Cyclic decomposition of k-permutations and eigenvalues of the arrangement graphs

Bai Fan Chen[1]   Ebrahim Ghorbani[2,3][*]   Kok Bin Wong[1]

[1] Institute of Mathematical Sciences
University of Malaya
50603 Kuala Lumpur, Malaysia

[2] Department of Mathematics
K.N. Toosi University of Technology
P.O. Box 16315-1618, Tehran, Iran

[3] School of Mathematics
Institute for Research in Fundamental Sciences (IPM)
P.O. Box 19395-5746, Tehran, Iran

tufofo1120@gmail.com   e_ghorbani@ipm.ir   kbwong@um.edu.my

Abstract

The (n, k)-arrangement graph $A(n, k)$ is a graph with all the k-permutations of an n-element set as vertices where two k-permutations are adjacent if they agree in exactly $k - 1$ positions. We introduce a cyclic decomposition for k-permutations and show that this gives rise to a very fine equitable partition of $A(n, k)$. This equitable partition can be employed to compute the complete set of eigenvalues (of the adjacency matrix) of $A(n, k)$. Consequently, we determine the eigenvalues of $A(n, k)$ for small values of k. Finally, we show that any eigenvalue of the Johnson graph $J(n, k)$ is an eigenvalue of $A(n, k)$ and that $-k$ is the smallest eigenvalue of $A(n, k)$ with multiplicity $\mathcal{O}(n^k)$ for fixed k.

Keywords: k-permutation, Cyclic decomposition, Arrangement graph, Eigenvalue of graph

1 Introduction

Let G be a simple graph with ν vertices. The *adjacency matrix* of G is a $\nu \times \nu$ matrix where its rows and columns are indexed by the vertex set of G and its (u, v)-entry is 1 if

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[*]corresponding author

the vertices u and v are adjacent and 0 otherwise. By the *eigenvalues* of G we mean the eigenvalues of its adjacency matrix.

For a positive integer ℓ, let $[\ell] := \{1, \ldots, \ell\}$. For positive integers k, n with $k \leqslant n$, a k-*permutation* of $[n]$ is an injective function from $[k]$ to $[n]$. When $k = n$, a k-permutation is a permutation. So any k-permutation π can be represented by a vector $(i_1, \ldots, i_k)$ where $\pi(j) = i_j$ for $j = 1, \ldots, k$. We denote the set of all k-permutations of $[n]$ by $V(n, k)$. The set $\{i_1, \ldots, i_k\}$ is the *image* of π and denoted $\mathrm{Im}(\pi)$. Unlike permutations which have a decomposition into cycles, k-permutations in general do not have such a decomposition. We observe that a decomposition of a k-permutation into cycles and 'paths' is possible where paths are the same as cycles except that in a path the last element is not mapped to the first element. We then define the cycle type for a k-permutation π to be the list of integers consisting of the lengths of the cycles and the paths appeared in the decomposition of π. This gives rise to the 'cycle-type partition' of $V(n, k)$ in which each class consists of all elements of $V(n, k)$ sharing the same cycle type. The reason for studying the cycle-type partition of k-permutations will become clear below.

The (n, k)-*arrangement graph* $A(n, k)$ has $V(n, k)$ as its vertices, and two k-permutations $\pi = (u_1, \ldots, u_k)$ and $\rho = (v_1, \ldots, v_k)$ are adjacent if they agree in exactly $k - 1$ positions, i.e. if for exactly one i_0, $u_{i_0} \neq v_{i_0}$ and for all $i \neq i_0$, $u_i = v_i$. The family of arrangement graphs was first introduced in [10] as an interconnection network model for parallel computation. In the interconnection network model, each processor has its own memory unit and communicates with the other processors through a topological network, i.e. a graph. For this purpose, the arrangement graphs possess many nice properties such as having small diameter, a hierarchical structure, vertex and edge symmetry, simple shortest path routing, high connectivity, etc. Many properties of arrangement graphs have been studied by a number of authors, see, e.g. [4, 5, 6, 7, 8, 19, 20]. Another family of graphs with the same nature as the arrangement graphs are the derangement graphs. The n-*derangement graph* is a graph whose vertices correspond to all the permutations of $[n]$ where two permutations are adjacent if they differ in all n positions. It is known that the eigenvalues of the derangement graph are integers (see [1, 11, 15, 17]). For other properties of the eigenvalues of the derangement graph, we refer the reader to [13, 14, 18].

As an application of the cycle-type partition of $V(n, k)$, we consider the problem of determining the eigenvalues of the arrangment graphs. It turns out that the cycle-type partition of $V(n, k)$ is indeed an equitable partition of the graph $A(n, k)$. Normally, the eigenvalues of equitable partitions of a graph give a subset of the set of eigenvalues of the graph. However, in view of a result of Godsil and McKay [12], the cycle-type partition of $V(n, k)$ is fine enough to give the complete set of eigenvalues as well as their multiplicities. Consequently, we will be able to determine the eigenvalues of $A(n, k)$ for small values of k. We also show that any eigenvalue of the Johnson graph $J(n, k)$ is an eigenvalue of $A(n, k)$. (Recall that the Johnson graph $J(n, k)$ has all the k-subsets of $[n]$ as vertices where two k-subsets are adjacent if they intersect in exactly $k - 1$ elements.) Finally, we prove that that $-k$ is the smallest eigenvalue of $A(n, k)$ with multiplicity $\mathcal{O}(n^k)$ for fixed k. We shall close the paper by some open problems on eigenvalues of the arrangement graphs.

2 Cyclic decomposition of a k-permutation and basic graphs

A permutation can be decomposed into disjoint cycles. However, in general such a decomposition does not exist for a k-permutation. In this section, we demonstrate how a decomposition of k-permutations is possible by taking into account a relaxation on cycles.

2.1 Decomposition of k-permutation into cycles and paths

We call a k-permutation π a *path* of length ℓ if for some $u_1, \ldots, u_\ell$ and v, $\pi(u_t) = u_{t+1}$ for $t = 1, \ldots, \ell - 1$ and $\pi(u_\ell) = v$ such that $u_t \in [k]$ for all $t = 1, \ldots, \ell$ and $v \notin [k]$. We denote such a path π by $(u_1 \ldots u_\ell\, v]$. As usual, $(u_1 \ldots u_\ell)$ denotes a cycle of length ℓ. The same method for decomposing permutations into disjoint cycles can be employed for decomposition of k-permutations into disjoint cycles and paths. We call this decomposition the *cyclic decomposition* of k-permutations. Here is examples of decompositions of some 5-permutations:

$$(1, 2, 3, i, j) = (1)(2)(3)(4\, i](5\, j],$$
$$(2, 3, 4, i, j) = (1\, 2\, 3\, 4\, i](5\, j],$$
$$(2, i, j, 5, 4) = (1\, 2\, i](3\, j](4\, 5),$$

where $i, j > 5$.

We put this observation formally in the following proposition. The straightforward proof is similar to the case of permutations.

Proposition 1. *Any k-permutation is a product of disjoint cycles and paths. This decomposition is unique up to the order in which the cycles and paths are written.*

In some applications, it would be useful to consider a graphical representation of a k-permutation π. This can be done simply by constructing a (directed) graph with vertices $[k] \cup \mathrm{Im}(\pi)$ and with the set of arcs $\{(u, \pi(u)) \mid u \in [k]\}$. We call the resulting graph, the *basic graph* of π and denote it by $\mathrm{BG}(\pi)$. It is seen that $\mathrm{BG}(\pi)$ consists of a union of directed cycle graphs and path graphs where the cycle and path graphs correspond to the cycles and paths in the cyclic decomposition of π, respectively. It turns out that in $\mathrm{BG}(\pi)$ all the edges in the cycles and paths have the same directions, the vertices of $[k] \cap \mathrm{Im}(\pi)$ have degree two, other vertices have degree one, and that any vertex of $\mathrm{Im}(\pi) \setminus [k]$ is the head of a (directed) path. In Table 1, all non-isomorphic basic graphs of 3-permutations are depicted where the edges are shown without directions and the black and white vertices represent elements of $[k]$ and $\mathrm{Im}(\pi) \setminus [k]$, respectively.

