A combinatorial proof for Cayley's identity

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Abstract

In a recent paper, Caracciolo, Sokal and Sportiello presented, *inter alia*, an algebraic/combinatorial proof for Cayley's identity. The purpose of the present paper is to give a "purely combinatorial" proof for this identity; i.e., a proof involving only combinatorial arguments. Since these arguments eventually employ a generalization of Laplace's Theorem, we present a "purely combinatorial" for this theorem, too.

1 Introduction

For $n \in \mathbb{N}$, denote by $[n]$ the set $\{1, 2, \ldots, n\}$ and let $X = X_n = (x_{i,j})_{(i,j)\in[n]\times[n]}$ be an $n \times n$ matrix of indeterminates. For $I \subseteq [n]$ and $J \subseteq [n]$, we denote

- the *submatrix* of X corresponding to the rows $i \in I$ and the columns $j \in J$ by $X_{I,J}$,
- the *complementary submatrix* of $X_{I,J}$ (which corresponds to the rows $i \in \overline{I} := [n] \setminus I$ and the columns $j \in \overline{J} := [n] \setminus J$) by $X_{\overline{I},\overline{J}}$.

Let $M = \{x_1 \leq x_2 \leq \cdots \leq x_m\}$ be a finite ordered set, and let $S = \{x_{i_1}, \ldots, x_{i_k}\} \subseteq M$ be a subset of M. We define

$$\operatorname{sgn}(S \trianglelefteq M) := (-1)^{\sum_{j=1}^{k} i_j}.$$

As pointed out in [2, Section 2.6], the following identity is conventionally but erroneously attributed to Cayley. (Muir [4, vol. 4, p. 479] attributes this identity to Vivanti [6].)

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Theorem 1 (Cayley's Identity)**.** *Consider* $X = (x_{i,j})_{(i,j)\in[n]\times[n]}$, *and let* $\partial = \left(\frac{\partial}{\partial x_{i,j}}\right)$ *be the corresponding* $n \times n$ *matrix of partial derivatives*[1]*. Let* $I, J \subseteq [n]$ *with* $|I| = |J| = k$*. Then we have for* $s \in \mathbb{N}$*:*

$$\det(\partial_{I,J})(\det(X))^s = \\ s \cdot (s+1) \cdots (s+k-1) \cdot (\det(X))^{s-1} \cdot \operatorname{sgn}(I \trianglelefteq [n]) \cdot \operatorname{sgn}(J \trianglelefteq [n]) \cdot \det\left(X_{\overline{I},\overline{J}}\right). \quad (1)$$

By the alternating property of the determinant, Cayley's Identity is in fact equivalent to the following special case of (1).

Corollary 1 (Vivanti's Theorem)**.** *Specialize* $I = J = [k]$ *for some* $k \leq n$ *in Theorem 1. Then we have for* $s \in \mathbb{N}$*:*

$$\det\left(\partial_{[k],[k]}\right)(\det(X))^s = s \cdot (s+1) \cdots (s+k-1) \cdot (\det(X))^{s-1} \cdot \det\left(X_{\overline{[k]},\overline{[k]}}\right). \quad (2)$$

2 Combinatorial proof of Vivanti's Theorem

We may *view* the determinant of X as the *generating function* of all permutations π in $\mathfrak{S}_n$, where the (signed) weight of a permutation π is given as $\omega(\pi) := \operatorname{sgn}\pi \cdot \prod_{i=1}^{n} x_{i,\pi(i)}$:

$$\det(X) = \sum_{\pi \in \mathfrak{S}_n} \omega(\pi).$$

2.1 View permutations as perfect matchings

For our considerations, it is convenient to *view* a permutation $\pi \in \mathfrak{S}_n$ as a *perfect matching* m_π of the complete bipartite graph $K_{n,n}$, where the vertices consist of two copies of $[n]$ which are arranged in their natural order; see Figure 1 for an illustration of this simple idea: In the picture, we show the *domain* of π as *lower* vertices and the *image* of π as *upper* vertices. It is easy to see that the edges of such perfect matching can be drawn in a way such that all *intersections* are of precisely two (and not more) edges, and that the number of these intersections equals the number of *inversions* of π, whence the sign of π is

$$\operatorname{sgn}(\pi) = (-1)^{\#(\text{intersections in } m_\pi)}.$$

This simple visualization of permutations and their inversions is already used in [1, §15, p.32]: We call it the *permutation diagram.* So assigning weight $x_{i,j}$ to the edge pointing from lower vertex i to upper vertex j and defining the weight $\omega(m_\pi)$ of the permutation diagram m_π to be the product of the edges belonging to m_π, we may write

$$\omega(\pi) = (-1)^{\#(\text{intersections in } m_\pi)} \cdot \omega(m_\pi).$$

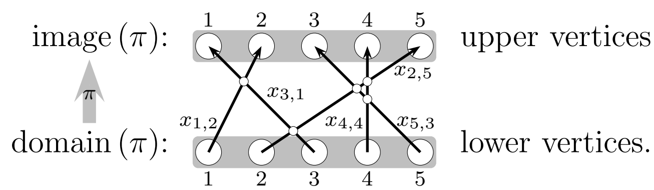
[1] ∂ is also known as *Cayley's* Ω*–process.*

Figure 1: Illustration: View permutations as perfect matchings.
View the permutation $\pi = 25143$ as the corresponding perfect matching m_π in the complete bipartite graph $K_{5,5}$.The intersections of edges are indicated by small circles; they correspond bijectively to π's inversions:

$$\#\left(\text{inversions of } \pi\right) = \left|\left\{(1,3),(2,3),(2,4),(2,5),(4,5)\right\}\right| = 5.$$

Assigning weight $x_{i,j}$ to the edge pointing from lower vertex i to upper vertex j gives the contribution of the permutation π to the determinant of X_5:

$$\omega\left(\pi\right) = \left(-1\right)^5 \cdot x_{1,2} \cdot x_{2,5} \cdot x_{3,1} \cdot x_{4,4} \cdot x_{5,3}.$$



Given this view, the combinatorial interpretation of the s-th power of the determinant $\det\left(X\right)$ is obvious: It is the generating function of all s-tuples $m = \left(m_{\pi_1}, \dots, m_{\pi_s}\right)$ of permutation diagrams, where the (signed) weight of such s-tuple m is given as

$$\omega\left(m\right) = \prod_{i=1}^{s} \left(-1\right)^{\#\left(\text{intersections in } m_{\pi_i}\right)} \cdot \omega\left(m_{\pi_i}\right).$$

(See Figure 2 for an illustration.)

