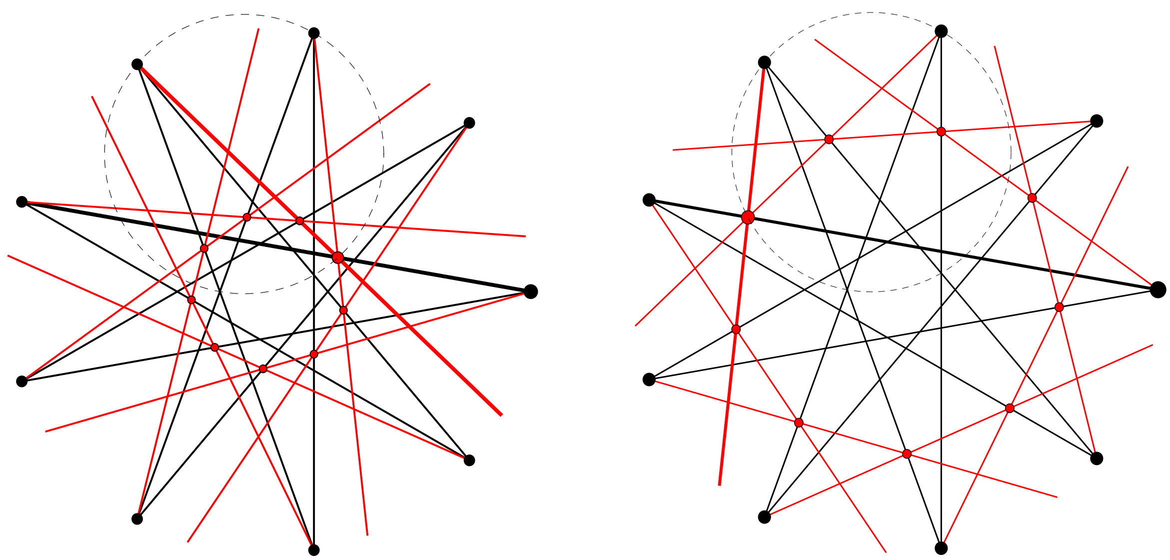


Figure 11: Constructing an astral configuration $7\#(3, 2; 1')$. The 0th element of each symmetry class is shown larger, and elements are cyclically labelled counterclockwise; the leftmost intersection of $\mathcal{C}$ with L_0 is chosen to define w_0.



(a) The configuration $9\#(4, 1; 3)$; w_0 uses the rightmost intersection of $\mathcal{C}$ with L_0.

(b) The configuration $9\#(4, 1; 3')$: w_0 uses the leftmost intersection.

Figure 12: Two chiral astral 3-configurations with $m = 9$, $a = 4$, $b = 3$, $t = 2$. In both configurations, the points v_i and lines L_i are black and the points w_i and lines N_i are red.

4.1 Multilateral 3-configurations

These configurations have a reduced Levi graph which alternates double-arcs and single arcs, shown in Figure 13; the configuration itself consists of mutually inscribed and circumscribed (possibly star) m-gons, where the inscribed m-gons need not be the same type. For example, Figure 14 shows two multilateral 4-astral 3-configurations with $m = 5$ and $h = 4$, along with the associated reduced Levi graphs; one uses only convex 5-gons, and the other uses a mixture of pentagrams and convex pentagons.

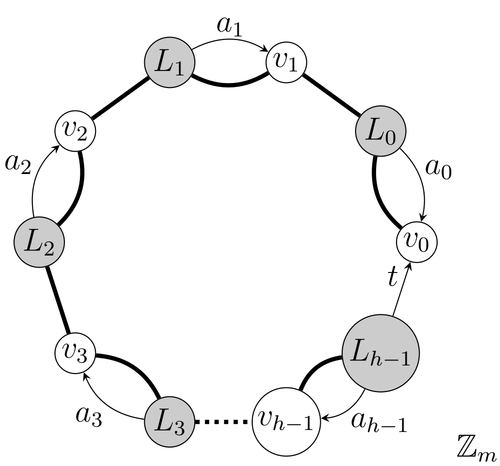


Figure 13: The reduced Levi graph for a multilateral h-astral 3-configuration

Multilateral 3-configurations were investigated carefully from an algebraic viewpoint (although not using that name) in Boben and Pisanski's article *Polycyclic Configurations* [3], where they were referred to as $\mathcal{C}_3(k,(p_0,p_1,\ldots,p_n),t)$, where n is the number of symmetry classes of points and lines and the symmetry group of the configuration is $\mathbb{Z}_k$. They are also discussed in [7, Section 5.2].

The construction for h-astral multilateral 3-configurations with symmetry group $\mathbb{Z}_m$ is as follows.

1. Begin with an arbitrary set of vertices v_0 forming the vertices of a regular m-gon. Typically,
$$(v_0)_i = \left(\cos\left(\frac{2\pi i}{m}\right), \sin\left(\frac{2\pi i}{m}\right)\right).$$

2. The first double-arc indicates that the first set of lines $(L_0)_i$ should be of span a_0 with respect to the points v_0. That is, construct $(L_0)_i = (v_0)_i \wedge (v_0)_{i+a_0}$.

3. The next, zero-labelled single arc means that we can place $(v_1)_i$ arbitrarily on line $(L_0)_i$. That is, we choose a value for a continuous parameter λ_1 (with $\lambda_1 \neq 0, 1$ to prevent degeneracy) so that
$$(v_1)_i = (1-\lambda_1)(v_0)_i + \lambda_1(v_0)_{i+a_0}.$$