We usually consider basic graphs without directions as the directions of the edges has no effect on the isomorphism type of them. However, in some applications such as the proof of Theorem 7 the direction of the edges might be useful. We remark that any (multi-)graph with k edges, maximum degree at most 2 and with no isolated vertices is a basic graph of some k-permutation. This is another description of the family of basic graphs.

3-permutation	Decomposition	Partition of 3	Basic graph
$(1,2,3)$	$(1)(2)(3)$	111	
$(1,3,2)$	$(1)(2\,3)$	12	
$(2,3,1)$	$(1\,2\,3)$	3	
$(1,2,i)$	$(1)(2)(3\,i]$	$111'$	
$(1,3,i)$	$(1)(2\,3\,i]$	$12'$	
$(2,3,i)$	$(1\,2\,3\,i]$	$3'$	
$(2,1,i)$	$(1\,2)(3,\,i]$	$21'$	
$(1,i,j)$	$(1)(2\,i](3\,j]$	$11'1'$	
$(2,i,j)$	$(1\,2\,i](3\,j]$	$1'2'$	
(i,j,ℓ)	$(1\,i](2\,j](3\,\ell]$	$1'1'1'$	

Table 1: The cycle structures of 3-permutations (here $i,j,\ell > 3$)

2.2 Cycle structure and partitions of k into parts of two kinds

To any cyclic decomposition of a permutation σ one may assign a list of integers consisting of the lengths of the cycles appearing in the decomposition. This list is called the *cycle structure* of σ. There is a one-to-one correspondence between the cycle structure of permutations on $[n]$ and the partitions of the integer n. For the case of k-permutations, we need to distinguish between cycles and paths. To this end, we take into account the partitions of k into parts of two kinds. More precisely, assume that there are integers of two kinds r and r' and we consider the ways to write n as a sum of integers of either kind where the order of terms in the sum does not matter. For instance, $k = 2$ has the following partitions into parts of two kinds: 11, $11'$, $1'1'$, 2, and $2'$. We define the *cycle structure* or *cycle type* of a k-permutation π to be the list consisting of the lengths of the cycles and the paths appearing in the cyclic decomposition of π where for cycles we write integers of the first kind and for paths we write integers of the second kind. For example, any 3-permutation of $[n]$ with $n \geqslant 6$ has one of the ten different types represented in Table 1 where for each type, an instance of a 3-permutation with that type together with its cyclic decomposition and the respective basic graph are demonstrated.

Proposition 2. *There is a one-to-one correspondence between the cycle structure of k-permutations and the partitions of k into parts of two kinds.*

Remark 3. We define $c(k)$ to be the number of partitions of k into parts of two kinds. The sequence consisting of $c(k)$'s, namely the sequence $1, 2, 5, 10, 20, 36, 65, 110, 185, 300, \ldots$

(starting with index $k = 0$) is the sequence A000712 of the OEIS [16]. This sequence has many interesting interpretations and nice properties (see [9, 16]), among which are the following two identities:

$$\sum_{k=0}^{\infty} c(k)x^k = \prod_{i=1}^{\infty} \frac{1}{(1 - x^i)^2},$$

and

$$c(k) = \sum_{i=0}^{k} p(i)p(k - i),$$

where $p(i)$ is the number of partitions of i. Here is yet another interpretation of A000712: $c(k)$ is equal to the number of non-isomorphic basic graphs of k-permutations, or in other words, the number of all multi-graphs with exactly k-edges and with vertex degrees 1 or 2.

The cycle type of a permutation of n is shown as $1^{a_1}2^{a_2}\ldots n^{a_n}$ where the superscripts $a_i \geqslant 0$ indicate multiplicities. It is known that the number of all permutations of $[n]$ of cycle type $1^{a_1}2^{a_2}\ldots n^{a_n}$ is equal to

$$\frac{n!}{a_1!a_2!2^{a_2}\cdots a_n!n^{a_n}}.$$

Here we determine the number of k-permutations sharing the same cycle structure. For integers $n \geqslant \ell \geqslant 0$ we use the *falling factorial* notation

$$(n)_\ell := n(n - 1)\cdots(n - \ell + 1),$$

with the convention that $(n)_\ell = 1$ if $\ell = 0$.

Theorem 4. *The number of k-permutations of $[n]$ of the cycle type*

$$1^{a_1}2^{a_2}\ldots k^{a_k}1'^{b_1}2'^{b_2}\ldots k'^{b_k} \tag{1}$$

is equal to

$$\frac{k!}{\prod_{i=1}^{k}(i^{a_i}a_i!b_i!)}(n - k)_s,$$

where $s = b_1 + \cdots + b_k$.

Proof. Note that $s = |\operatorname{Im}(\pi) \setminus [k]|$ for any k-permutation π of cycle type (1). Any k-permutation with cycle type (1) can be obtained from an arbitrary permutation σ of $[k]$ by first inserting $a_1 + \cdots + a_k$ pairs of parentheses to make the cycles and inserting s pairs consisting of a parenthesis and a bracket and then putting s integers from $[n] \setminus [k]$ alongside the brackets to make the paths. We count the number of ways that this can be done. There are $k!$ ways to fill in the permutation σ. The integers inserted alongside the s brackets can be regarded as an s-permutation of $[n] \setminus [k]$ and can be chosen in $(n - k)_s$ ways. These give the number $k!(n - k)_s$; but we must correct our overcounting. Each of

the a_i cycles of length i can be rotated around i ways and be the same cycle, so we should divide by i^{a_i} for $i = 1, \ldots, k$. (Note that this is not the case for the paths; because in any path $(u_1 \ldots u_\ell \, v]$ the last element v always comes from $[n] \setminus [k]$ and rotating around the u_i's yields different paths.) There are a_i cycles of length i and b_i paths of length i which can be permuted around in $a_i!$ and $b_i!$ ways, respectively, so we divide by $a_i! b_i!$ for $i = 1, \ldots, k$. $\qquad \square$

3 The cycle-type partition

We partition $V(n, k)$ according to the cycle type of k-permutations. So $V(n, k)$ is partitioned into $c(k)$ cells (cf. Remark 3) where the k-permutations of each cell/part share the same cycle type. Equivalently, two k-permutations belong to the same cell if they have isomorphic basic graphs. We call this partition the *cycle-type partition* of $V(n, k)$. For instance, the cycle-type partition of 3-permutations of $[n]$ together with the corresponding cycle type of each cell are demonstrated in Table 2.

Type	Cell
111	$V_1 = \{(1, 2, 3)\}$
12	$V_2 = \{(1, 3, 2), (2, 1, 3), (3, 2, 1)\}$
3	$V_3 = \{(2, 3, 1), (3, 1, 2)\}$
111′	$V_4 = \{(1, 2, i), (1, i, 3), (i, 2, 3) \mid 4 \leqslant i \leqslant n\}$
21′	$V_5 = \{(2, 1, i), (3, i, 1), (i, 3, 2) \mid 4 \leqslant i \leqslant n\}$
11′1′	$V_6 = \{(1, i, j), (i, 2, j), (i, j, 3) \mid 4 \leqslant i, j \leqslant n, i \neq j\}$
12′	$V_7 = \{(1, 3, i), (3, 2, i), (1, i, 2), (2, i, 3), (i, 2, 1), (i, 1, 3) \mid 4 \leqslant i \leqslant n\}$
1′1′1′	$V_8 = \{(i, j, k) \mid 4 \leqslant i, j, k \leqslant n, i \neq j \neq k \neq i\}$
1′2′	$V_9 = \{(2, i, j), (3, i, j), (i, 1, j), (i, 3, j), (i, j, 1), (i, j, 2) \mid 4 \leqslant i, j \leqslant n, i \neq j\}$
3′	$V_{10} = \{(2, 3, i), (3, 1, i), (3, i, 2), (2, i, 1), (i, 3, 1), (i, 1, 2) \mid 4 \leqslant i \leqslant n\}$

Table 2: The cycle-type partition of 3-permutations

An *equitable partition* of a graph G is a partition $\Pi = (V_1, \ldots, V_m)$ of the vertex set such that each vertex in V_i has the same number q_{ij} of neighbors in V_j for any i, j (and possibly $i = j$). The *quotient matrix* of Π is the $m \times m$ matrix $Q = (q_{ij})$. It is well-known that every eigenvalue of the quotient matrix Q is an eigenvalue of G (see [2, p. 24]). Under certain circumstances, the converse of this is also true as the next lemma shows.