2.2 Action of the determinant of partial derivatives

Next we need to describe combinatorially the *action* of the determinant $\det\left(\partial_{[k],[k]}\right)$ of partial derivatives. Let $m = \left(m_{\pi_1}, \dots, m_{\pi_s}\right)$ be an s-tuple of permutation diagrams counted in the generating function $\left(\det\left(X\right)\right)^s$, and let $\tau \in \mathfrak{S}_k$: Then the summand

$$\partial_\tau := \operatorname{sgn}\left(\tau\right) \cdot \prod_{i=1}^{k} \frac{\partial}{\partial x_{i,\tau(i)}}$$

applied to $\omega\left(m\right)$ yields

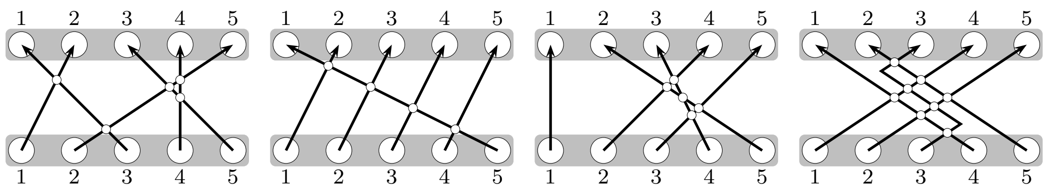
$$\operatorname{sgn}\left(\tau\right) \cdot \left(\prod_{i=1}^{k} \frac{\partial}{\partial x_{i,\tau(i)}}\right) \omega\left(m\right) = \operatorname{sgn}\left(\tau\right) \cdot c_{\tau,m} \cdot \frac{\omega\left(m\right)}{\prod_{i=1}^{k} x_{i,\tau(i)}},$$

where $c_{\tau,m}$ is the number of ways to *choose* the set of k edges $\left\{\left(i \to \tau\left(i\right)\right) : i \in [k]\right\}$ from *all* the edges in m (this number, of course, might be zero). We may visualize the action of δ_τ as "erasing the edges constituting τ in m"; see Figure 3 for an illustration.

Figure 2: Illustration: Objects counted by the generating function of a power of a determinant.
For $n = 5$, the picture shows a typical object of weight

$$+x_{1,1}x_{1,2}^2x_{1,4}x_{2,3}x_{2,4}x_{2,5}^2x_{3,1}x_{3,3}x_{3,4}x_{3,5}x_{4,1}x_{4,3}x_{4,4}x_{4,5}x_{5,1}x_{5,2}^2x_{5,3},$$

which is counted by the generating function $\det(X)^4$. (The edge connecting lower vertex 3 to upper vertex 3 in the 4–th (right–most) matching is drawn as zigzag-line, just to avoid intersections of more than two edges in a single point.)



Hence we have:

$$\det\left(\partial_{[k],[k]}\right)\left(\det\left(X\right)\right)^s = \sum_{m\in\mathfrak{S}_n^s}\omega\left(m\right)\sum_{\tau\in\mathfrak{S}_k} c_{\tau,m}\cdot\frac{\operatorname{sgn}\left(\tau\right)}{\prod_{i=1}^k x_{i,\tau(i)}}. \tag{3}$$

2.3 Double counting

For our purposes, it is convenient to interchange the summation in (3). This application of *double counting* amounts here to a simple change of view: Instead of counting the ways to *choose* the set of edges corresponding to τ from all the edges corresponding to some *fixed* s-tuple m, we fix τ and consider the set of m's from which τ's edges might be chosen. This will involve two considerations:

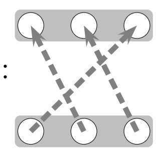
- In how many ways can the edges corresponding to τ be *distributed* on s copies of the bipartite graph $K_{n,n}$?
- For each such distribution, what is the set of compatible s-tuples of permutation diagrams?

For example, if $k = 3$ and $s = 4$ (as in Figure 3), there clearly

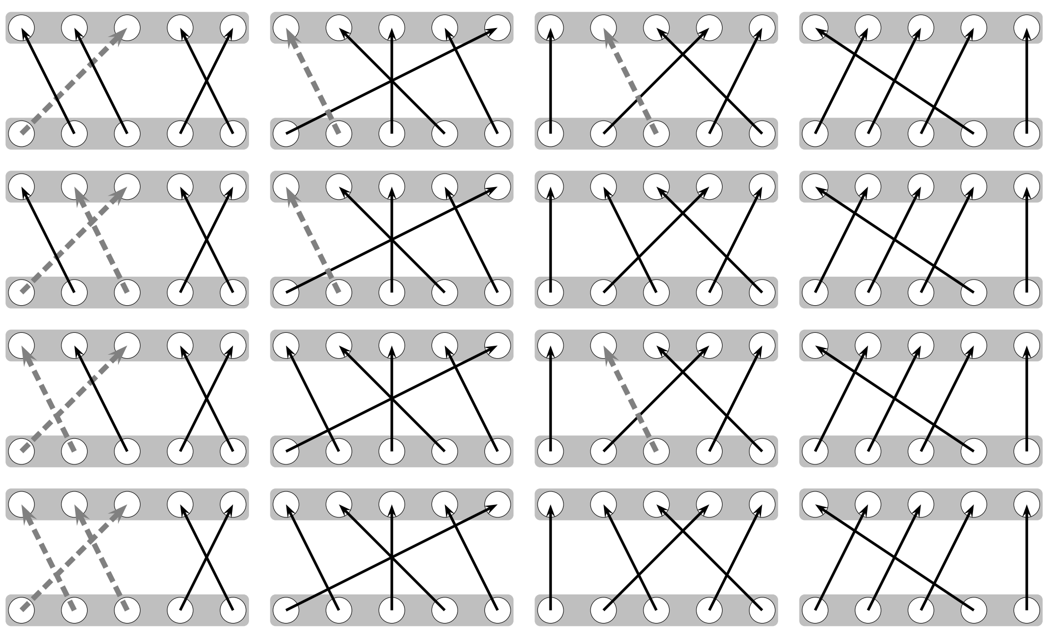
- is 1 way to distribute the three edges on a *single* copy of the 4 bipartite graphs (see the fourth row of pictures in Figure 3), and there are 4 ways to choose such single copy,
- are 3 ways to distribute the three edges on *precisely two* copies of the 4 bipartite graphs (see the second and third row of pictures in Figure 3), and there are $4 \cdot 3$ ways to choose such pair of copies (whose order is relevant),