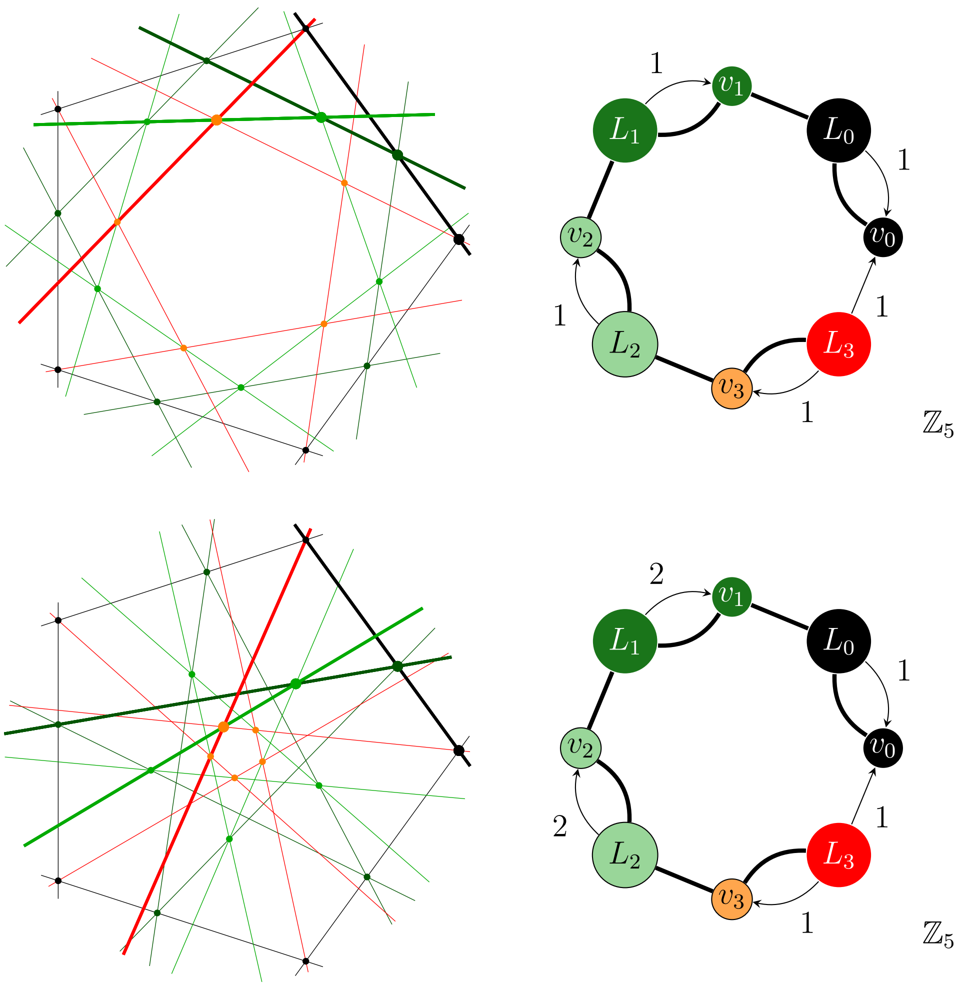


Figure 14: Two multilateral 3-configurations, with $m = 5$. In both configurations, the first vertex class v_0 is colored black, and the last vertex class v_3 is colored orange; the first line class L_0 is black and the last line class L_3 is red. Top: $\lambda_0 = \lambda_1 = \lambda_2 = .4$; Bottom: $\lambda_0 = .4, \lambda_1 = \lambda_2 = .3$.

4. This process continues iteratively: for $1 \leqslant j \leqslant h-2$,

$$(v_j)_i = (1-\lambda_j)(v_{j-1})_i + \lambda_j (v_{j-1})_{i+a_{j-1}}$$

and

$$(L_j)_i = (v_j)_i \wedge (v_j)_{i+a_j}$$

5. Finally, to construct the h-th orbit of points and lines (that is, the lines L_{h-1} and the points v_{h-1}), note that in the reduced Levi graph we have a double-arc between L_{h-1} and v_{h-1}, labelled a_{h-1} and 0, and a single arc labelled t between L_{h-1} and v_0. This labelling indicates that the lines L_{h-1} should be of span a_{h-1} with respect to the vertices v_{h-1}, and moreover, line $(L_{h-1})_0$ must pass through point $(v_0)_t$. We use

the configuration construction lemma and the lines L_{h-1} to construct the required points v_{h-1}: point $(v_{h-1})_0$ is an intersection of the line $(L_{h-1})_0$ with the circumcircle $\mathcal{C}$ passing through $\mathcal{O}$, $(v_0)_t$ and $(v_0)_{t-a_{h-1}}$. The rest of the points $(v_{h-1})_i$ are the rotated images, and $(L_{h-1})_i := (v_{h-1})_i \wedge (v_{h-1})_{i+a_{h-1}}$.

Note that for a given multilateral h-astral 3-configuration, if $h > 2$ then there are $h-2$ continuous parameters $\lambda_1, \ldots, \lambda_{h-2}$: non-astral multilateral configurations are movable! Also, if $h = 2$ then the construction produces the chiral astral 3-configurations already described.

4.2 Caterpillar 3-configurations

These have the reduced Levi graphs of the form shown in Figure 15. Note that there is a "wiggle" of zero-labels along the center of the configuration.

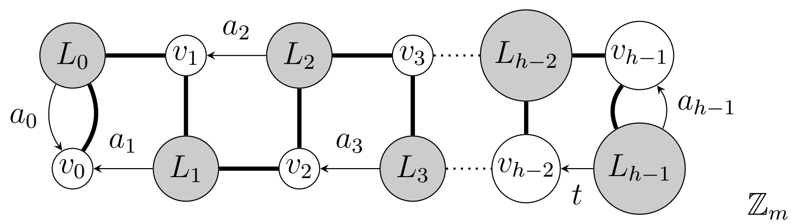


Figure 15: A reduced Levi graph for a caterpillar 3-configuration

To construct a caterpillar configuration:

1. Construct vertices $(v_0)_i$ as the vertices of a regular convex m-gon. Construct lines $(L_0)_i$ as lines of span a_0; that is, construct $(L_0)_i := (v_0)_i \vee (v_0)_{i+a_0}$.

2. For each $i = 1, 2, \ldots, h-2$, place vertex $(v_j)_0$ arbitrarily on line $(L_{j-1})_0$, and construct the rest of the vertices $(v_j)_i$ as the image under rotation by $\frac{2\pi i}{m}$ of vertex $(v_j)_0$. Then construct lines $(L_j)_i$ as $(L_j)_i := (v_j)_i \vee (v_{j-1})_{i+a_j}$. Note that these lines $(L_j)_i$ for $j > 1$ connect vertices from two consecutive symmetry classes! More precisely, there are continuous parameters $\lambda_1, \ldots, \lambda_{h-2}$ so that $(v_1)_0 = (1-\lambda_1)(v_0)_0 + \lambda_1(v_0)_{a_0}$, while if $j = 2, \ldots, h-2$, $(v_j)_i = (1-\lambda_j)(v_{j-1})_i + \lambda_j(v_{j-2})_{i+a_j}$.

3. To complete the configuration, we need to construct a class of vertices $(v_{h-1})_i$ and a class of lines $(L_{h-1})_i$ so that $(v_{h-1})_i$ lies on $(L_{h-2})_i$, $(L_{h-1})_i = (v_{h-2})_i \vee (v_{h-2})_{i+a_{h-1}}$, and $(L_{h-1})_i$ passes through vertex $(v_{h-2})_t$. To do so, we apply the configuration construction lemma. Construct the circumcircle passing through $(v_{h-2})_t$, $(v_{h-2})_{t-a_{h-1}}$, and the center of the configuration. Define $(v_{h-1})_0$ to be one of the two intersections of this circle with line $(L_{h-2})_0$. The configuration construction lemma says that if we now construct line $(L_{h-1})_0 := (v_{h-1})_0 \vee (v_{h-1})_{a_{h-1}}$, then this line passes through $(v_h)_t$; by symmetry, if line $(L_h)_i$ is the image under rotation by $\frac{2\pi i}{m}$, then each line $(L_h)_i$ passes through $(v_h)_i$, $(v_h)_{i+a_{h-1}}$, and $(v_{h-1})_t$, as desired.

An interesting example of a caterpillar configuration is shown in Figure 16; it is a geometric realization of a famous (15_3) configuration known as the Cremona-Richmond configuration (compare with the drawing in [7, p. 2]).

Note that an h-astral caterpillar configuration has h discrete parameters, corresponding to the labels $a_0, a_1, \ldots, a_{h-1}, t$ in the reduced Levi graph, and $h-2$ *continuous* parameters, corresponding to the arbitrary placement of vertex $(v_j)_0$ on line $(L_{j-1})_0$ for $j = 1, 2, \ldots, h-2$ in the construction of the configuration. If $h = 2$, the caterpillar construction produces again a chiral astral 3-configuration, which has no continuous parameters.