We recall that a graph G is *walk-regular* if for every positive integer r, the number of closed walks of length r starting at a vertex v is independent of the choice of v. Clearly, vertex-transitive graphs are walk-regular and so are the arrangement graphs $A(n, k)$.

Lemma 5 ([12]). *Let G be a walk-regular graph with ν vertices. Let $\Pi = (V_1, \ldots, V_m)$ be an equitable partition of G with $|V_1| = 1$ and let Q be the quotient matrix of Π.*

(i) *Every eigenvalue of G is an eigenvalue of Q.*

(ii) *Let $S = \mathrm{diag}(\sqrt{|V_1|}, \sqrt{|V_2|}, \ldots, \sqrt{|V_m|})$ and $P = SQS^{-1}$. If $\{\mathbf{x}_1, \ldots, \mathbf{x}_\ell\}$ is a full set of orthonormal eigenvectors of P for the eigenvalue λ, then the multiplicity of λ as an eigenvalue of G is*

$$\nu \sum_{i=1}^{\ell} (\mathbf{x}_i)_1^2,$$

where $(\mathbf{x}_i)_1$ denotes the first coordinate of $\mathbf{x}_i$.

In the next theorem we show that the cycle-type partition is indeed an equitable partition for the arrangement graphs. This partition is fine enough to meet the condition of Lemma 5 as the cell of type 1^k contains a single k-permutation, namely $(1, \ldots, k)$.

Theorem 6. *The cycle-type partition of $V(n,k)$ is an equitable partition of $A(n,k)$.*

Proof. Let X and Y (possibly $X = Y$) be two cells of the cycle-type partition of $V(n,k)$. Let $\pi, \rho \in X$. We show that there is a one-to-one correspondence between $N_Y(\pi)$ (the set of neighbors of π in Y) and $N_Y(\rho)$. Let

$$\pi = \alpha_1 \cdots \alpha_\ell \beta_1 \cdots \beta_m,$$
$$\rho = \gamma_1 \cdots \gamma_\ell \delta_1 \cdots \delta_m,$$

be the cycle decompositions where α_i and γ_i are cycles of the same length and β_i and δ_i are paths of the same length. Let $\alpha_i = (a_{i1} \ldots a_{ir_i})$, $\gamma_i = (a'_{i1} \ldots a'_{ir_i})$, $\beta_i = (b_{i1} \ldots b_{i(s_i-1)} \, b_{is_i}]$, and $\delta_i = (b'_{i1} \ldots b'_{i(s_i-1)} \, b'_{is_i}]$. Note that $b_{is_i}, b'_{is_i} \in [n] \setminus [k]$ and $a_{ij}, a'_{ij}, b_{ie}, b'_{ie} \in [k]$ for all i, j and $1 \leqslant e \leqslant s_i - 1$.

Let σ_1 be the permutation on $[k]$ that maps $a_{ij} \to a'_{ij}$ and $b_{ie} \to b'_{ie}$ for all i, j and $1 \leqslant e \leqslant s_i - 1$ and $\sigma_2 : [n] \setminus ([k] \cup \mathrm{Im}(\pi)) \to [n] \setminus ([k] \cup \mathrm{Im}(\rho))$ be an arbitrary bijection. Let

$$\sigma = \sigma_1 \sigma_2 \prod_{i=1}^{m} (b_{is_i} \quad b'_{is_i}).$$

Now, σ is a permutation on $[n]$ and $\sigma \pi \sigma^{-1} = \rho$. Let $\zeta \in N_Y(\pi)$. Clearly, $\sigma \zeta \sigma^{-1} \in Y$. We have $\zeta(u_0) \neq \pi(u_0)$ for exactly one $u_0 \in [k]$ and $\zeta(u) = \pi(u)$ for all $u \in [k] \setminus \{u_0\}$. Now, $\sigma \pi \sigma^{-1}$ and $\sigma \zeta \sigma^{-1}$ coincides at u for all $u \in [k] \setminus \{u_0\}$ and differ at u_0. Hence, $\sigma \zeta \sigma^{-1} \in N_Y(\rho)$. The map $\psi : N_Y(\pi) \to N_Y(\rho)$ defined by $\psi(\zeta) = \sigma \zeta \sigma^{-1}$ is a bijection, so the proof follows. $\square$

Next, we explicitly give the quotient matrix of the cycle-type partition of $A(n,k)$. To state this result, we need more notations. We may denote the cycle type of a k-permutation by the notation $[A, B]$, where A and B represent multisets of unprimed and primed integers, respectively. Thus, elements of A represent cycles while elements of B represent paths. For a list L and $i \in L$, we denote the resulting list of removing (one) i from L by L_i and the resulting list of adding r to L by L^r. We may combine these,

so, e.g. $L_{i,j}^{r,t}$ is the list obtained from L by removing i, j and adding r, t. The number of elements of L counting multiplicities is denoted $|L|$.

In the next theorem, given a representative π of the cycle type $[A, B]$, we determine the cycle types of the neighbors of π which are described in the mutually exclusive subcases of the theorem. We also count the number of neighbors of π of each type.

Theorem 7. *Suppose that $\pi \in V(n, k)$ has the cycle type $[A, B]$. Then the neighbors of π are as follows.*

(i) *For any $i \in A$ with multiplicity a,*

   (i.1) *π has $ia(n - k - |B|)$ neighbors in $[A_i, B^i]$;*

   (i.2) *for any $j \in B$ with multiplicity b, π has abi neighbors in $[A_i, B_j^{i+j}]$.*

(ii) *For any $j \in B$ with multiplicity b and for any ℓ with $1 \leqslant \ell \leqslant j$,*

   (ii.1) *π has b neighbors in $[A^\ell, B_j^{j-\ell}]$;*

   (ii.2) *for any $m \in B_j$ with multiplicity c and $m + \ell \neq j$, in $[A, B_{j,m}^{m+\ell, j-\ell}]$, π has $2bc$ neighbors if $m \neq j$ and $m - j + \ell \geqslant 1$ and has bc neighbors otherwise;*

   (ii.3) *in $[A, B_j^{\ell, j-\ell}]$, π has $2b(n - k - |B|)$ neighbors if $j \neq 2\ell$ and $b(n - k - |B|)$ neighbors if $j = 2\ell$.*

(iii) *If $B = j_1^{b_1} \ldots j_h^{b_h}$, then π has $|B|(n - k - |B|) + \sum_{1 \leqslant r < t \leqslant h} b_r b_t$ neighbors in $[A, B]$ (in particular, if $B = \emptyset$, then π has no neighbor in $[A, B]$).*

Proof. Let $G = \mathrm{BG}(\pi)$. The vertex $\pi' \in V(n, k)$ is adjacent to π if there is a unique $u \in [k]$ such that $\pi(u) \neq \pi'(u)$. Considering the basic graphs, that means one can obtain $\mathrm{BG}(\pi')$ from G by changing the arc $(u, \pi(u))$ to $(u, \pi'(u))$. It follows that we can identify all the neighbors of π in $A(n, k)$ by determining all basic graphs obtained from G by changing an arc (u, v) to (u, w). Notice that we are allowed to do this only by changing the *head* of the arc and not the *tail* of the arc. Also as the resulting graph must be a basic graph, we have necessarily either $w \in [k]$ and it is a degree 1 vertex of G or $w \in [n] \setminus V(G)$.