Figure 3: Let $n = 5$, $s = 4$ and $k = 3$ in Corollary 1. The picture shows four possible ways of "erasing" the edges constituting $\tau \in \mathfrak{S}_3$ from the 4-tuple $(m_{\pi_1}, m_{\pi_2}, m_{\pi_3}, m_{\pi_4})$ of matchings, where $(\pi_1, \pi_2, \pi_3, \pi_4) \in \mathfrak{S}_5^4$ is $(31254, 51324, 14253, 23415)$. The erased edges are shown as grey dashed lines.

$\tau = (312) \in \mathfrak{S}_3$:



$$\delta_\tau = \frac{\partial}{\partial x_{1,3}} \cdot \frac{\partial}{\partial x_{2,1}} \cdot \frac{\partial}{\partial x_{3,2}}$$



- is 1 way to distribute the three edges on *precisely three* copies of the 4 bipartite graphs (see the first row of pictures in Figure 3), and there are $4 \cdot 3 \cdot 2$ ways to choose such triple of copies (whose order is relevant).

2.4 Partitioned permutations

A distribution of the edges corresponding to $\tau \in \mathfrak{S}_k$ on s copies of the bipartite graph $K_{n,n}$ may be viewed (see Figure 3)

- as an s-tuple of *matchings* (some of which may be empty; to stress the fact that these matchings are *not* perfect, we also call them *partial* matchings) of $K_{k,k}$
- such that the union of these s *partial* matchings gives the *perfect matching* m_τ of $K_{k,k}$.

Clearly, to each such partial matching corresponds a *partial permutation* τ_i, which we may write in two-line notation as follows:

- the lower line shows the *domain* of τ_i in its natural order,
- the upper line shows the *image* of τ_i,
- the *ordering* of the upper line represents the permutation τ_i.

We say that each of these τ_i is a *partial permutation* of τ, and that τ is a *partitioned permutation*. We write in short:

$$\tau = \tau_1 \star \tau_2 \star \ldots \star \tau_s.$$

For example, the rows of pictures in Figure 3 correspond to the partitioned permutations (written in the aforementioned two-line notation)

- $\binom{3}{1} \star \binom{1}{2} \star \binom{2}{3} \star ()$ for the first row,
- $\binom{3\,2}{1\,3} \star \binom{1}{2} \star () \star ()$ for the second row,
- $\binom{3\,1}{1\,2} \star () \star \binom{2}{3} \star ()$ for the third row,
- $\binom{3\,1\,2}{1\,2\,3} \star () \star () \star ()$ for the fourth row.

2.5 Equivalence relation for partitioned permutations

For any partitioned permutation $\tau = \tau_1 \star \tau_2 \star \ldots \star \tau_s$, consider the s-tuple of the *upper rows* (in the aforementioned two-line notation) *only*: We call this s-tuple of *permutation words* the *partition scheme* of τ and denote it by $[\tau]$. We say that $\tau = \tau_1 \star \tau_2 \star \ldots \star \tau_s$ *complies* to its partition scheme $[\tau] = [\tau_1 \star \tau_2 \star \ldots \star \tau_s]$ and denote this by $\tau \subseteq [\tau_1 \star \tau_2 \star \ldots \star \tau_s]$.

Now consider the following equivalence relation on the set of partitioned permutations:

$$\mu = \mu_1 \star \ldots \star \mu_s \sim \nu = \nu_1 \star \ldots \star \nu_s :\Longleftrightarrow [\mu] = [\nu] .$$

By definition, the corresponding equivalence classes are *indexed* by a partition scheme, and $\mu = \mu_1 \star \mu_2 \star \ldots \star \mu_s$ belongs to the equivalence class of $\tau = \tau_1 \star \tau_2 \star \ldots \star \tau_s$ iff $\mu \subseteq [\tau]$. (For $s > 1$, a partitioned permutation τ is *not* uniquely determined by $[\tau]$.)

It is elementary to determine the *number* of these equivalence classes: Think of filling in *successively* the entries $1, 2, \ldots, k$ into the partition scheme $[\tau_1 \star \tau_2 \star \ldots \star \tau_s]$. Starting with the empty scheme $[\star \ldots \star \star]$, we find s possibilities to fill in 1, giving $[\star \ldots \star 1 \star \ldots \star]$. Now there are $s+1$ possibilities to fill in 2, etc.: So the number of these equivalence classes is $s \cdot (s+1) \cdots (s+k-1)$, which is precisely the factor in (2). Our proof will be complete if we manage to show that the generating functions of *each* of these equivalence classes are the *same*, namely

$$(\det(X))^{s-1} \cdot \det\left(X_{\overline{I},\overline{J}}\right).$$

2.6 Accounting for the signs

A necessary first step for this task is to investigate how the sign of a permutation π is changed by *removing* a given partial permutation π_1: We view this as *erasing* all the edges belonging to π_1's permutation diagram m_{π_1} from π's permutation diagram m_π; see again Figure 3.