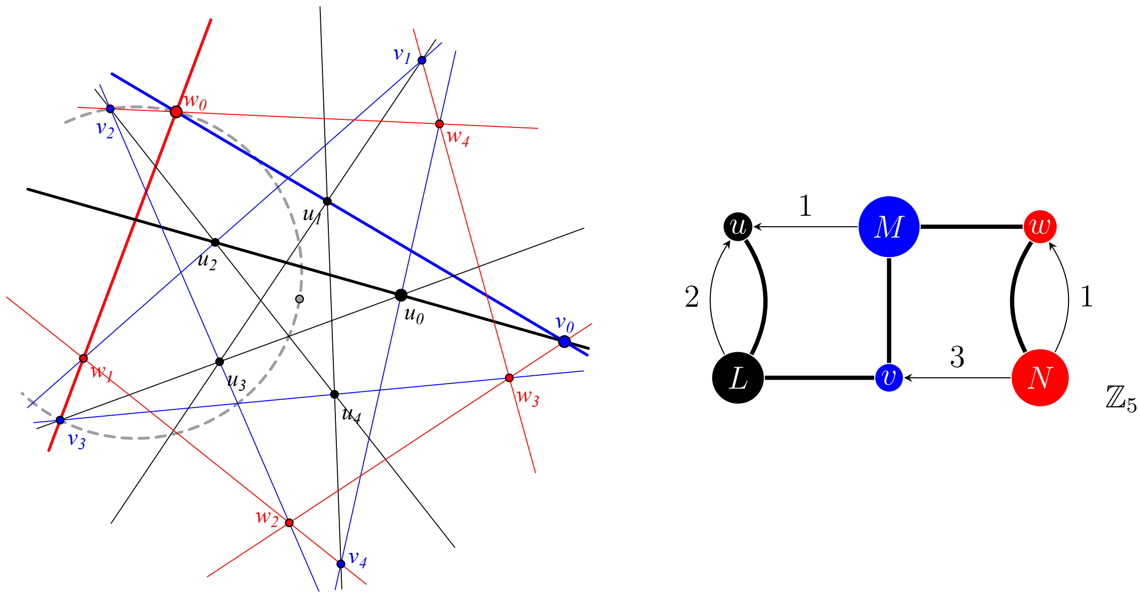


Figure 16: The Cremona-Richmond configuration analyzed as a caterpillar configuration, along with the corresponding reduced Levi graph.

5 Other symmetric 3-configurations

There is a single underlying reduced Levi graph that corresponds to chiral 3-configurations with two orbits of points and lines, which we discussed in Section 3, shown in Figure 10. In *Polycyclic Configurations* [3, Figures 8 and 9], Boben and Pisanski present all possible (unlabelled) 3-valent multigraphs which might be the underlying graph for a reduced Levi graph of 3-configurations with 3 and 4 orbits of points and lines; their figures are reproduced in Figures 17 and 18. Of these, seven have at least one double-arc, and using variants of the techniques discussed previously, it is possible to develop geometric constructions to produce configurations with these as their underlying reduced Levi graphs.

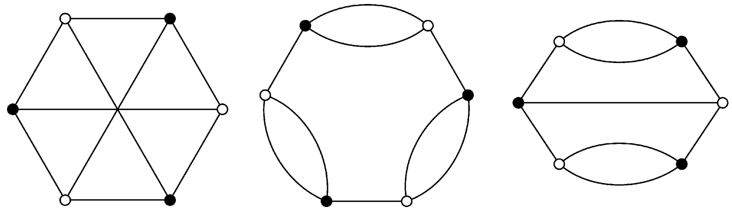


Figure 17: The possible reduced Levi graphs corresponding to 3-configurations with three orbits of points and of lines (image taken from [3]). Note that the middle graph corresponds to 3-astral multilateral configurations and the rightmost graph corresponds to 3-astral caterpillar configurations.

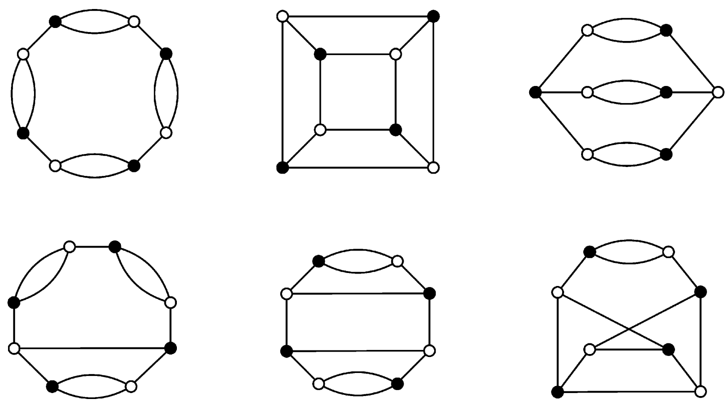
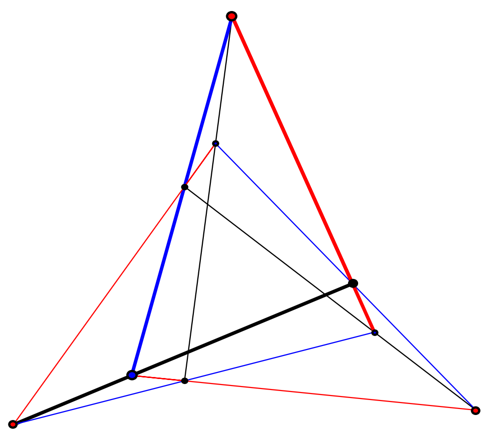


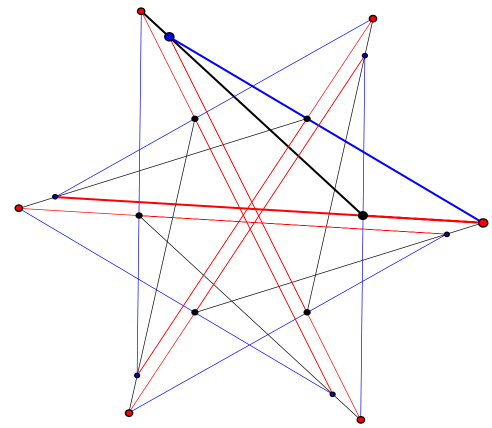
Figure 18: The possible reduced Levi graphs corresponding to 3-configurations with four orbits of points and of lines (image taken from [3]), i.e., 4-astral 3-configurations. The graph in the (1,1)-entry corresponds to a multilateral configuration and the (2,2)-entry to a caterpillar configuration. Configurations corresponding to the graphs in the $(1,3)$-, $(2,1)$-, and $(2,3)$-entries may be constructed using variants of the construction techniques described in this article, with examples shown in Figures 20a,b and Figure 21 respectively. The construction methods used to produce these configurations are discussed in Examples 3 and 4.

The single reduced Levi graph with 3 orbits which has no double-arcs corresponds to "pappus-like" configurations (Figure 19b); the Pappus configuration has a realization with 3-fold rotational symmetry with such a graph as its underlying reduced Levi graph, shown in Figure 19a. Similar h-astral configurations for odd h (which Boben and Pisanski mentioned briefly [3] as configurations which had realizations over odd Möbius graphs) are straightforward to construct using algebraic techniques but no purely geometric construction technique is yet known. It is also possible using algebraic techniques to construct h-astral 3-configurations for even h in which the corresponding reduced Levi graphs have no double-arcs and are even prism graphs instead of odd Möbius graphs (since the reduced Levi graphs must be bipartite). Figure 19d shows a configuration whose reduced

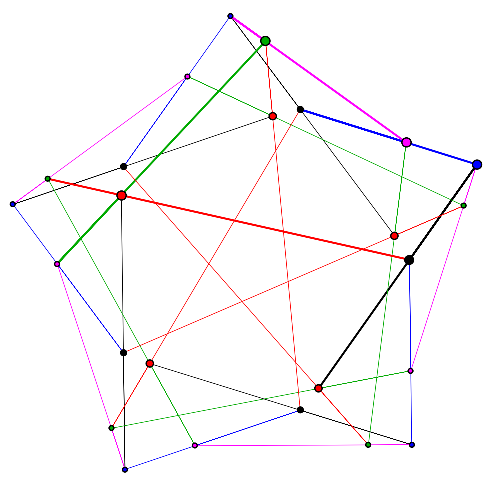
Levi graph is the cubical (4-prism) graph, i.e., the (1,2)-entry in Figure 18, but again, no purely geometric construction is known.