We describe all the basic graphs obtained from G through the above procedure in what follows.

(i) Let C be a cycle of length i in G. The only way to obtain a basic graph from G by changing an arc (u, v) of C to (u, w) is if either (a) $w \in [n] \setminus V(G)$ or (b) $w \in [k]$ and w is the tail of a path of length j, say. For the case (a), the resulting graph is isomorphic to the one obtained from G by replacing C with a path of length i, and thus it is isomorphic to the basic graph H of $[A_i, B^i]$. We have $(n - k - |B|)$ different choices for w and also we may choose any of ai edges of the i-cycles to obtain a basic graph isomorphic to H. Thus π has $ai(n - k - |B|)$ neighbors in $[A_i, B^i]$, proving (i.1). For the case (b), the resulting graph is isomorphic to the basic graph H of $[A_i, B_j^{i+j}]$. If there exist b paths of length j in G, then we have abi different ways to obtain a basic graph isomorphic to H. This proves (i.2).

(ii) Let $(u_1 \ldots u_j \, u_{j+1}]$ be a path of length j in G. Consider the basic graph H obtained from G by changing the arc $(u_\ell, u_{\ell+1})$, $1 \leqslant \ell \leqslant j$, to (u, w). There are three possibilities, namely: (a) $w = u_1$, (b) $w \in [k] \setminus \{u_1\}$, or (c) $w \in [n] \setminus V(G)$. In the case (a), H is isomorphic to the basic graph of $[A^\ell, B_j^{j-\ell}]$. From any path of length j we may obtain exactly one graph isomorphic to H. It follows that π has b neighbors in $[A^\ell, B_j^{j-\ell}]$, proving (ii.1). In the case (b), w is necessarily the tail of a path of length m, say. Thus H is isomorphic to the basic graph of $[A, B_{j,m}^{m+\ell,j-\ell}]$. This gives bc neighbors for π in $[A, B_{j,m}^{m+\ell,j-\ell}]$. Note that if $m \neq j$ and $r := m - j + \ell \geqslant 1$, then we may obtain more neighbors for π in $[A, B_{j,m}^{m+\ell,j-\ell}]$. As $m \neq j$ and $m \in B_j$, we have $j \in B_m$. With the same reasoning as above, π has bc neighbors in $[A, B_{m,j}^{j+r,m-r}]$. However, $B_{m,j}^{j+r,m-r} = B_{j,m}^{m+\ell,j-\ell}$. Therefore, if $m \neq j$ and $m - j + \ell \geqslant 1$, the total number of neighbors of π in $[A, B_{j,m}^{m+\ell,j-\ell}]$ is $2bc$. This proves (ii.2). In the case (c), we have H isomorphic to the basic graph of $[A, B_j^{\ell,j-\ell}]$. We have b paths of length j and also $(n - k - |B|)$ different choices for w. This gives $b(n - k - |B|)$ neighbors for π in $[A, B_j^{\ell,j-\ell}]$. But, if $j \neq 2\ell$, then by interchanging ℓ and $j - \ell$ we obtain another $b(n - k - |B|)$ copies of H from G. This proves (ii.3).

(iii) There are only two ways to change an arc (u, v) to (u, w) in G and obtain a basic graph isomorphic to G, namely (a) by choosing v to be the head of some path in G and $w \in [n] \setminus V(G)$, (b) by the same way as in (ii.2) but with $m + \ell = j$. For (a), v can be any of the heads of the $|B|$ paths of G and w any of the $(n - k - |B|)$ elements of $[n]$ not appearing in G. This gives $|B|(n - k - |B|)$ different copies of G. Now we count the rest of the neighbors of π coming from (b). For any $m, j \in B$ with $m < j$, letting $\ell = j - m$, we have $B_{j,m}^{m+\ell,j-\ell} = B$. There are $\sum_{1 \leqslant r < t \leqslant h} b_r b_t$ different ways to choose pairs of $m, j \in B$ with $m < j$. The proof now follows by summing up the number of neighbors given in the cases (a) and (b). $\qquad \square$

4   Eigenvalues of $A(n, k)$

In this section we determine the eigenvalues of $A(n, k)$ for $k \leqslant 7$. As mentioned before, the arrangement graphs are vertex-transitive and thus walk-regular. Furthermore, the cycle-type partition of $A(n, k)$ contains a cell of cardinality 1. So Lemma 5 can be applied to derive the eigenvalues of of $A(n, k)$. For the graphs $A(n, 2)$, though, we employ a different method taking into account a characterization which directly gives the eigenvalues.

Since $A(n, k)$ is a $k(n - k)$-regular graph, the largest eigenvalue is $k(n - k)$. When $n = k$, the edge set is an empty set. So, $A(k, k)$ has one eigenvalue only, which is 0. So we shall assume that $n > k$. When $k = 1$, $A(n, 1)$ is the complete graph with n vertices. Therefore the eigenvalues of $A(n, 1)$ are $(n - 1)$ with multiplicity 1, and -1 with multiplicity $n - 1$.

4.1   Eigenvalues of $A(n, 2)$

We use the following well-known lemma (see [2, p. 10]) to derive the eigenvalues of $A(n, 2)$.

Lemma 8. *If G is an r-regular graph ($r \geqslant 2$) with ν vertices and ε edges, and eigenvalues λ_i, $i = 1, \ldots, \nu$, then the line graph of G is $(2r - 2)$-regular with eigenvalues $\lambda_i + r - 2$, $i = 1, \ldots, \nu$ together with $\varepsilon - \nu$ times -2.*

Proposition 9. *Let $n \geqslant 3$. The eigenvalues of the arrangement graph $A(n, 2)$ are*

$$(-2)^{[n^2 - 3n + 1]}, \ (n - 4)^{[n-1]}, \ (n - 2)^{[n-1]}, \ (2n - 4)^{[1]},$$

where the superscripts indicate multiplicities.

Proof. Let H_n denote the graph obtained from the complete bipartite graph $K_{n,n}$ after removing a perfect matching. Assume that $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ is the vertex set and $\{a_i b_j \mid 1 \leqslant i, j \leqslant n, i \neq j\}$ is the edge set of H_n. In the line graph $L(H_n)$ of H_n, two distinct edges $a_i b_j$ and $a_r b_s$ are adjacent if and only if either $i = r$ or $s = j$. This shows that the map which sends the vertex (i, j) of $A(n, 2)$ to the vertex $a_i b_j$ of $L(H_n)$ defines an isomorphism between $A(n, 2)$ and $L(H_n)$.

Now, it suffices to determine the eigenvalues of $L(H_n)$. Since the adjacency matrix of H_n is $(J_2 - I_2) \otimes (J_n - I_n)$, with '$\otimes$' denoting the Kronecker product, the eigenvalues of H_n are $\pm(n - 1)$ with multiplicity 1 and ± 1 with multiplicity $n - 1$. The result now follows by applying Lemma 8. $\square$

4.2 Eigenvalues of $A(n, 3)$ and $A(n, 4)$

Throughout this subsection, we use the notation n_i for $(n - i)$ for saving space in tables and arrays.

Theorem 10. *For $n \geqslant 4$, the eigenvalues of $A(n, 3)$ are*

$$(-3)^{[n(n-2)(n-4)-1]}, \ (n - 7)^{[n(n-3)/2]}, \ (n - 6)^{[(n-2)(n-1)]}, \ (n - 4)^{[n(n-3)]},$$
$$(n - 3)^{[(n-1)(n-2)/2]}, \ (2n - 9)^{[n-1]}, \ (2n - 6)^{[2(n-1)]}, \ (3n - 9)^{[1]}.$$

Proof. The eigenvalues of $A(4, 3)$ and $A(5, 3)$ are determined by a computer; these are

$$\{-3^{[1]}, \ -2^{[6]}, \ -1^{[3]}, \ 0^{[4]}, \ 1^{[3]}, \ 2^{[6]}, \ 3^{[1]}\} \quad \text{and} \quad \{-3^{[14]}, \ -2^{[5]}, \ -1^{[12]}, \ 1^{[14]}, \ 2^{[6]}, \ 4^{[8]}, \ 6^{[1]}\},$$

respectively, which agree with the assertion. (Note that letting $n = 4$ in the assertion, the sum of the multiplicities of -3 and $n - 7$ equals 1.) Now, we may assume that $n \geqslant 6$.