Lemma 1. *Let $\pi \in \mathfrak{S}_n$ be a partitioned permutation $\pi = \pi_1 \star \pi_2$, where π_1 is the partial permutation*

$$\pi_1 = \begin{pmatrix} \pi(i_1) & \pi(i_2) & \cdots & \pi(i_k) \\ i_1 & i_2 & \cdots & i_k \end{pmatrix}$$

(with $\{i_1 \leq i_2 \leq \cdots \leq i_k\} \subseteq [n]$). Clearly, π_2 is the permutation corresponding to the matching m_π with edges $(i_1, \pi(i_1)), (i_2, \pi(i_2)), \ldots, (i_k, \pi(i_k))$ erased, which we also denote by $\pi \setminus \pi_1$. Then we have

$$\operatorname{sgn}(\pi) = (-1)^{\sum_{j=1}^{k} \pi(i_k) - i_k} \cdot \operatorname{sgn}(\pi_1) \cdot \operatorname{sgn}(\pi_2).$$

If we denote $I = \{i_1, \ldots, i_k\}$ and $J = \{\pi(i_1), \ldots, \pi(i_k)\}$, we may rewrite this as

$$\operatorname{sgn}(\pi) = \operatorname{sgn}(I \trianglelefteq [n]) \cdot \operatorname{sgn}(J \trianglelefteq [n]) \cdot \operatorname{sgn}(\pi_1) \cdot \operatorname{sgn}(\pi \setminus \pi_1). \tag{4}$$

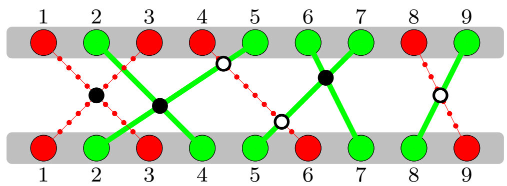
Proof. View the permutation diagram m_π of $\pi = \pi_1 \star \pi_2$ as a *bicoloured* perfect matching of $K_{n,n}$, where the edges and vertices corresponding to π_1 are coloured green and the edges and vertices corresponding to π_2 are coloured red (see Figure 4). Clearly,

- the set I is the set of (the labels of the) *lower* green vertices,
- the set J is the set of (the labels of the) *upper* green vertices.

Figure 4: Illustration: View partitioned permutations as bicoloured permutation diagrams.
The picture shows the *bicoloured* permutation diagram m_π corresponding to the partitioned permutation $\pi = \pi_1 \star \pi_2$, where

$$\pi_1 = \binom{52769}{24578} \text{ and } \pi_2 = \binom{3148}{1369}.$$

The edges corresponding to π_1 are shown as green (solid) lines, the edges corresponding to π_2 are shown as red (dotted) lines. The *inactive intersections* (of green/red edges) are indicated by small white circles, the other intersections (of green/green or red/red edges) are indicated by small black circles.



Note that the intersections in m_π come in three flavours:

- intersections of two green edges (which are accounted for in $\operatorname{sgn}(\pi_1)$),
- intersections of two red edges (which are accounted for in $\operatorname{sgn}(\pi_2)$),
- intersections of a green and a red edge: Since they do not contribute to the signs, let us call them the *inactive intersections*.

We will prove (4) by showing the following two statements:

1. The parity of the number of inactive intersections depends *only* on the sets I and J, i.e., on the *positions* of the (lower and upper) green vertices.
2. The number of inactive intersections equals $\sum_{j=1}^{k} |\pi(i_j) - i_j|$ (which, of course, is equal to $\sum_{j=1}^{k} \pi(i_j) + \sum_{j=1}^{k} i_j$ modulo 2) in the case that $\pi_1 : I \to J$ and $\pi_2 : ([n] \setminus I) \to ([n] \setminus J)$ are the unique *order–preserving* bijections (i.e, there are *only* inactive intersections; see Figure 6).

For the first statement, consider two edges e_1 and e_2 of the *same* colour, where $e_1 = (a, d)$ and $e_2 = (b, c)$ connect lower vertices a and b with upper vertices d and c, respectively, and look at the effect of replacing these edges by $e_1' = (a, c)$ and $e_2' = (b, d)$: It is easy to see that an edge e of the *other* colour

- has an *even* number of intersections with e_1 and e_2 (i.e., intersects neither of them or both of them) if and only if it has an *even* number of intersections with e_1' and e_2',

Figure 5: The left picture shows the green edges (shown as solid lines) $e_1 = (a, d)$ and $e_2 = (b, c)$, which are replaced by the green edges $e'_1 = (a, c)$ and $e'_2 = (b, d)$ in the right picture: Observe that for every red edge e (shown as dotted line) the *parities* of the numbers of intersections with $\{e_1, e_2\}$ and $\{e'_1, e'_2\}$ are the same. (Some edges are drawn as curved lines here for graphical reasons.)

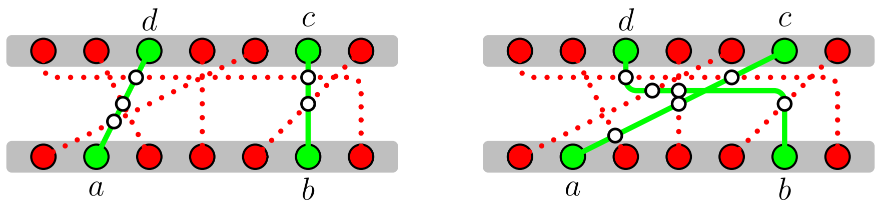
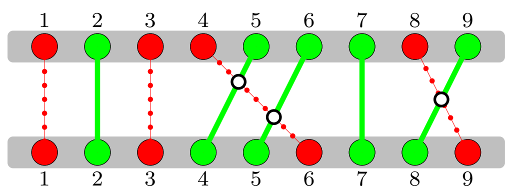


Figure 6: Partitioned permutation $\pi' = \pi'_1 \star \pi'_2$, where both π'_1 and π'_2 have no inversions; i.e., there are only *inactive* intersections in the bicoloured permutation diagram $m_{\pi'_1 \star \pi'_2}$.



- has an *odd* number of intersections with e_1 and e_2 (i.e., intersects exactly one of them) if and only if it has an *odd* number of intersections with e'_1 and e'_2.