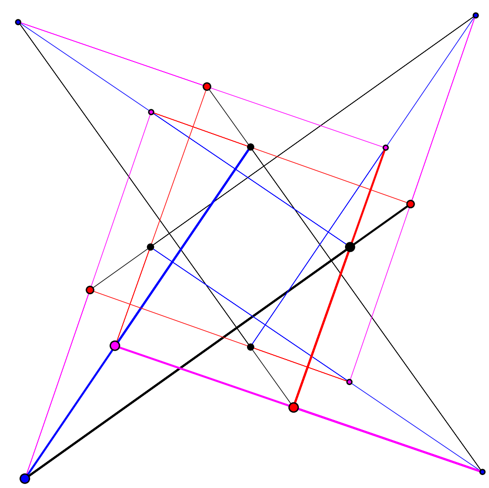
(a) A chiral realization of the Pappus configuration



(b) A "pappus-like" configuration, with $m = 6$



(c) A configuration whose unlabelled reduced Levi graph is the 5-Möbius graph, over $\mathbb{Z}_5$

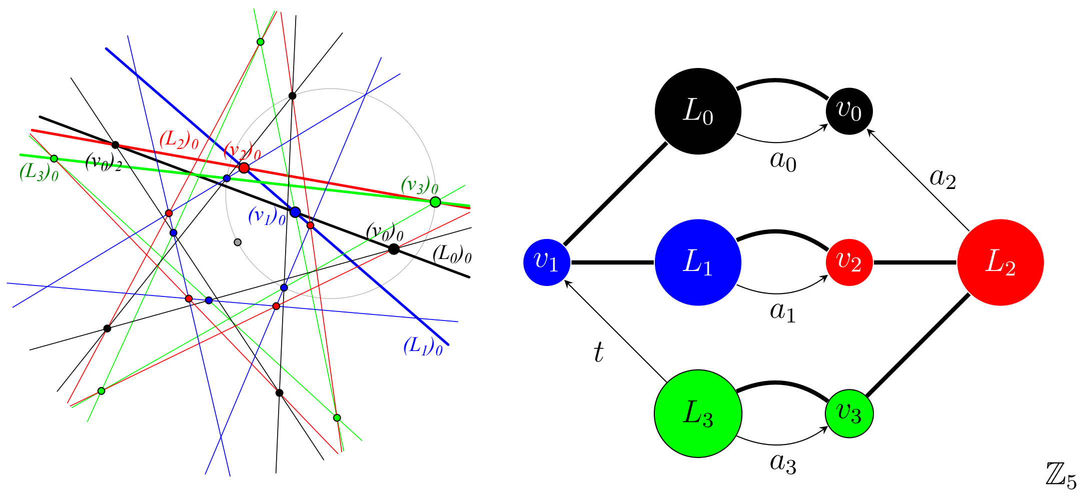


(d) A configuration whose unlabelled reduced Levi graph is the cubical graph, over $\mathbb{Z}_4$

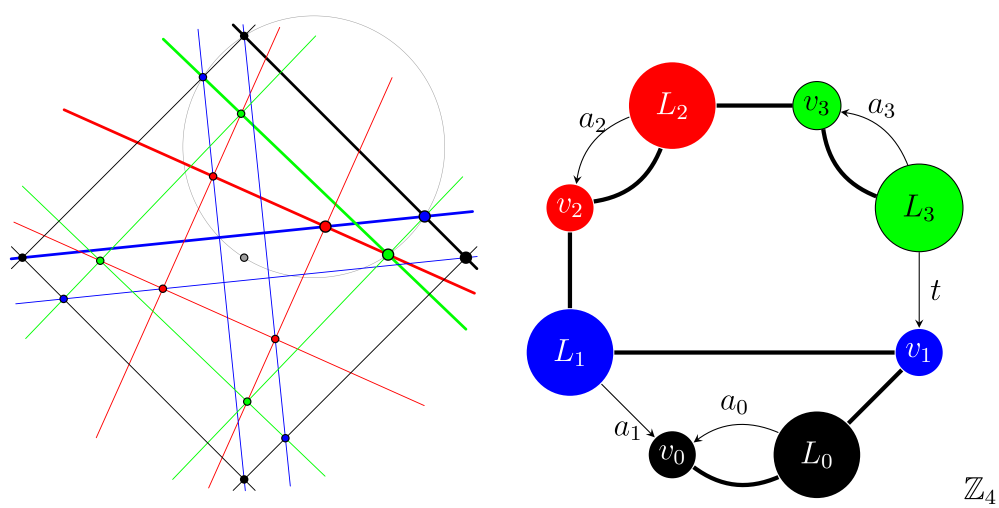
Figure 19: Various small 3-configurations whose associated reduced Levi graphs contain no double-arcs.

Example 3. *Constructing unusual 4-astral 3-configurations with two double-arcs in the reduced Levi graph.*

To construct configurations whose underlying reduced Levi graph has two double-arcs, first identify a spanning path in the reduced Levi graph that begins at one double-arc and ends at the other. Assign labels to edges that don't participate in the spanning path, and use the path to determine the construction order. (Of course, in general finding Hamiltonian paths in graphs is a hard problem. However, for small graphs, it is straightforward.) Figure 20 shows two examples, whose construction is described below.



(a) $m = 5$ and $(a_0, a_1, a_2, a_3, t) = (2, 1, 1, 2, 1)$. Important vertices and lines are labelled.



(b) Here, $(a_0, a_1, a_2, a_3, t) = (1, 2, 1, 1, 1)$ and $m = 4$. The 0th element of each symmetry class is shown larger; colors between the configuration and the reduced Levi graph correspond.

Figure 20: Two unusual 4-astral 3-configurations, and the corresponding general reduced Levi graphs.

As an example, consider the spanning path shown in the reduced Levi graph in Figure 20a, with $m = 5$ and $(a_0, a_1, a_2, a_3, t) = (2, 1, 1, 2, 1)$. To construct the corresponding configuration:

1. Construct the vertices $(v_0)_i$ as the vertices of a regular convex 5-gon.

2. Following the thick path, we encounter a double-arc, so we construct lines $(L_0)_i$ of span $a_0 = 2$ with respect to the $(v_0)_i$.

3. We encounter (v_1) in the reduced Levi graph, so we place $(v_1)_0$ arbitrarily on line $(L_0)_0$, and construct the rest of the $(v_1)_i$ by cyclically rotating $(v_1)_0$ (we "spin the points around").

4. We construct line $(L_1)_0$ **arbitrarily** (that is, at an arbitrary angle) through point $(v_1)_0$, and construct the rest of the symmetry class by spinning the lines around.

5. Next, we encounter a double-arc, so we need to construct point $(v_2)_0$ to lie on the intersection of lines $(L_1)_0$ and $(L_1)_{a_1} = (L_1)_1$: that is, $(v_2)_0 = (L_1)_0 \wedge (L_1)_{a_1}$. Construct the rest of the $(v_2)_i$ by spinning.