By Theorem 7, the quotient matrix of the cycle-type partition of $A(n, 3)$ is the following where the cells are indexed as in Table 2:

$$Q = \begin{pmatrix} 0 & 0 & 0 & 3n_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_3 & 0 & 2n_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3n_3 \\ 1 & 0 & 0 & n_4 & 0 & 2n_4 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & n_4 & 0 & 0 & 0 & 2n_4 & 2 \\ 0 & 0 & 0 & 2 & 0 & 2n_5 & 2 & n_5 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & n_4 & n_4 & 0 & n_4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3n_6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & n_5 & 2n-9 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 2n_4 & n_4 \end{pmatrix}.$$

By computation, we worked out the eigenvalues and eigenvectors of Q as given in Table 3.

Eigenvalue	Eigenvector
-3	$[-3n_3 n_4,\ n_3 n_4,\ 0,\ 3n_4,\ -n_4,\ -2,\ -n_4,\ 0,\ 1,\ 0]$
-3	$[-n_3 n_4 n_5,\ 0,\ 0,\ n_4 n_5,\ 0,\ -n_5,\ 0,\ 1,\ 0,\ 0]$
-3	$[-n_3,\ n_3,\ -n_3,\ 1,\ -1,\ 0,\ -1,\ 0,\ 0,\ 1]$
$n-7$	$[3n_3 n_4,\ 3n_3 n_4,\ 3n_3 n_4,\ n_4 n_7,\ n_4 n_7,\ 22-4n,\ n_4 n_7,\ 18,\ 22-4n,\ n_4 n_7]$
$n-6$	$[6n_3,\ 0,\ -3n_3,\ 2n_6,\ -2n_3,\ -6,\ n_3,\ 0,\ 3,\ -n_6]$
$n-4$	$[-6n_3,\ 0,\ 3n_3,\ -2n_4,\ -2n_1,\ 2,\ n_1,\ 0,\ -1,\ n_4]$
$n-3$	$[3,\ -3,\ 3,\ 1,\ -1,\ 0,\ -1,\ 0,\ 0,\ 1]$
$2n-9$	$[3n_3,\ 3n_3,\ 3n_3,\ 2n-9,\ 2n-9,\ n_9,\ 2n-9,\ -9,\ n_9,\ 2n-9]$
$2n-6$	$[-6,\ 0,\ 3,\ -4,\ 2,\ -2,\ -1,\ 0,\ 1,\ 2]$
$3n-9$	$[1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1]$

Table 3: The eigenvalues and the (transposed) eigenvectors of the quotient matrix of $A(n,3)$

Let

$$S = \operatorname{diag}\left(\sqrt{1}, \sqrt{3}, \sqrt{2}, \sqrt{3(n-3)}, \sqrt{3(n-3)}, \sqrt{3(n-3)(n-4)}, \sqrt{6(n-3)},\right.$$
$$\left.\sqrt{(n-3)(n-4)(n-5)}, \sqrt{6(n-3)(n-4)}, \sqrt{6n-18}\right),$$

and $P = SQS^{-1}$. Note that $\mathbf{v}$ is an eigenvector of Q for the eigenvalue λ if and only if $S\mathbf{v}$ is an eigenvector of P for the eigenvalue λ. For any eigenvalue λ of Q with multiplicity 1 and with eigenvector $\mathbf{v}$, setting $\mathbf{w} = S\mathbf{v}$, the multiplicity of λ as an eigenvalue of $A(n,3)$ is obtained by Lemma 5 as

$$n(n-1)(n-2)\frac{(\mathbf{w})_1^2}{\mathbf{w}^\top \mathbf{w}}.$$

For instance,

$$\mathbf{v} = [3(n-3), 3(n-3), 3(n-3), 2n-9, 2n-9, n-9, 2n-9, -9, n-9, 2n-9]^\top$$

is an eigenvector of Q for the 1-fold eigenvalue $2n-9$. Hence,

$$\mathbf{w} = S\mathbf{v} = \left[3n_3, 3\sqrt{3}n_3, 3\sqrt{2}n_3, (2n-9)\sqrt{3n_3}, (2n-9)\sqrt{3n_3}, (n-9)\sqrt{3n_3 n_4},\right.$$
$$\left.(2n-9)\sqrt{6n_3}, -9\sqrt{n_3 n_4 n_5}, (n-9)\sqrt{6n_3 n_4}, (2n-9)\sqrt{6n-18}\right]^\top,$$

is an eigenvector of P for the eigenvalue $2n-9$. The multiplicity of $2n-9$ as an eigenvalue of $A(n,3)$ is then equal to

$$n(n-1)(n-2)\frac{(\mathbf{w})_1^2}{\mathbf{w}^\top \mathbf{w}} = n(n-1)(n-2)\frac{9(n-3)^2}{9n(n-2)(n-3)^2} = n-1.$$

By similar calculations, the multiplicities of other eigenvalues of $A(n,3)$ can be determined (except for the eigenvalue -3). The multiplicity of -3 is $n(n-1)(n-2)$ minus the sum of all the multiplicities of the rest of the eigenvalues. $\square$

Theorem 11. *For $n \geqslant 5$, the eigenvalues of $A(n,4)$ are as follows:*

$$
\begin{array}{lll}
(-4)^{[n(n-3)(n^2-7n+8)+1]} & (n-10)^{[n(n-1)(n-5)/6]} & (n-9)^{[n(n-2)(n-4)]} \\
(n-8)^{[(n-1)(n-2)(n-3)/2]} & (n-7)^{[2n(n-2)(n-4)/3]} & (n-6)^{[n(n-1)(n-5)/2]} \\
(n-5)^{[n(n-2)(n-4)]} & (n-4)^{[(n-1)(n-2)(n-3)/6]} & (2n-14)^{[n(n-3)/2]} \\
(2n-12)^{[3(n-1)(n-2)/2]} & (2n-10)^{[3n(n-3)/2]} & (2n-8)^{[5n(n-3)/2+3]} \\
(3n-16)^{[n-1]} & (3n-12)^{[3(n-1)]} & (4n-16)^{[1]}.
\end{array}
$$

Proof. The eigenvalues of $A(5,4)$, $A(6,4)$ and $A(7,4)$ are determined by a computer; these are

$$
\{-4^{[1]}, -3^{[12]}, -2^{[28]}, -1^{[4]}, 0^{[30]}, 1^{[4]}, 2^{[28]}, 3^{[12]}, 4^{[1]}\},
$$
$$
\{-4^{[42]}, -3^{[48]}, -2^{[39]}, -1^{[32]}, 0^{[45]}, 1^{[48]}, 2^{[42]}, 4^{[48]}, 6^{[15]}, 8^{[1]}\}, \text{ and}
$$
$$
\{-4^{[225]}, -3^{[14]}, -2^{[105]}, -1^{[60]}, 0^{[84]}, 1^{[42]}, 2^{[150]}, 3^{[20]}, 4^{[42]}, 5^{[6]} 6^{[73]}, 9^{[18]}, 12^{[1]}\},
$$

respectively, which agree with the assertion. So we may assume that $n \geqslant 6$.

