

# Small snarks with large oddness

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## Abstract

We estimate the minimum number of vertices of a cubic graph with given oddness and cyclic connectivity. We prove that a bridgeless cubic graph  $G$  with oddness  $\omega(G)$  other than the Petersen graph has at least  $5.41 \cdot \omega(G)$  vertices, and for each integer  $k$  with  $2 \leq k \leq 6$  we construct an infinite family of cubic graphs with cyclic connectivity  $k$  and small oddness ratio  $|V(G)|/\omega(G)$ . In particular, for cyclic connectivity 2, 4, 5, and 6 we improve the upper bounds on the oddness ratio of snarks to 7.5, 13, 25, and 99 from the known values 9, 15, 76, and 118, respectively. In addition, we construct a cyclically 4-connected snark of girth 5 with oddness 4 on 44 vertices, improving the best previous value of 46.

**Keywords:** cubic graph, snark, oddness, resistance

## 1 Introduction

Cubic graphs—and more generally all graphs with maximum degree 3—naturally fall into two classes depending on whether they do or do not admit a 3-edge-colouring. Accordingly, graphs from the former class are often called *colourable* while those in the latter class are called *uncolourable*. Although most cubic graphs are known to be colourable [16], the problem of determining whether a cubic graph has a 3-edge-colouring is NP-complete [7].

In the study of uncolourable graphs the following lemma plays a fundamental role.

**Parity Lemma.** *Let  $G$  be a cubic graph endowed with a proper 3-edge-colouring with colours 1, 2, and 3. If a cutset consisting of  $m$  edges contains  $m_i$  edges of colour  $i$  for  $i \in \{1, 2, 3\}$ , then*

$$m_1 \equiv m_2 \equiv m_3 \equiv m \pmod{2}.$$

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An immediate consequence of the Parity Lemma is that every cubic graph with a bridge is uncolourable. Besides this trivial family of uncolourable cubic graphs there exist more interesting examples, called *snarks*. These are 2-connected cubic uncolourable graphs, sometimes required to satisfy additional conditions, such as cyclic 4-edge-connectivity and girth at least five, to avoid triviality. Snarks are quintessential to many important problems and conjectures in graph theory including the 4-colour theorem, Tutte's 5-flow conjecture, the cycle double cover conjecture, and many others. While most of these problems are trivial for 3-edge-colourable graphs, they are exceedingly difficult for snarks in general. On the other hand, for those which are close to being colourable they are usually tractable. For example, the 5-flow conjecture has been verified for snarks with oddness at most 2 (Jaeger [10]), and the cycle double cover conjecture has been verified for snarks with oddness at most 4 (Huck and Kochol [8], Häggkvist and McGuinness [6]). Snarks with large oddness thus remain potential counterexamples to these conjectures and therefore deserve further study.

Oddness is a natural measure of uncolourability of a cubic graph based on the fact that every bridgeless cubic graph has a 1-factor [15] and consequently also a 2-factor. It is easy to see that a cubic graph is 3-edge-colourable if and only if it has a 2-factor that only consists of even circuits. In other words, snarks are those cubic graphs which have an odd circuit in every 2-factor. The minimum number of odd circuits in a 2-factor of a bridgeless cubic graph  $G$  is its *oddness*, and is denoted by  $\omega(G)$ . Since every cubic graph has even number of vertices, its oddness must also be even.

Another natural measure of uncolourability of a cubic graph is based on minimising the use of the fourth colour in a 4-edge-colouring of a cubic graph. Alternatively, one can ask how many edges have to be deleted in order to get a 3-edge-colourable graph. Somewhat surprisingly, the required number of edges to be deleted is the same as the number of vertices that have to be deleted in order to get a 3-edge-colourable graph (see [18, Theorem 2.7]). This quantity is called the *resistance* of  $G$ , and will be denoted by  $\rho(G)$ . Observe that  $\rho(G) \leq \omega(G)$  for every bridgeless cubic graph  $G$  since deleting one edge from each odd circuit in a 2-factor leaves a colourable graph. On the other hand, the Parity Lemma implies that  $\rho(G)$  never equals 1, which in turn yields that  $\rho(G) = 2$  if and only if  $\omega(G) = 2$ . The difference between  $\omega(G)$  and  $\rho(G)$  can be arbitrarily large in general [19], nevertheless, resistance can serve as a convenient lower bound for oddness, because it is somewhat easier to handle.

The purpose of this article is to study how large the oddness of a cubic graph can be compared to its order. The results of this comparison strongly depend on the cyclic connectivity of the graphs in question. We are therefore interested in the smallest possible value of the ratio  $|V(G)|/\omega(G)$  for a snark  $G$  within the class of cyclically  $k$ -connected snarks. Recall that a cubic graph  $G$  is *cyclically  $k$ -connected* if no set of fewer than  $k$  edges can separate two circuits of  $G$  into distinct components; the *cyclic connectivity*  $\zeta(G)$  of  $G$  is the largest  $k$  such that  $G$  is cyclically  $k$ -connected. It is useful to realise that the values of vertex-connectivity, edge-connectivity, and cyclic connectivity of a cubic graph coincide for  $k \leq 3$  but cyclic connectivity may be arbitrarily large (see [14]).

So far, only trivial lower bounds for the oddness ratio  $|V(G)|/\omega(G)$  have been known;

as regards the upper bounds, there are various constructions of snarks which are probably not optimal. Since the oddness ratio of the Petersen graph equals 5, it is meaningless to attempt improving this absolute lower bound. In Section 2 we therefore adopt an asymptotical approach similar to that taken by Steffen [19] and by Hägglund [5]. We summarise the previously known results as well as our improvements in Table 1. We only consider cyclic connectivity  $k \in \{2, 3, 4, 5, 6\}$  because no cyclically 7-connected snarks are known. In fact, Jaeger [9] conjectured that such snarks do not exist. We prove, in particular, that the oddness ratio of every snark is bounded above by 7.5. We conjecture that this general bound is best possible as we believe that all snarks with oddness  $\omega$  have at least  $7.5\omega - 5$  vertices (see Section 6 for details).