6. Since L_2 in the reduced Levi graph is adjacent to both v_0 (with a label of a_2) and v_2 (with a label of a_1), we construct $(L_2)_0 = (v_2)_{a_1} \wedge (v_0)_{a_2}$.

7. Finally, we are close to our final double-arc. We need to construct the $(v_3)_i$ and the $(L_3)_i$ so that $(v_3)_0$ lies on $(L_2)_0$ and also so that a line through $(v_3)_0$ and $(v_3)_{a_3}$ passes through $(v_1)_t$. To do so, we apply the configuration construction lemma and define $(v_3)_0$ to be an intersection of the circumcircle through $(v_1)_t$, $(v_1)_{t-a_3}$ and $\mathcal{O}$ with line $(L_2)_0$. In the example in Figure 20a we constructed a circumcircle passing through $(v_1)_1$, $\mathcal{O}$, and $(v_1)_{1-2} = (v_1)_{-1}$ and chose the farther intersection with $(L_2)_0$ as the point $(v_3)_0$. Construct the rest of the $(v_3)_i$ as usual, and construct lines $(L_3)_i$ to be of span a_3 with respect to these $(v_3)_i$.

The construction technique for the second type of configuration (shown in Figure 20b) is slightly simpler. Construct the $(v_0)_i$ as the vertices of a regular convex m-gon. Following the thick path in the reduced Levi graph, construct the lines $(L_0)_i$ as lines of span a_0. Construct the point $(v_1)_0$ arbitrarily on line $(L_0)_0$ and construct the rest of the $(v_1)_i$ as rotated images. Construct $(L_1)_0 = (v_1)_0 \vee (v_0)_{a_1}$ and the rest of the $(L_1)_i$ as rotated images. Construct $(v_2)_0$ arbitrarily on $(L_1)_0$, and the $(L_2)_i$ of span a_2. Finally, we need the lines $(L_3)_i$ to be span a_3 with respect to the $(v_3)_i$ and pass through $(v_1)_t$, so we construct the circumcircle through $(v_1)_t$, $(v_1)_{t-a_3}$ and the center, and choose an intersection with $(L_2)_0$ to construct $(v_3)_0$; the configuration construction lemma says if we construct the $(v_3)_i$ in this way, then the lines $(L_3)_i$ constructed to be span a_3 with respect to the $(v_3)_i$ pass through $(v_1)_{i+t}$.

Example 4. *Constructing a 3-configuration whose reduced Levi graph has only a single double-arc.*

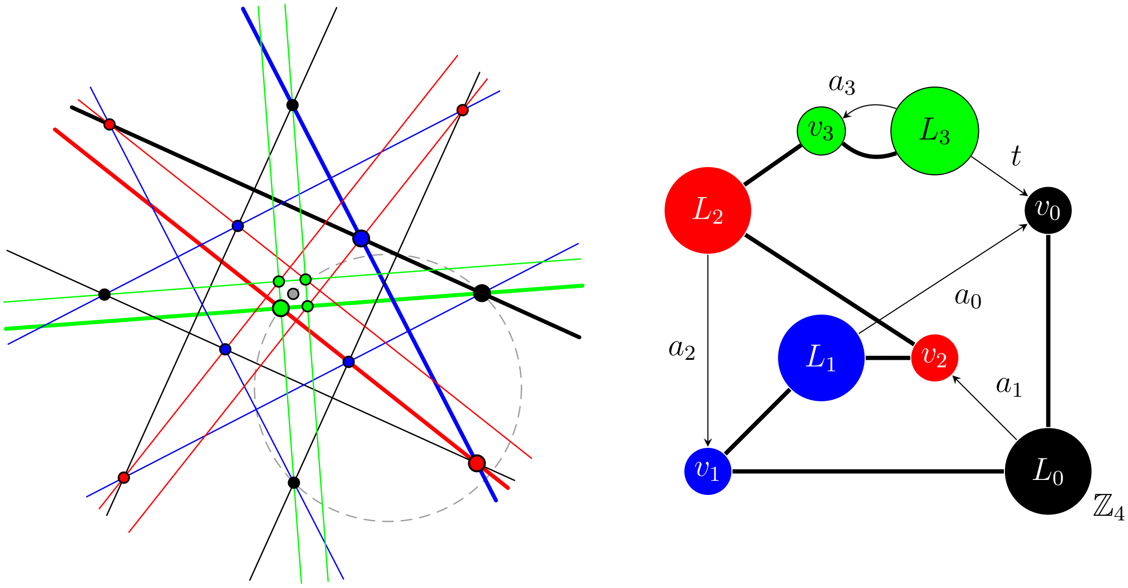


Figure 21: A 4-astral 3-configuration with a single double-arc in the reduced Levi graph. Here, $m = 4$ and $(a_0, a_1, a_2, a_3, t) = (1, 2, 3, 1, 0)$. The 0-th element of each symmetry class is shown larger, and colors indicate correspondence between the configuration and the reduced Levi graph.

While it is easier to construct configurations with a pair of non-adjacent double-arcs, in fact, it is possible to construct configurations with only a single double-arc in the reduced Levi graph, by choosing a 0-labelled path in the reduced Levi graph that *ends* at the double-arc. An example of the construction technique for the configuration shown in Figure 21 follows, with $m = 4$ and $(a_0, a_1, a_2, a_3, t) = (1, 2, 3, 1, 0)$. This is the final configuration whose underlying reduced Levi graph is listed in Figure 18 and contains a double-arc.

1. Begin by constructing the vertices $(v_0)_i$ as the vertices of a regular convex m-gon. Construct line $(L_0)_0$ **arbitrarily** through vertex $(v_0)_0$ and construct the rest of the $(L_0)_i$.

2. Construct $(v_1)_0$ arbitrarily on line $(L_0)_0$, and construct the rest of the $(v_1)_i$.

3. Next, construct $(L_1)_i = (v_1)_i \vee (v_0)_{i+a_0}$.

4. Then, construct the points $(v_2)_i$ as the intersection $(L_1)_i \wedge (L_0)_{i+a_1}$.

5. Construct $(L_2)_i = (v_2)_i \vee (v_1)_{i+a_2}$.

6. Finally, we use the circumcircle construction lemma to construct $(v_3)_i$ and $(L_3)_i$ so that $(L_3)_i$ are lines of span a_3 with respect to the points (v_3) that pass through the points $(v_0)_{i+t}$, by constructing the circumcircle through the center of the m-gon,

$(v_0)_t$, and $(v_0)_{t-a_3}$ and taking the intersection of the circumcircle and line $(L_2)_0$ to be point $(v_3)_0$.

Note each of the classes of configurations described in Examples 3 and 4, as well as the 4-astral multilateral and caterpillar 3-configurations, have two continuous parameters, corresponding to arbitrary placement of certain points and/or lines.

6 Dihedrally symmetric 3-configurations

Any of the reduced Levi graphs used to construct configurations with chiral symmetry discussed in the previous sections may also be used to construct configurations with dihedral symmetry, by a simple modification of the construction technique; we say that we "dihedralize" the original configuration. Suppose you have a chirally symmetric configuration whose reduced Levi graph contains a double-arc at the end of a 0-labelled path whose nodes, say w and N, are attached by single arcs to two other nodes in the reduced Levi graph, say L and v respectively. Delete w and N and the attached arcs from the reduced Levi graph, and duplicate the remaining subgraph, attaching superscripts of $+$ to all the node labels in one copy and $-$ to all the node labels in the other copy (see Figure 22).