We consider the following order for the cycle types of 4-permutations:

$$
1111, 112, 22, 13, 4, 1111', 121', 31', 111'1', 21'1', 112',
$$
$$
22', 11'1'1', 11'2', 13', 1'1'1'1', 1'1'2', 2'2', 1'3', 4'.
$$

We order the cells of the cycle-type partition according to the above ordering. From Theorem 7, the quotient matrix Q of the cycle-type partition of $A(n,4)$ is computed as:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 4n_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2n_4 & 0 & 0 & 0 & 2n_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4n_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & n_4 & 0 & 0 & 0 & 0 & 0 & 0 & 3n_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4n_4 \\
1 & 0 & 0 & 0 & 0 & n_5 & 0 & 0 & 3n_5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & n_5 & 0 & 0 & n_5 & 0 & 1 & 0 & 2n_5 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & n_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3n_5 & 3 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2n_6 & 0 & 2 & 0 & 2n_6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2n_6 & 0 & 2 & 0 & 0 & 0 & 0 & 2n_6 & 0 & 4 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & n_5 & 0 & n_5 & 0 & 0 & 2n_5 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & n_5 & 0 & n_5 & 0 & 0 & 0 & 0 & 0 & 2n_5 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3n_7 & 6 & 0 & n_7 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & n_6 & 2n-11 & 2 & 0 & n_6 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 2n_5 & n_5 & 0 & 0 & 0 & n_5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 4n_8 & 12 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & n_7 & 3n-19 & 2 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 2n_6 & 2n_6 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 2n_6 & 1 & 2n-11 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & n_5 & 2n_5 & n_5 \\
\end{pmatrix}
.
$$

By computation, we obtain the eigenvalues and eigenvectors of Q as given in Table 4.

Eigenvalue	Eigenvector
-4	$[5n_4n_5n_6, 0, -n_4n_5n_6, -n_4n_5n_6, n_4n_5n_6, -5n_5n_6, 0, n_5n_6, 5n_6, -n_6, 0, n_5n_6, -3, -n_6, n_5n_6, 0,$ $1, 0, 0, -n_5n_6]$
-4	$[2n_4n_5, 0, 0, -n_4n_5, n_4n_5, -2n_5, 0, n_5, 2, 0, 0, 0, 0, -1, n_5, 0, 0, 1, 0, -n_5]$
-4	$[2n_4n_5, 0, -2n_4n_5, -n_4n_5, 2n_4n_5, -2n_5, 0, n_5, 2, -2, 0, 2n_5, 0, -1, n_5, 0, 0, 0, 1, -2n_5]$
-4	$[n_4, -n_4, n_4, n_4, -n_4, -1, 1, -1, 0, 0, 1, -1, 0, 0, -1, 0, 0, 0, 0, 1]$
-4	$[n_4n_5n_6n_7, 0, 0, 0, 0, -n_5n_6n_7, 0, 0, n_6n_7, 0, 0, 0, -n_7, 0, 0, 1, 0, 0, 0, 0]$
$3n-16$	$[4n_4, 4n_4, 4n_4, 4n_4, 4n_4, 3n-16, 3n-16, 3n-16, -2n_8, -2n_8, 3n-16, 3n-16, n_{16}-2n_8,$ $3n-16, -16, n-16, -2n_8, -2n_8, 3n-16]$
$n-10$	$[-4n_4n_5n_6, -4n_4n_5n_6, -4n_4n_5n_6, -4n_4n_5n_6, -4n_4n_5n_6, -n_5n_6n_{10}, -n_5n_6n_{10}, -n_5n_6n_{10}, 4n_6n_8,$ $4n_6n_8, -n_5n_6n_{10}, -n_5n_6n_{10}, 132-18n, 4n_6n_8, -n_5n_6n_{10}, 96, 132-18n, 4n_6n_8, 4n_6n_8, -n_5n_6n_{10}]$
$n-9$	$\left[-12n_4n_5, -4n_4n_5, 4n_4n_5, 0, 4n_4n_5, -3n_5n_9, 4n_5, 3(n-5)^2/2, 10n-66, -2n_5, -2n_5n_7,\right.$ $\left. n_5n_9, -24, 3n-19, -(n-5)^2/2, 0, 8, -4n_7, 19-3n, n_5n_9\right]$
$n-8$	$[-12n_4, 4n_4, 4n_4, 0, -4n_4, -3n_8, 3n-16, -3n_4, 8, -8, -n, n_8, 0, -4, n_4, 0, 0, 0, 4, -n_8]$
$n-7$	$[4n_4, 0, 4n_4, -2n_4, 0, n_7, 0, -n_7/2, -2, -2, 0, n_7, 0, 1, -n_7/2, 0, 0, -2, 1, 0]$
$n-6$	$[-12n_4n_5, -4n_4n_5, 4n_4n_5, 0, 4n_4n_5, -3n_5n_6, -n_5(3n-10), -3n_2n_5, 4n_6, 4n_2, (n+2)n_5,$ $n_5n_6, -6, -4, n_2n_5, 0, 2, -4n_4, 4, n_5n_6]$
$n-5$	$[-12n_4, 4n_4, 4n_4, 0, -4n_4, -3n_5, -4, n_3/2, 2, -2, 2n_3, n_5, 0, -1, -n_1/2, 0, 0, 0, 1, -n_5]$
$n-4$	$[-4, 4, -4, -4, 4, -1, 1, -1, 0, 0, 1, -1, 0, 0, -1, 0, 0, 0, 0, 1]$
$2n-14$	$[-6n_4n_5, -6n_4n_5, -6n_4n_5, -6n_4n_5, -6n_4n_5, -3n_5n_7, -3n_5n_7, -3n_5n_7, \tau, \tau, -3n_5n_7,$ $-3n_5n_7, 9n_3, \tau, -3n_5n_7, -72, 9n_3, \tau, \tau, -3n_5n_7]$ where $\tau = 21n - n^2 - 92$
$2n-12$	$[-6n_4, -2n_4, 2n_4, 0, 2n_4, 18-3n, 2, 3n_{12}/2, -n_{12}, n_4, -2n_5, n_6, 6, -n_8/2, -n_4/2, 0, -2, -4, n_4/2, n_6]$
$2n-10$	$[6n_4, 2n_4, -2n_4, 0, -2n_4, 3n_5, n_3, 3, n_8, n, n_7, -n_5, -3, -2, -1, 0, 1, -n_4, 2, -n_5]$
$2n-8$	$[12, -2, 4, -3, 2, 6, -2, 0, 2, 0, 0, 2, 0, -1, -2, 0, 0, 1, 0, 1]$
$2n-8$	$[12, -4, -4, 0, 4, 6, -4, 3, 2, -2, 0, -2, 0, -1, -1, 0, 0, 0, 1, 2]$
$3n-12$	$[12, 4, -4, 0, -4, 9, 1, -3, 6, -2, 5, -3, 3, 2, 1, 0, -1, -2, -2, -3]$
$4n-16$	$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$

Table 4: The eigenvalues and the (transposed) eigenvectors of the quotient matrix of $A(n, 4)$

Let
$$S = \operatorname{diag}\Big(\sqrt{1}, \sqrt{6}, \sqrt{3}, \sqrt{8}, \sqrt{6}, \sqrt{4n_4}, \sqrt{12n_4}, \sqrt{8n_4}, \sqrt{6n_4n_5}, \sqrt{6n_4n_5}, \sqrt{12n_4}, \sqrt{12n_4}, \sqrt{4n_4n_5n_6},$$
$$\sqrt{24n_4n_5}, \sqrt{24n_4}, \sqrt{n_4n_5n_6n_7}, \sqrt{12n_4n_5n_6}, \sqrt{12n_4n_5}, \sqrt{24n_4n_5}, \sqrt{24n_4}\Big),$$

and $P = SQS^{-1}$. For any eigenvalue λ of Q with multiplicity 1 and with eigenvector $\mathbf{v}$, setting $\mathbf{w} = S\mathbf{v}$, the multiplicity of λ as an eigenvalue of $A(n, 4)$ is obtained by Lemma 5 as

$$n(n-1)(n-2)(n-3)\frac{(\mathbf{w})_1^2}{\mathbf{w}^\top \mathbf{w}}.$$

However, if the multiplicity of λ for Q is larger than 1, we need to find an orthogonal set

of eigenvectors for λ and P. Besides -4, only $2n-8$ is such an eigenvalue. Set

$$\mathbf{w}_1 = S\left[12, -2, 4, -3, 2, 6, -2, 0, 2, 0, 0, 2, 0, -1, -2, 0, 0, 1, 0, 1\right]^{\top},$$
$$\mathbf{w}_2 = S\left[12, -4, -4, 0, 4, 6, -4, 3, 2, -2, 0, -2, 0, -1, -1, 0, 0, 0, 1, 2\right]^{\top}.$$