See Figure 5 for an illustration: Note that replacing edges e_1, e_2 by e'_1, e'_2 corresponds to multiplying π_1 with the transposition (c, d), and by multiplying with a sequence of appropriate transpositions, we can remove all inversions from π_1 and π_2; and this operation does not change the parity of the number of inactive intersections.

For the second statement, simply have a look at Figure 6 and observe that in the case where neither π_1 nor π_2 have inversions, $|\pi(i_j) - i_j|$ is the number of intersections of the j–the green edge with red edges. □

2.7 Sums of (signed) products of minors

Now consider a fixed equivalence class in the sense of section 2.5, which is indexed by a *partition-scheme*

$$[\tau_1 \star \tau_2 \star \cdots \star \tau_s].$$

We want to compute the generating function $G_{[\tau]}$ of this equivalence class: Clearly, we may concentrate on the *nonempty* partial permutations; so w.l.o.g. we have to consider the partition-scheme

$$[\tau_1 \star \tau_2 \star \cdots \star \tau_m]$$

which consists only of *nonempty* partial permutations τ_j for $1 \leq j \leq m \leq s$. For any $\sigma \in \mathfrak{S}_k$ with $\sigma \subseteq [\tau_1 \star \tau_2 \star \cdots \star \tau_m]$, such partition scheme corresponds to a unique *ordered partition* of the image of σ:

$$\text{image}\,(\sigma) = [k] = (\text{image}\,(\tau_1)) \dot\cup (\text{image}\,(\tau_2)) \dot\cup \cdots \dot\cup (\text{image}\,(\tau_m)) = J_1 \dot\cup J_2 \dot\cup \cdots \dot\cup J_m,$$

and any specification of a *compatible ordered partition* $\mathbf{I}_{[J]} = (I_1, I_2, \ldots, I_m)$, i.e.,

$$[k] = I_1 \dot\cup I_2 \dot\cup \cdots \dot\cup I_m \text{ where } |I_l| = |J_l|\,, l = 1, \ldots, m,$$

uniquely determines such σ, which we denote by $\sigma\left(\mathbf{I}_{[J]}, [\tau]\right)$.

Equation (4) gives the sign-change caused by erasing the edges corresponding to τ_l (with respect to *any* permutation in $\mathfrak{S}_n$ which contains τ_l as a partial permutation), whence we can write the generating function as

$$G_{[\tau]} = \det(X)^{s-m} \times \sum_{\mathbf{I}_{[J]}} \operatorname{sgn}\left(\sigma\left(\mathbf{I}_{[J]}, [\tau]\right)\right) \cdot \prod_{l=1}^{m} \left(\operatorname{sgn}(\tau_l) \cdot \operatorname{sgn}(I_l \trianglelefteq [n]) \cdot \operatorname{sgn}(J_l \trianglelefteq [n]) \cdot \det\left(X_{\overline{I_l}, \overline{J_l}}\right)\right),$$

where the sum is over all compatible partitions $\mathbf{I}_{[J]}$. (The factor $\operatorname{sgn}\left(\sigma\left(\mathbf{I}_{[J]}, [\tau]\right)\right)$ comes from the *determinant* of partial derivatives.) Clearly,

$$\left(\prod_{l=1}^{m} \operatorname{sgn}(I_l \trianglelefteq [n])\right) \cdot \left(\prod_{l=1}^{m} \operatorname{sgn}(J_l \trianglelefteq [n])\right) = 1,$$

so it remains to show that

$$\sum_{\mathbf{I}_{[J]}} \operatorname{sgn}\sigma\left(\mathbf{I}_{[J]}, [\tau]\right) \cdot \prod_{l=1}^{m} \left(\operatorname{sgn}(\tau_l) \cdot \det\left(X_{\overline{I_l}, \overline{J_l}}\right)\right) = \det(X)^{m-1} \det\left(X_{\overline{[k]}, \overline{[k]}}\right). \tag{5}$$

This, of course, is true for $m = 1$. We proceed by induction on m.

For any ordered partition $S_1 \dot\cup S_2 \dot\cup \ldots \dot\cup S_m = [k]$, we introduce the shorthand notation

$$\mathbf{S}_l := [k] \setminus (S_1 \dot\cup S_2 \dot\cup \ldots \dot\cup S_l)\,.$$

Moreover, write $d_{I_j} := \det\left(X_{\overline{I_j}, \overline{J_j}}\right)$ for short. Then the lefthand-side of (5) may be written as the $(m-1)$-fold sum

$$\sum_{\substack{I_1 \subseteq \mathbf{I}_0 \\ |I_1| = |J_1|}} \operatorname{sgn}(\tau_1)\, d_{I_1} \sum_{\substack{I_2 \subseteq \mathbf{I}_1 \\ |I_2| = |J_2|}} \operatorname{sgn}(\tau_2)\, d_{I_2} \cdots \sum_{\substack{I_{m-1} \subseteq \mathbf{I}_{m-2} \\ |I_{m-1}| = |J_{m-1}|}} \operatorname{sgn}(\tau_{m-1})\, d_{I_{m-1}} \operatorname{sgn}(\tau_m)\, d_{I_m} \cdot \operatorname{sgn}(\sigma)\,, \tag{6}$$

where $I_m = \mathbf{I}_{m-2} \setminus I_{m-1}$ and $\sigma = \sigma\left(\mathbf{I}_{[J]}, [\tau]\right)$.