We reflect this labeling in the reduced Levi graph within the original configuration by first constructing a subconfiguration consisting of the vertices and lines corresponding to the $+$ superscripted sub-reduced-Levi graph, and consider the labels of the vertices and lines to be superscripted with $+$ in the obvious way. Next, choose an arbitrary mirror $\mathcal{M}$ passing through the center of the subconfiguration. Finally, reflect the vertices and lines over $\mathcal{M}$, and superscript the reflected images with $-$ (see Figures 24a and 26a for examples).

Now construct new nodes w^+ and N^+ and new arcs in the reduced Levi graph, by using the configuration construction lemma in the configuration, by defining w_0^+ to be the intersection of the circumcircle through v_d^+, $\mathcal{O}$ and v_{d-c}^+ with line L_0^-, and then define N_0^+ to be the line through w_0^+, w_c^+, and v_d^+; construct the other vertices w_i^+ and the lines N_i^+ in the configuration in the appropriate way. Finally, reflect all the lines N^+ and points w^+ over $\mathcal{M}$ to construct N^- and w^-; this mirroring is indicated by green dashing in Figure 22. By construction, line N_0^- contains points w_0^-, w_c^- and v_d^-, and point w_0^- lies on lines L_0^+, N_0^-, and N_c^-.

As indicated by the reduced Levi graphs, if the chirally symmetric 3-configurations have h symmetry classes of points and lines, then the dihedralized versions have $2h$ symmetry classes of points and lines under the symmetry group $\mathbb{Z}_m$. However, by construction, the elements w^+ and w^- lie in the same symmetry class under the symmetry group D_m, since by construction w_0^+ reflects to w_0^-, etc. As explicit examples, we dihedralize a chiral astral 3-configuration, to produce a dihedrally symmetric astral 3-configuration, which gives a new geometric construction for the dihedral astral DD configurations discussed in [7, Section 2.8] that does not rely on solving some quadratic equation. We also dihedralize the Cremona-Richmond configuration shown in Figure 16.

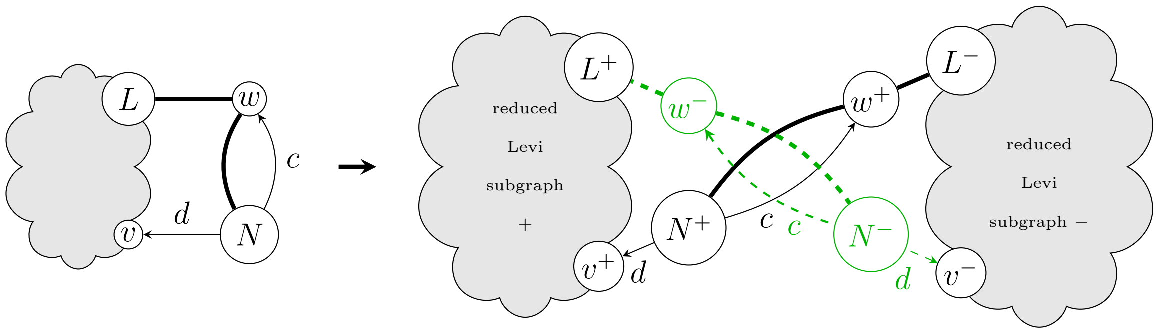


Figure 22: Beginning with a reduced Levi graph for a chirally symmetric 3-configuration that includes a double-arc, it is possible to convert the reduced Levi graph to a reduced Levi graph for a dihedrally symmetric configuration.The final mirroring is indicated by green dashing in the reduced Levi graph.

Example 5. *Constructing a dihedral astral 3-configuration*

The dihedralized chiral astral 3-configuration has the reduced Levi graph shown in Figure 23. Figure 24 shows four steps in the construction of the corresponding configuration, in the case where $m = 3$ and $a = b = t = 1$. Begin by constructing vertices v_i^+ and their mirror images v_i^- and the span a lines L_i^+ and L_i^-. Next, construct the circle $\mathcal{O}$ through v_t^+, v_{t-b}^+ and $\mathcal{O}$, and intersect that with line L_0^- to form the point w_0^+; construct the rest of the points w_i^+ as rotated images. The lines N_i^+ are constructed as span b lines with respect to the w_i. Finally, reflect the elements w_i^+ and N_i^+ over the mirror to form w_i^- and N_i^- finish the construction.

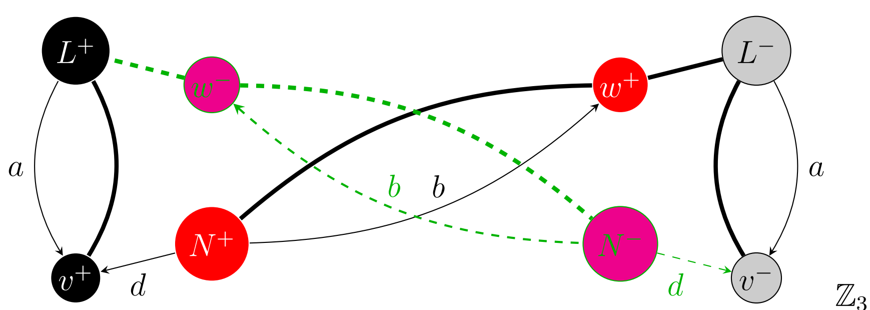
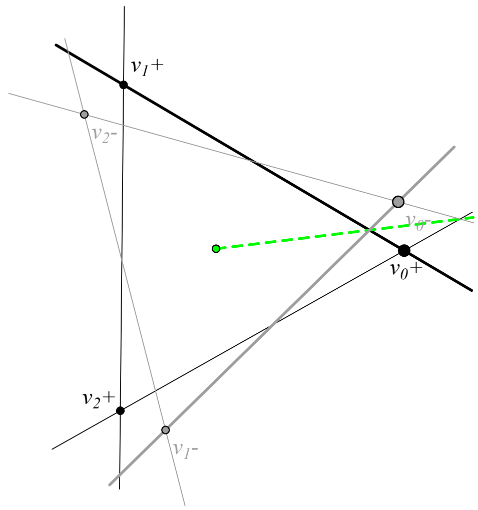
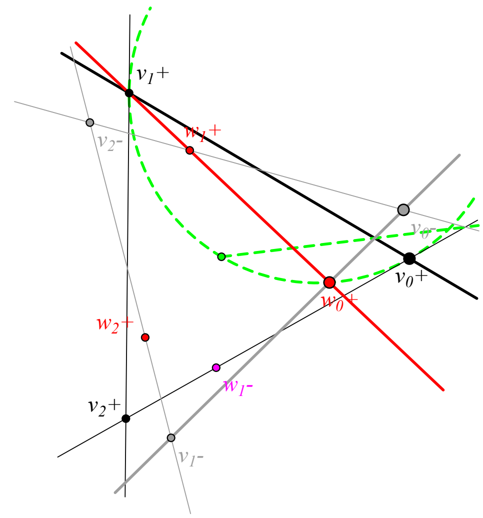


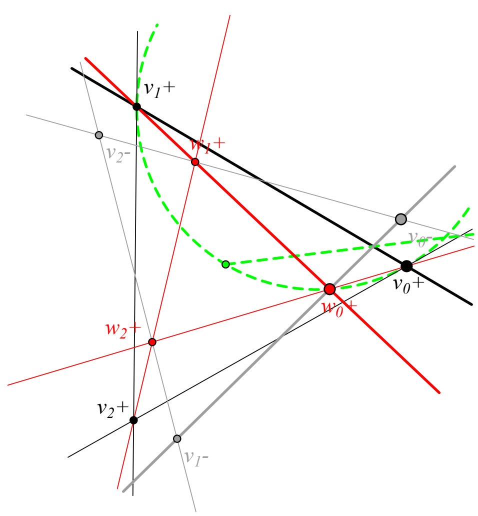
Figure 23: The reduced Levi graph corresponding to the dihedralized chiral astral 3-configuration whose reduced Levi graph is shown in Figure 10. The node colors correspond to the colors of the points and lines in the configuration shown in Figure 24 (in which $m = 3$, $a = b = d = 1$).