Now the vectors $\mathbf{u}_1, \mathbf{u}_2$ with $\mathbf{u}_1 = \mathbf{w}_1$ and $\mathbf{u}_2 = \mathbf{w}_2 - \frac{\mathbf{w}_2^{\top}\mathbf{w}_2}{\mathbf{w}_1^{\top}\mathbf{w}_2}\mathbf{w}_1$ forms a set of orthogonal eigenvectors of P for $2n-8$. By Lemma 5, the multiplicity of $2n-8$ as an eigenvalue of $A(n,4)$ is

$$n(n-1)(n-2)(n-3)\left(\frac{(\mathbf{u}_1)_1^2}{\mathbf{u}_1^{\top}\mathbf{u}_1} + \frac{(\mathbf{u}_2)_1^2}{\mathbf{u}_2^{\top}\mathbf{u}_2}\right)$$
$$= n(n-1)(n-2)(n-3)\left(\frac{12}{6-15n+5n^2} + \frac{(n^2-3n+6)^2}{2n(n-1)(n-2)(n-3)(5n^2-15n+6)}\right)$$
$$= \frac{5n(n-3)}{2} + 3.$$

Now, the multiplicity of -4 is $n(n-1)(n-2)(n-3)$ minus the sum of all the multiplicities of the rest of the eigenvalues. $\qquad\square$

4.3 Eigenvalues of $A(n,k)$ for $k = 5, 6, 7$

In a similar fashion as for $A(n,3)$ and $A(n,4)$, we are able to determine the complete set of eigenvalues of more families of the arrangement graphs. The eigenvalues of $A(n,k)$ for $k = 5, 6, 7$ are given in Tables 5, 6 and 7. We would like to point out that by using our method it is possible to compute the eigenvalues of the graphs $A(n,k)$ for some larger values of $k > 7$.

$(-5)^{[n^5-15n^4+75n^3-145n^2+89n-1]}$	$(n-13)^{[n(n-1)(n-2)(n-7)/24]}$	$(n-12)^{[n(n-1)(n-3)(n-6)/2]}$
$(n-11)^{[n(n-5)(7n)^2-35n+37)/6]}$	$(n-10)^{[(n-1)(n-2)(n-3)(n-4)/6]}$	$(n-9)^{[5n(n-3)(n^2-7n+8)/4]}$
$(n-8)^{[n(n-1)(n-2)(n-7)/6]}$	$(n-7)^{[n(n-1)(7n)^2-63n+131)/6]}$	$(n-6)^{[n(n-2)(n-3)(n-5)/2]}$
$(n-5)^{[(n-1)(n-2)(n-3)(n-4)/24]}$	$(2n-19)^{[n(n-1)(n-5)/6]}$	$(2n-17)^{[4n(n-2)(n-4)/3]}$
$(2n-15)^{[(n-1)(n-2)(n-3)]}$	$(2n-14)^{[n(7n)^2-42n+50)/2]}$	$(2n-12)^{[2n(n-2)(n-4)]}$
$(2n-11)^{[5n(n-1)(n-5)/6]}$	$(2n-10)^{[(n-2)(7n^2-28n+6)/3]}$	$(3n-23)^{[n(n-3)/2]}$
$(3n-20)^{[2(n-1)(n-2)]}$	$(3n-18)^{[2n(n-3)]}$	$(3n-15)^{[(11n^2-33n+12)/2]}$
$(4n-25)^{[n-1]}$	$(4n-20)^{[4n-4]}$	$(5n-25)^{[1]}$

Table 5: The eigenvalues of $A(n,5)$

$(-6)^{[n^6-21n^5+160n^4-545n^3+814n^2-415n+1]}$	$(n-16)^{[n(n-1)(n-2)(n-3)(n-9)/120]}$	$(n-15)^{[n(n-1)(n-2)(n-4)(n-8)/6]}$
$(n-14)^{[n(n-1)(n-7)(7n^2-49n+78)/8]}$	$(n-13)^{[n(n-3)(n-6)(n^2-6n+6)]}$	$(n-12)^{[(n-1)(n-2)(n-5)(n^2-7n+2)/4]}$
$(n-11)^{[n(n-4)(7n^3-77n^2+217n-162)/4]}$	$(n-10)^{[n(n-1)(n-3)(n-4)(n-7)/4]}$	$(n-9)^{[n(n-1)(n-2)(n^2-12n+34)]}$
$(n-8)^{[n(n-1)(n-3)(7n^2-77n+202)/8]}$	$(n-7)^{[n(n-2)(n-3)(n-4)(n-6)/6]}$	$(n-6)^{[(n-1)(n-2)(n-3)(n-4)(n-5)/120]}$
$(2n-24)^{[n(n-1)(n-2)(n-7)/24]}$	$(2n-22)^{[5n(n-1)(n-3)(n-6)/8]}$	$(2n-20)^{[n(n-5)(2n-3)(2n-7)/2]}$
$(2n-18)^{[(7n^3-63n^2+136n-40)(n-1)/4]}$	$(2n-17)^{[2n(n-2)(n-3)(n-5)]}$	$(2n-16)^{[15n(n-1)(n-3)(n-6)/8]}$
$(2n-15)^{[4n(n-1)(n-4)(n-5)/3]}$	$(2n-14)^{[n(n-2)(13n^2-104n+171)/8]}$	$(2n-13)^{[2n(n-1)(n-3)(n-6)]}$
$(2n-12)^{[(7n^4-70n^3+217n^2-210n+20)/4]}$	$(3n-30)^{[n(n-1)(n-5)/6]}$	$(3n-27)^{[5n(n-4)(n-2)/3]}$
$(3n-23)^{[3n(n-2)(n-2)]}$	$(3n-24)^{[5(n-4)(n-1)^2/2]}$	$(3n-21)^{[10n(n-2)(n-4)/3]}$
$(3n-20)^{[3n(n-1)(n-5)/2]}$	$(3n-18)^{[(n-4)(47n^2-94n+15)/6]}$	$(4n-34)^{[n(n-3)/2]}$
$(4n-30)^{[5(n-1)(n-2)/2]}$	$(4n-28)^{[5n(n-3)/2]}$	$(4n-24)^{[(19n^2-57n+20)/2]}$
$(5n-36)^{[n-1]}$	$(5n-30)^{[5n-5]}$	$(6n-36)^{[1]}$