Assume $\mathbf{J}_{m-2} = \{j_1 \leq \cdots \leq j_a\}$, $\mathbf{I}_{m-2} = \{i_1 \leq \cdots \leq i_a\}$ and $J_m = \{j_{s_1} \leq \cdots \leq j_{s_b}\}$. Then the special choice $I'_m = \{i_{s_1} \leq \cdots \leq i_{s_b}\}$ (i.e., with respect to the relative ordering, "I'_m is the same subset as J_m ") and $I'_{m-1} = \mathbf{I}_{m-2} \setminus I'_m$ determines *uniquely* a partial permutation τ'_{m-1}

$$\tau'_{m-1} : \mathbf{I}_{m-2} \to \mathbf{J}_{m-2}.$$

According to (4), by construction we have

$$\operatorname{sgn}\left(\tau'_{m-1}\right) = \operatorname{sgn}\left(\tau_{m-1}\right) \cdot \operatorname{sgn}\left(\tau_m\right). \tag{7}$$

Now consider $\sigma = \sigma\left(\mathbf{I}_{[J]}, [\tau]\right)$ in the innermost sum of (6): *Erasing* the edges corresponding to τ_{m-1} and τ_{m-2} from m_σ and *replacing* them by the edges corresponding to τ'_{m-1} yields a permutation $\sigma' = \tau_1 \star \cdots \tau_{m-2} \star \tau'_{m-1}$ (which, of course, complies to the partition scheme $[\tau'] = \left[\tau_1 \star \cdots \tau_{m-2} \star \tau'_{m-1}\right]$). Since by (4) together with (7) we have

$$\operatorname{sgn}\left(\tau'_{m-1}\right) = \operatorname{sgn}\left(\tau_{m-1} \star \tau_m\right) \cdot \operatorname{sgn}\left(I_m \trianglelefteq \mathbf{I}_{m-2}\right) \cdot \operatorname{sgn}\left(J_m \trianglelefteq \mathbf{J}_{m-2}\right)$$

and (clearly)

$$\sigma \setminus \left(\tau_{m-1} \star \tau_m\right) = \sigma' \setminus \tau'_{m-1},$$

we also have (again by (4))

$$\operatorname{sgn}(\sigma) = \operatorname{sgn}\left(I_m \trianglelefteq \mathbf{I}_{m-2}\right) \cdot \operatorname{sgn}\left(J_m \trianglelefteq \mathbf{J}_{m-2}\right) \cdot \operatorname{sgn}\left(\sigma'\right).$$

Hence the innermost sum of (6) can be written as

$$\operatorname{sgn}\left(\tau'_{m-1}\right) \cdot \left(\sum_{\substack{I_{m-1} \subseteq \mathbf{I}_{m-2} \\ |I_{m-1}| = |J_{m-1}|}} \operatorname{sgn}\left(I_m \trianglelefteq \mathbf{I}_{m-2}\right) \cdot \operatorname{sgn}\left(J_m \trianglelefteq \mathbf{J}_{m-2}\right) \cdot d_{I_{m-1}} \cdot d_{I_m} \right) \cdot \operatorname{sgn}\left(\sigma'\right).$$

If we can show that this last sum equals $\det(X) \cdot \det\left(X_{\overline{\mathbf{I}_{m-2}}, \overline{\mathbf{J}_{m-2}}}\right)$, then (5) follows by induction, since the $(m-1)$–fold sum in (6) thus reduces to an $(m-2)$–fold sum, which corresponds to the partition-scheme $[\tau'] = \left[\tau_1 \star \tau_2 \star \ldots, \tau_{m-2} \star \tau'_{m-1}\right]$.

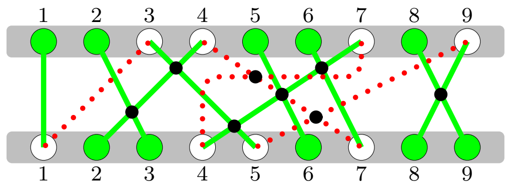
2.8 (A generalization of) Laplace's theorem

Luckily, a generalization (see [5, section 148]) of Laplace's Theorem serves as the closer for our argumentation:

Theorem 2. *Let X be an $(m+k) \times (m+k)$-matrix, and let $1 \leq i_1 < i_2 < \cdots < i_m \leq m+k$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq m+k$ be (the indices of) k fixed rows and k fixed columns of X. Denote the set of these (indices of) rows and columns by R and C, respectively. Consider some fixed subset $I \subseteq R$. Then we have:*

$$\det(X) \cdot \det\left(X_{\overline{R}, \overline{C}}\right) = \sum_{\substack{J \subseteq C, \\ |J| = |I|}} \operatorname{sgn}\left(I \trianglelefteq R\right) \cdot \operatorname{sgn}\left(J \trianglelefteq C\right) \cdot \det\left(X_{\overline{R \setminus I}, \overline{C \setminus J}}\right) \cdot \det\left(X_{\overline{I}, \overline{J}}\right). \tag{8}$$

Figure 7: Terms in the expansion of the product of determinants may be viewed as "overlays" of the permutation diagrams of two partial permutations. The red (dotted) edge connecting lower vertex 4 to upper vertex 7 is drawn as curved line for graphical reasons only. The active intersections of edges are indicated by small circles.



A combinatorial proof for this identity (using an interpretation of determinants as non–intersecting lattice paths) is implicit in [3, proof of Theorem 6], but we shall give a combinatorial argument which employs the ideas presented in this paper.

Proof. Denote by **lhs** (**rhs**) the set of signed and weighted objects corresponding to the left–hand side (right–hand side) of (8). We will prove (8) by showing

- that there is a weight–preserving and sign–preserving *injection* $\phi : \mathbf{lhs} \to \mathbf{rhs}$,
- and that there is as weight–preserving but sign–reversing *involution* ψ on the set $\mathbf{rhs} \setminus \phi(\mathbf{lhs})$.

Overlays of green/red (partial) matchings: In the same sense as presented in section 2.1, we may view both **lhs** and **rhs** as families of *pairs* of matchings (m_π, m_σ), where we may draw the first matching m_π (with green edges) upon the second one m_σ (with red edges), so that the pairs appear as *overlays* of green and red matchings.

Figure 7, which serves as running example in our proof, shows such overlay of matchings belonging to **lhs** for $m = 5$, $k = 4$, $R = \{2, 3, 6, 8, 9\}$ and $C = \{1, 2, 5, 6, 8\}$ (whence $\overline{R} = [9] \setminus R = \{1, 4, 5, 7\}$ and $\overline{C} = [9] \setminus C = \{3, 4, 7, 9\}$). More precisely, the picture shows the permutation diagrams m_π and m_σ corresponding to the pair of partial permutations (π, σ), where (in 2–line notation)

$$\pi = \binom{142735698}{123456789} \text{ and } \sigma = \binom{3794}{1457}.$$

The green edges belonging to m_π are shown as solid lines, and the red edges belonging to m_σ are shown as dotted lines.