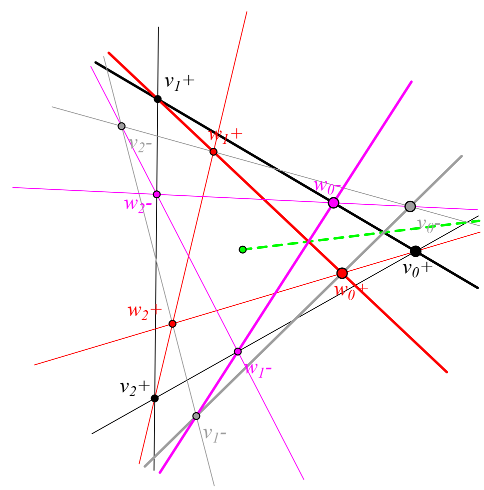
(a) The signed subconfigurations, which consist of vertices v^+ and v^- and lines L^+ and L^-



(b) The point w_0^+ is the intersection of L_0^- and $\mathcal{C}$.



(c) Constructing the rest of the w_i^+ and the lines N_i^+



(d) Mirroring to construct the w_i^- and N_i^-

Figure 24: Constructing a dihedrally symmetric astral (12_3) configuration. The green dashed line is the mirror $\mathcal{M}$. The initial subconfiguration in black is the $+$ and the one in gray is the $-$; the new points w^+ and new lines N^+ are red, and their reflected images are in magenta.

Unlike the chiral astral 3-configurations, these dihedral astral 3-configurations are movable: they have a continuous parameter corresponding to the angle between $\overline{\mathcal{O}v_0}$ and $\mathcal{M}$, as well as three discrete parameters.

Example 6. *Constructing a dihedral version of a caterpillar configuration.*

Here, we will construct a dihedralized version of the Cremona-Richmond configuration, which was shown previously in Figure 16. The reduced Levi graph is shown in Figure 25, and the various steps of the construction are shown in Figure 26. We begin by constructing the subconfiguration corresponding to vertex nodes u and v and line nodes L and M, shown in black and blue, and reflect them over the mirror $\mathcal{M}$ shown in green; the reflected subconfiguration is shown in Figure 26a in lighter colors (gray and cyan, respectively). Next, we need to construct new nodes w^+, which should be 0-adjacent to L^-, and N^+ in the reduced Levi graph so that N^+ is of span 1 with respect to the w_i^+ and adjacent to v^+ with a label of 3. To do so, we apply the configuration construction lemma, which says that if we construct w_0^+ to be the intersection of the circumcircle $\mathcal{C}$ through v_3^+, $\mathcal{O}$, and $v_{(3-1)}^+ = v_2^+$ with line M_0^- (note that you may need to vary the placement of the mirror so that this intersection point exists), then the points w_0^+, w_1^+ and v_3^+ will all be collinear (and we call that line N_0^+), while w_0^+ will lie on M_0^-. Finally, after constructing all the points w_i^+ and lines N_i^+, we construct the reflected images w_i^- and N_i^- to complete the configuration.

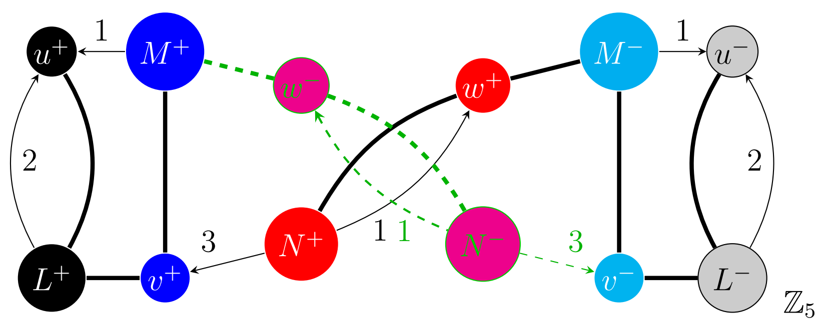
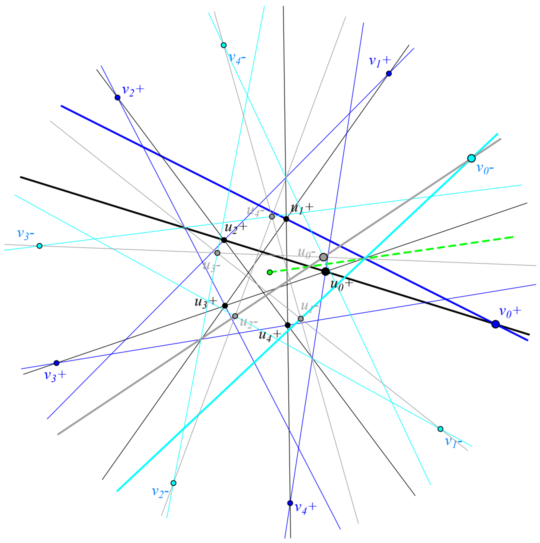


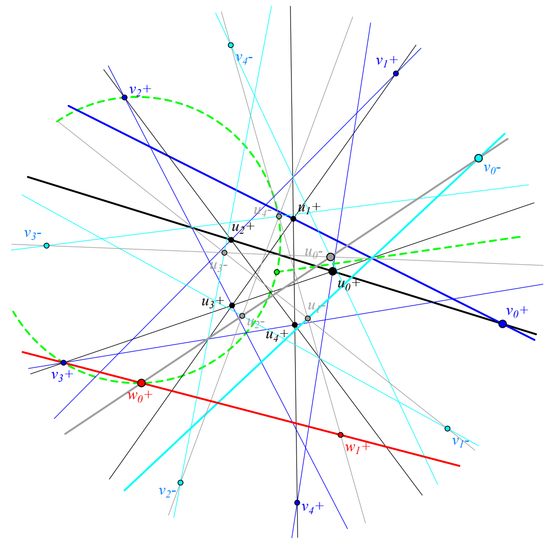
Figure 25: The reduced Levi graph for the dihedralized Cremona-Richmond configuration.

7 Constraints on labels; expansion of the constructions

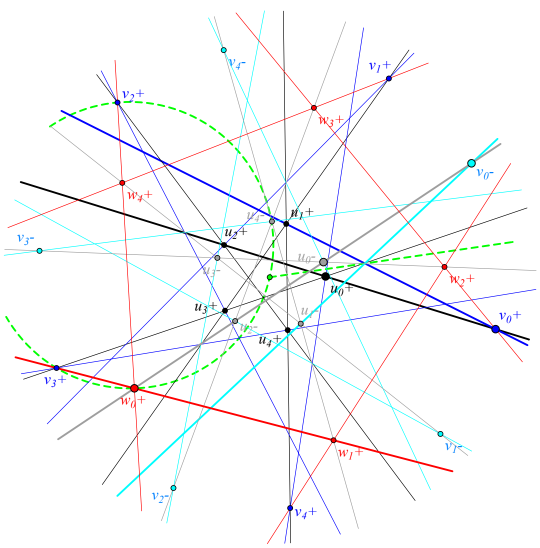
In the previous sections, the constructions have all been based on the notion that we are working with a reduced Levi graph that has valid labels; that is, following the construction techniques using the given labels, a geometric configuration is produced. However, we have been silent on what allowable labels are. Because the reduced Levi graphs must correspond to geometric configurations, there are some immediate constraints on allowable labels. For example, if a and 0 are labels on a double-arc in a reduced Levi graph over $\mathbb{Z}_m$, then $0 < a < \frac{m}{2}$, since we do not allow lines of span a to be diameters of a configuration (and because the Levi graph of the configuration (which the reduced Levi graph covers) can contain no 4-cycles), and by convention we assume that we take the smallest possible span measurement.