Table 6: The eigenvalues of $A(n,6)$

$(-7)^{[n^7-28n^6+301n^5-1575n^4+4179n^3-5243n^2+2372n-1]}$	$(n-19)^{[n(n-1)(n-2)(n-3)(n-4)(n-11)/720]}$
$(n-18)^{[n(n-1)(n-2)(n-3)(n-5)(n-10)/24]}$	$(n-17)^{[n(n-1)(n-2)(n-9)(23n^2-207n+439)/60]}$
$(n-16)^{[n(n-1)(n-8)(83n^3-996n^2+3691n-4182)/72]}$	$(n-15)^{[n(n-7)(11n^4-154n^3+739n^2-1400n+844)/16]}$
$(n-14)^{[(n-1)(n-2)(n-6)(13n^3-156n^2+401n-10)/20]}$	$(n-13)^{[7n(n-5)(n^4-16n^3+80n^2-151n+89)/6]}$
$(n-12)^{[n(n-1)(n-4)(13n^3-208n^2+1003n-1348)/20]}$	$(n-11)^{[(11n^2-165n+592)(n-1)(n-2)(n-3)n/16]}$
$(n-10)^{[n(n-1)(n-2)(83n^3-1494n^2+8593n-15822)/72]}$	$(n-9)^{[n(n-1)(n-3)(n-4)(23n^2-299n+941)/60]}$
$(n-8)^{[n(n-2)(n-3)(n-4)(n-5)(n-7)/24]}$	$(n-7)^{[(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)/720]}$
$(2n-29)^{[n(n-1)(n-2)(n-3)(n-9)/120]}$	$(2n-27)^{[n(n-1)(n-2)(n-4)(n-8)/5]}$
$(2n-25)^{[n(n-1)(n-7)(8n^2-56n+89)/6]}$	$(2n-23)^{[n(n-3)(n-6)(17n^2-102n+101)/8]}$
$(2n-22)^{[n(n-1)(n-2)(11n^2-132n+367)/10]}$	$(2n-21)^{[5(n-1)(n-3)(n-4)(3n^2-21n+2)/8]}$
$(2n-20)^{[5n(n-2)(n-4)(n^2-9n+15)/3]}$	$(2n-19)^{[7(n-7)(11n^2-77n+122)n(n-1)/20]}$
$(2n-18)^{[n(n-1)(n-3)(n-4)(n-7)]}$	$(2n-17)^{[7n(n-1)(7n^3-98n^2+427n-568)/20]}$
$(2n-16)^{[5n(n-2)(n-4)(n^2-9n+11)/3]}$	$(2n-15)^{[7n(n-1)(n-7)(3n^2-21n+34)/8]}$
$(2n-14)^{[(n-3)(11n^4-132n^3+469n^2-438n+20)/10]}$	$(3n-37)^{[n(n-1)(n-2)(n-7)/24]}$
$(3n-34)^{[3n(n-1)(n-3)(n-6)/4]}$	$(3n-31)^{[n(n-5)(73n^2-365n+382)/24]}$
$(3n-30)^{[n(n-1)(n-2)(n-7)/4]}$	$(3n-29)^{[7n(n-1)(n-3)(n-6)/4]}$
$(3n-28)^{[5(n-1)(n-2)(n-3)(n-4)/6]}$	$(3n-27)^{[5n(n-3)(5n^2-35n+44)/4]}$
$(3n-26)^{[7n(n-1)(n-3)(n-6)n/4]}$	$(3n-25)^{[7n(n-1)(n^2-9n+19)/2]}$
$(3n-24)^{[5n(n-2)(n-3)(n-5)/2]}$	$(3n-22)^{[7n(n-1)(n-2)(n-7)/12]}$
$(3n-23)^{[35n(n-1)(n-3)(n-6)/8]}$	$(3n-21)^{[(75n^4-750n^3+2233n^2-1958n+120)/8]}$
$(4n-43)^{[n(n-1)(n-5)/6]}$	$(4n-39)^{[2n(n-2)(n-4)]}$
$(4n-36)^{[n(n-1)(n-5)]}$	$(4n-35)^{[5(n-1)(n-2)(n-3)/2]}$
$(4n-34)^{[14n(n-2)(n-4)/3]}$	$(4n-32)^{[5n(n-2)(n-4)]}$
$(4n-31)^{[7n(n-1)(n-5)/3]}$	$(4n-28)^{[(52n^3-312n^2+425n^2-60)/3]}$
$(5n-47)^{[n(n-3)/2]}$	$(5n-42)^{[3(n-1)(n-2)]}$
$(5n-40)^{[3n(n-3)]}$	$(5n-35)^{[(29n^2-87n+30)/2]}$
$(6n-49)^{[n-1]}$	$(6n-42)^{[6(n-1)]}$
$(7n-49)^{[1]}$	

Table 7: The eigenvalues of $A(n,7)$

4.4 Connection with the Johnson graph

We first recall the eigenvalues of the Johnson graph (see [2, p. 179]).

Lemma 12. *The eigenvalues of $J(n,k)$ are*

$$(k-i)(n-k-i) - i \quad \text{with multiplicity} \quad \binom{n}{i} - \binom{n}{i-1}, \quad \text{for} \ \ i = 0, \ldots, k.$$

Let $\alpha_1, \ldots, \alpha_{\binom{n}{k}}$ be all the k-subsets of $[n]$. Let V_i be the set of all permutations of α_i. If $|\alpha_i \cap \alpha_j| = k - 1$, then each $\pi \in V_i$ is adjacent to exactly one $q \in V_j$ and if $|\alpha_i \cap \alpha_j| < k - 1$, then no vertex of V_i has a neighbor in V_j. It follows that $(V_1, \ldots, V_n)$ is an equitable partition of $A(n,k)$ where its quotient matrix is the adjacency matrix of $J(n,k)$. So we come up with the following.

Proposition 13. *For every $i = 0, \ldots, k$, $(k-i)(n-k-i) - i$ is an eigenvalue of $A(n,k)$ with multiplicity at least $\binom{n}{i} - \binom{n}{i-1}$.*

4.5 The smallest eigenvalue

From Proposition 13 it follows that $-k$ is an eigenvalue of $A(n,k)$ with multiplicity at least $\binom{n}{k} - \binom{n}{k-1}$. In this section we establish that $-k$ is indeed the smallest eigenvalue, but with a much larger multiplicity.

Theorem 14. *If $n \geqslant 2k$, then $-k$ is the smallest eigenvalue of $A(n,k)$ with multiplicity at least $n(n-1)\cdots(n-k+2)(n-2k+1)$.*

Proof. Consider the complete k-partite graph $K_{n,\ldots,n}$ and let $a_{i1}, \ldots, a_{in}$ be the vertices of the ith part for $i = 1, \ldots, k$. We remove the edges $\{a_{ij}a_{rj} \mid 1 \leqslant j \leqslant n, 1 \leqslant i, r \leqslant k\}$ and denote the resulting graph by $H_{n,k}$. A set of vertices $\{a_{1j_1}, \ldots, a_{kj_k}\}$ forms a k-clique in $H_{n,k}$ if and only if $(j_1, \ldots, j_k)$ is a k-permutation of $[n]$. Hence there is a one-to-one correspondence between the k-cliques of $H_{n,k}$ and the k-permutations of $[n]$. For a k-permutation π we denote the corresponding k-clique by $C(\pi)$. Now, two k-permutations π_1 and π_2 are adjacent in $A(n,k)$ if and only if $|C(\pi_1) \cap C(\pi_2)| = k - 1$. Let M denote the incidence matrix of $(k-1)$-cliques versus k-cliques of $H_{n,k}$, that is the rows and columns of M are indexed by the $(k-1)$-cliques and the k-cliques of $H_{n,k}$, respectively, where $M(C, C') = 1$ if $C \subset C'$ and $M(C, C') = 0$ otherwise. It is straightforward to see that the adjacency matrix of $A(n,k)$ is equal to $M^\top M - kI$. It follows that $-k$ is the smallest eigenvalue of $A(n,k)$. As M has $n(n-1)\cdots(n-k+2)k$ rows and $n(n-1)\cdots(n-k+1)$ columns, it follows that the multiplicity of $-k$ as an eigenvalue of $A(n,k)$ is at least $n(n-1)\cdots(n-k+2)(n-2k+1)$. $\qquad\square$

We close the paper with some open problems on the eigenvalues of the arrangement graphs.

The main problem we would like to put forward is the following:

Problem 1. What are the eigenvalues of the arrangement graph $A(n,k)$?

Our observations allow us to narrow down Problem 1 to the following more specific conjectures:

Conjecture 2. The eigenvalues of the arrangement graphs $A(n, k)$ consist entirely of integers.[1]

Conjecture 3. For any integer k, there is an integer n_0 such that for all $n \geqslant n_0$, $-k$ is the only negative eigenvalue of $A(n, k)$.

Acknowledgments

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[1]Recently, Conjecture 2 was settled affirmatively in [3].

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