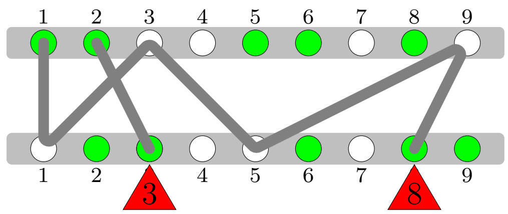
This pair (π, σ) corresponds to the term

$$(-1)^6 \cdot (x_{1,1}x_{2,4}x_{3,2}x_{4,7}x_{5,3}x_{6,5}x_{7,6}x_{8,9}x_{9,8}) \cdot (-1)^2 \cdot (x_{1,3}x_{4,7}x_{5,9}x_{7,4}),$$

which occurs in the expansion of the product of the minors

$$\det(X) \cdot \det\left(X_{\overline{R},\overline{C}}\right)$$

Figure 8: Bicoloured paths (drawn here as thick gray lines) starting in lower vertices 3 and 8 (see the running example in Figure 7).



for $X = (x_{i,j})_{i,j=1}^{9}$. Obviously, the *lower vertices* correspond to the *rows* of X, while the *upper vertices* correspond to the *columns* of X in Figure 7.

Note that for an overlay of matchings, an intersection of two edges does only contribute to the sign if the edges are of the *same* colour (both red or both green): We call such intersections *active*; all other intersections (of edges of different colours) are called *inactive* (recall section 2.1).

Green, red and uncoloured vertices: In every overlay of matchings in $\mathbf{lhs} \cup \mathbf{rhs}$, all *upper* vertices labelled with numbers from the set $\overline{C}$ and all *lower* vertices labelled with numbers from the set $\overline{R}$ are incident with a red edge *and* with a green edge: We call these the *uncoloured* vertices. All the other vertices are incident with precisely one (either green or red) edge: We assign to them the colour of this single incident edge and call them the *coloured* (i.e., either green or red) vertices. (All coloured vertices are green in Figure 7.)

Bicoloured paths: Obviously, an overlay of matchings constitutes a bipartite graph (with double edges allowed). The *connected components* of this bipartite graph are either double edges (one green and one red, see the edges connecting lower vertex 4 to upper vertex 7 in Figure 7) or *paths*

- whose endpoints are *coloured* points,
- and whose edges *alternate* in colour.

We call these components *bicoloured paths*: Figure 8 shows the two bicoloured paths connecting the lower vertices labelled 3 and 8 with the upper vertices labelled 2 and 1, respectively, in our running example.

Obviously, if a bicoloured path connects vertices x and y, then

- x and y are on *different* levels (i.e., one lower and one upper vertex) if and only if x and y have the *same* colour,
- x and y have *different* colours (i.e., one green and one red vertex) if and only if x and y are on the *same* level.

Swapping of colours in bicoloured paths: Observe that for any overlay of matchings, the *swapping* of the colours (red and green) for all edges in some bicoloured path p simply yields another overlay of matchings *with the same absolute weight* (since only the *colour* of edges and vertices does change) and with the same set of uncoloured vertices: We call this operation the *swapping of colours in the bicoloured path* p. (Figure 9 shows the effect of swapping colours in both bicoloured paths from Figure 8: Observe that now there are also red vertices.)

Note that a bicoloured path p might have "inner intersections" (i.e., p may contain intersecting edges), but the swapping of colours in p does not change the status (active or inactive) of such "inner intersections". On the other hand, for *every* "outer intersection" (of some edge e_1 belonging to p with some edge e_2 belonging to *another* bicoloured path), the status is changed (from active to inactive and vice versa) by swapping colours. So swapping colours in p effects a sign change $(-1)^k$, where k is the number of intersections of (edges of) p with (edges of) *other* bicoloured paths.

The injection ϕ: For $(m_\pi, m_\sigma) \in \mathbf{lhs}$, we define $\phi(m_\pi, m_\sigma)$ by swapping colours in *all* (distinct!) bicoloured paths starting in the lower vertices labelled by the numbers from $I \subseteq R$. It is easy to see that ϕ is an *injective* mapping $\mathbf{lhs} \to \mathbf{rhs}$ (in fact, it is an involution $\mathbf{lhs} \to \phi(\mathbf{lhs})$) which preserves (absolute) weights.

In our running example, choose $I = \{3, 8\}$: Figure 9 shows the result of swapping colours in the bicoloured paths shown in Figure 8. The *lower* vertices with labels in I are now *red*, and the subset of labels of *upper* red vertices is $J = \{1, 2\}$. The overlay of matchings $(m_{\pi'}, m_{\sigma'})$ corresponds to the partial permutations

$$\pi' = \binom{3479568}{1245679} \text{ and } \sigma' = \binom{127349}{134578},$$

which correspond to the term

$$(-1)^5 \cdot (x_{1,3}x_{2,4}x_{4,7}x_{5,9}x_{6,5}x_{7,6}x_{9,8}) \cdot (-1)^2 \cdot (x_{1,1}x_{3,2}x_{4,7}x_{5,3}x_{7,4}x_{8,9})$$

occuring in the expansion of the product of the minors

$$\det\left(X_{\overline{R\setminus I}, \overline{C\setminus J}}\right) \cdot \det\left(X_{\overline{I}, \overline{J}}\right).$$

The mapping ϕ is not injective, so we really need the involution ψ: So far we found an injection $\phi : \mathbf{lhs} \to \mathbf{rhs}$ which preserves *absolute* weights: We need to show yet

- that the change of sign effected by ϕ equals $\operatorname{sgn}(I \trianglelefteq R) \cdot \operatorname{sgn}(J \trianglelefteq C)$,
- and that the total weight of $\mathbf{rhs} \setminus \phi(\mathbf{lhs})$ equals 0.