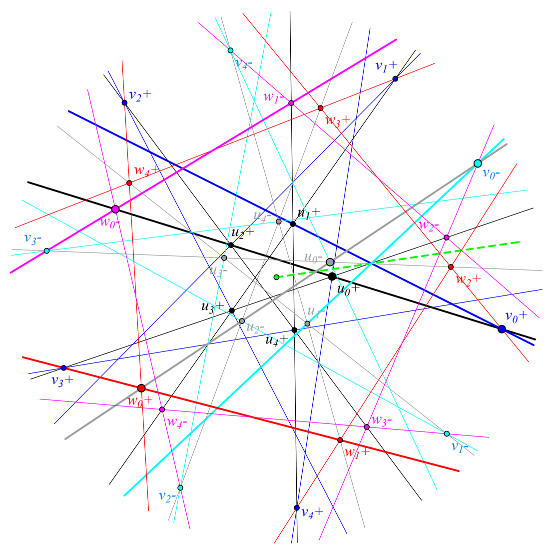
(a) The subconfiguration corresponding to deleting the nodes w and N and the adjacent arcs, plus its mirror image (shown with lighter colors).



(b) Constructing w_0 as the intersection of $\mathcal{C}$ with M_0^- (thick and gray), and N_0.



(c) Constructing the rest of the w_i and N_i



(d) Mirroring constructs the w_i^- and N_i^-

Figure 26: Constructing a (30_3) configuration as a dihedralized version of the Cremona-Richmond configuration.

However, in general, fully describing allowable labels for these construction methods for 3-configurations is beyond the scope of this paper and will be discussed in a subsequent work.

In addition to possible choices of labels that would lead to reduced Levi graphs that do not correspond to actual constructable configurations (perhaps the circumcircle does not intersect the necessary line, for example), it is also possible for the construction methods to fail because they produce too many incidences. For example, Figure 27 shows a configuration corresponding to $12\#(4,4;1)$, a chiral astral 3-configuration with $m = 12, a = 4, b = 4, d = 1$ which is a 4-configuration rather than a 3-configuration. A complete classification of when and why these unintended incidences occur in the case of chiral astral 3-configurations will also be addressed in a subsequent work.

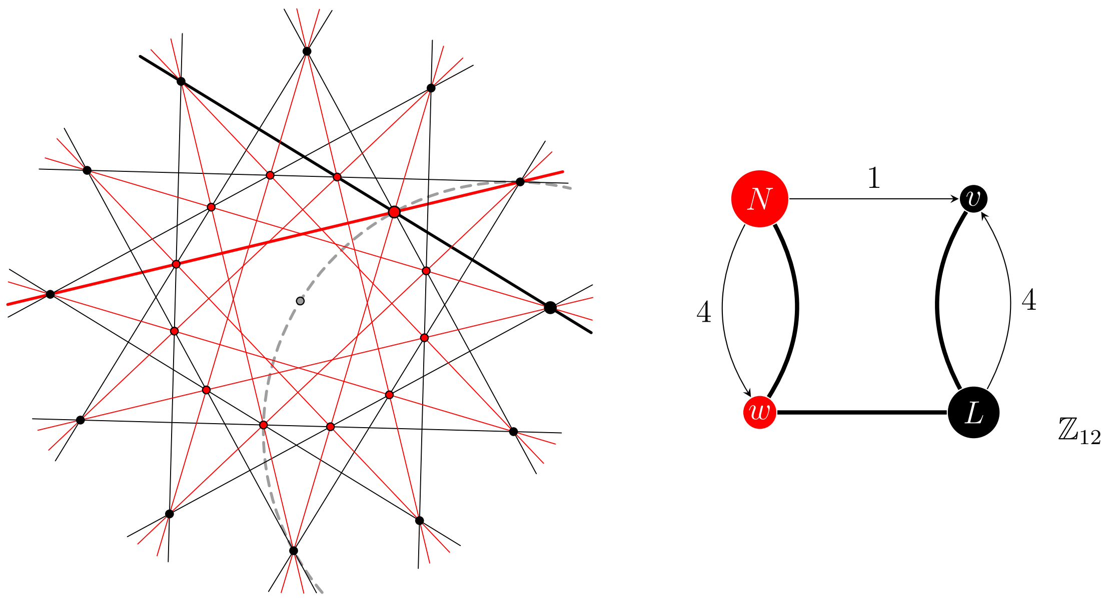


Figure 27: The reduced Levi graph for $12\#(4,4;1)$ and the corresponding configuration, which has too many incidences to be a 3-configuration.

It is also possible to use the configuration construction lemma to produce geometric constructions for more highly incident configurations. For example, it is possible to repeat the construction of astral chiral 3-configurations multiple times with the same initial polygon to construct k-configurations for arbitrary k. This construction is discussed in more detail in [1].

8 Open Questions

Question 1. *What are allowable labels for the caterpillar graphs? Are there classifications of labels that generate configurations with too many incidences?*

Question 2. *Are there other interesting infinite families of 3-configurations, similar to the multilateral and caterpillar 3-configurations?*

Question 3. *It is possible to use the construction of chiral astral 3-configurations multiple times, with an iterative process, to produce more highly incident configurations (e.g., 4-, 5- and 6-configurations); this construction is discussed in [1]. Can other 3-configurations be used to construct highly incident configurations?*

Question 4. *Given two chirally symmetric 3-configurations and their associated reduced Levi graph, is it possible to connect the reduced Levi graphs together to produce a new chiral configuration, similar to how the dihedralization technique produced a new configuration beginning with two copies of a single reduced Levi graph?*

Question 5. *Is there a purely geometric construction method for Pappus-like configurations or configurations whose reduced Levi graphs are the even prism graphs or the odd Möbius graphs?*

Acknowledgements

The author thanks Jill Faudree and Tomaž Pisanski for helpful comments, and the students in her Fall 2010 *Math 665: Configurations* graduate topics course for stimulating some of the initial ideas.

References

[1] L. W. Berman and J. R. Faudree. Highly incident configurations with chiral symmetry. *Discrete & Computational Geometry*, 49(3):671–694, April 2013.

[2] A. Betten, G. Brinkmann, and T. Pisanski. Counting symmetric configurations v_3. In *Proceedings of the 5th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1997)*, volume 99, pages 331–338, 2000.

[3] M. Boben and T. Pisanski. Polycyclic configurations. *European J. Combin.*, 24(4):431–457, 2003.

[4] J. L. Gross and T. W. Tucker. *Topological graph theory.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1987. A Wiley-Interscience Publication.

[5] B. Grünbaum. Configurations of points and lines. In *The Coxeter legacy*, pages 179–225. Amer. Math. Soc., Providence, RI, 2006.

[6] B. Grünbaum. Musings on an example of Danzer's. *European J. Combin.*, 29(8):1910–1918, 2008.

[7] B. Grünbaum. *Configurations of points and lines*, volume 103 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.