Figure 10 demonstrates that (in general) ϕ is not surjective (whence $\mathbf{rhs} \setminus \phi(\mathbf{lhs}) \neq \emptyset$): The overlay of matchings $(m_{\pi''}, m_{\sigma''})$ depicted there corresponds to the partial permutations

$$\pi'' = \binom{5346789}{1245679} \text{ and } \sigma'' = \binom{123479}{134578},$$

Figure 9: Overlay of matchings obtained by swapping colours in the bicoloured paths (see Figure 8) starting in lower vertices 3 and 8 (see the running example in Figure 7).

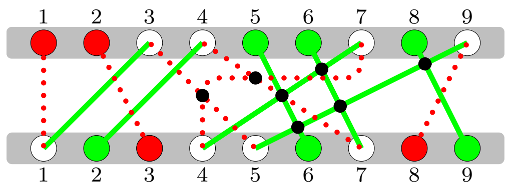
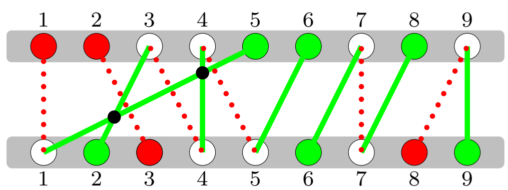


Figure 10: An overlay of two matchings belonging to **rhs** which does not belong to $\phi(\mathbf{lhs})$.



which correspond to the term

$$(-1)^2 \cdot (x_{1,5}x_{2,3}x_{4,4}x_{5,6}x_{6,7}x_{7,8}x_{9,9}) \cdot (-1)^0 \cdot (x_{1,1}x_{3,2}x_{4,3}x_{5,4}x_{7,7}x_{8,9})$$

also occuring in the expansion of the product of the minors

$$\det\left(X_{\overline{R\setminus I},\overline{C\setminus J}}\right) \cdot \det\left(X_{\overline{I},\overline{J}}\right),$$

but *not* occurring as $\phi(z)$ for any $z \in \mathbf{lhs}$, since there is a bicoloured path starting in the upper vertex with label 1, which also *ends* in an upper vertex (with label 5): This cannot happen in overlays belonging to **lhs**. It is easy to see that *every* element of $\mathbf{rhs} \setminus \phi(\mathbf{lhs})$ contains a bicoloured path starting *and* ending in upper vertices, so the involution ψ suggests itself: Identify the leftmost upper vertex which is connected to another upper vertex by a bicoloured path p, and swap colours in p. This clearly defines an involution preserving *absolute* weights: It remains to show that ψ is *sign–reversing*, so that the total weight of $\mathbf{rhs} \setminus \phi(\mathbf{lhs})$ equals 0.

Sign changes effected by swapping colours in bicoloured paths: Bicoloured paths are *connections* of coloured points, which we may simply indicate by *corresponding edges* (see Figures 11 and 12). Observe that two such edges corresponding to (different) bicoloured paths p_1 and p_2 have an intersection if and only if p_1 and p_2 have an *odd* number of intersections, and recall that swapping the colours in some bicoloured path p yields a sign change of $(-1)^k$, where k is the number of "outer intersections" of p (i.e., intersections with *other* bicoloured paths).

Figure 11: The edges corresponding to the connections by bicoloured paths in Figure 7.

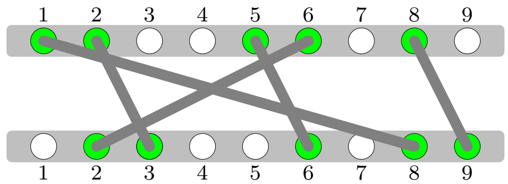
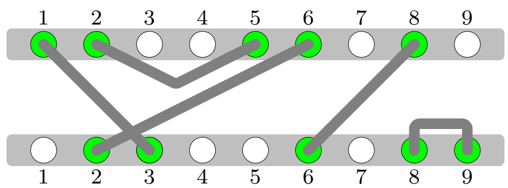


Figure 12: The edges corresponding to the connections by bicoloured paths in Figure 10.



If we *forget* the uncoloured points in Figure 11, we recognize a permutation diagram: It is clear that for *every* overlay of matchings from **lhs**, we obtain such permutation diagram.

Denote by B the set of *all* bicoloured paths in an overlay of matchings from **lhs** and consider some *subset* $S \subset B$ and its complement $\overline{S} = B \setminus S$. By swapping colours of *all* bicoloured paths in S, the status (active/inactive) of every "inner intersection" of S (i.e., an intersection of some $p \in S$ with another $p' \in S$, where $p = p'$ is possible) is *unchanged*, while the status of every "outer intersection" of S (i.e., an intersection of some path $p \in S$ with some path $q \in \overline{S}$) is swapped (from active to inactive and vice versa): So the sign change equals $(-1)^k$, where k is the number of "outer intersections" of S. But this is the *same* sign change we encounter if we partition the permutation corresponding to B into the two partial permutations corresponding to S and $\overline{S}$, respectively, in the sense of Lemma 1: So by (4), this sign equals $\operatorname{sgn}(I \trianglelefteq R) \cdot \operatorname{sgn}(J \trianglelefteq C)$, and the injection ϕ is sign–preserving.

By the same reasoning, we see that the sign change effected by swapping some bicoloured path connecting two *upper* vertices x and y (see Figure 13) is $(-1)^k$, where k is the number of coloured vertices lying *between* x and y. Assuming that x is the i–th and y is the $(i + k + 1)$–the element of the ordered set C (which is the set of upper coloured vertices), such swapping replaces factor $(-1)^i$ by $(-1)^{i+k+1}$ (or vice versa) in $\operatorname{sgn}(J \trianglelefteq C)$, which gives sign change $(-1)^{k+1}$: Altogether, such swapping yields sign change $(-1)^{2k+1} = -1$, whence the involution ψ is sign–reversing. $\square$

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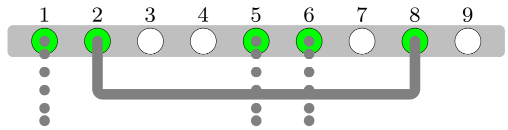
Figure 13: An edge corresponding to a connection between two *upper* vertices.



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