Table 1: Upper and lower bounds on oddness ratio  $|V|/\omega$ .

connectivity	lower bound	current upper bound	previous upper bound
2	5.41	7.5	9 (Steffen [19])
3	5.52	9	9 (Steffen [19])
4	5.52	13	15 (Hägglund [5])
5	5.83	25	76 (Steffen [19])
6	7	99	118 (Kochol [13])

Besides general bounds, we are also interested in identifying the smallest snark with oddness 4, addressing a long-standing open problem restated as Problem 4 in [1]. We show that the smallest order of a snark with oddness 4 is 28 and present one with cyclic connectivity 3 (see Figure 2 right). We further construct a cyclically 4-connected snark of girth 5 with oddness 4 and resistance 3 on 44 vertices (see Figure 4), improving by 2 the value established in [5]. We believe that 44 is the smallest possible order of a non-trivial snark of oddness 4.

## 2 Oddness and resistance ratios

The *oddness ratio* of a snark  $G$  is the quantity  $|V(G)|/\omega(G)$ , and its *resistance ratio* is the quantity  $|V(G)|/\rho(G)$ . In order to derive some relevant information about these parameters we also examine their asymptotic behaviour. To this end, we define

$$A_\omega = \liminf_{|V(G)| \rightarrow \infty} \frac{|V(G)|}{\omega(G)}$$

and

$$A_\rho = \liminf_{|V(G)| \rightarrow \infty} \frac{|V(G)|}{\rho(G)}.$$

Since the oddness ratio is at least as large as its resistance ratio, we have  $A_\omega \leq A_\rho$ .

The oddness and resistance ratios heavily depend on the cyclic connectivity of a graph in question. This suggests to study analogous values  $A_\omega^k$  and  $A_\rho^k$  obtained under the assumption that the class of snarks is restricted to those with cyclic connectivity at least  $k$ . Note that  $A_\omega^k \leq A_\rho^k$  for every  $k \geq 2$  and that  $A_\omega^2 = A_\omega$  and  $A_\rho^2 = A_\rho$ .

Similar ideas were pursued by Steffen [19] who asked the following question: For which values  $a_\rho$  the inequality  $n \geq a_\rho \rho(G)$  has only finitely many counterexamples? He proved (Theorem 2.4 and Lemma 3.4 in [19]) that the resistance ratio of every snark of order  $n > 14$  is at least 8 and constructed a family of snarks for which  $\rho(G) \geq n/9$ . It follows that  $8 \leq A_\rho \leq 9$  and therefore  $A_\omega \leq A_\rho \leq 9$ . Since the snarks that he constructed are cyclically 3-connected, we also have  $A_\omega = A_\omega^2 \leq A_\omega^3 \leq A_\rho^3 \leq 9$ . An earlier construction of Rosenfeld [17] provided the bounds  $A_\omega \leq A_\rho \leq 10$ .

In this paper we concentrate on upper and lower bounds for  $A_\omega$  and  $A_\omega^k$ , while their resistance counterparts  $A_\rho$  and  $A_\rho^k$  will only play an auxiliary role. To start with, let us discuss the value  $A_\omega^6$ . Since every odd circuit in a cyclically 6-connected snark has length at least 7, every such snark of oddness  $\omega$  has at least  $7\omega$  vertices. Thus  $A_\omega^6 \geq 7$ , which fills in the entry in the second column of the last line of Table 1.

### 3 Reduction lemmas

In the study of various properties of snarks it is often convenient to avoid snarks having short circuits or small edge-cuts, because such snarks can either be considered trivial or lack the desired properties. There are well-known reductions that remove these structures from snarks such as contraction of a 3-circuit or a 2-circuit and suppression of a 2-valent vertex whenever it arises [20]. Similar reductions will also be needed in our further investigation.

We begin with the observation that in the search for small snarks with large oddness we can ignore graphs with parallel edges or triangles.

**Lemma 1.** *For every snark  $G$  there exists a snark  $G'$  of order not exceeding that of  $G$  such that  $\omega(G') \geq \omega(G)$ ,  $\zeta(G') \geq \zeta(G)$ , and the girth of  $G'$  is at least 4.*

*Proof.* Let  $G_0$  be the snark obtained from  $G$  by the standard reduction of a single circuit  $C$  of length 2 or 3: the graph  $G_0$  arises from  $G$  simply by the contraction of  $C$  into a single vertex and by suppressing a vertex of degree 2, if it arises. It is immediate that  $\zeta(G_0) \geq \zeta(G)$ .

For each 2-factor  $F_0$  of  $G_0$  there exists a corresponding 2-factor  $F$  in  $G$  such that every circuit of  $F_0$  extends to a unique circuit of  $F$  with the same parity of the number of vertices and at least the same length. Since  $F$  contains no circuits other than those corresponding to the circuits of  $F_0$ , we have  $\omega(G_0) \geq \omega(G)$ .

The required graph  $G'$  is now obtained by repeating a similar procedure with any circuit of length 2 or 3 in  $G_0$ .  $\square$

The standard reduction of a 4-circuit, which consists in removing a pair of opposite edges and suppressing the resulting vertices of degree 2, does not work here because it

may decrease cyclic connectivity, produce a bridge, or even create a disconnected graph. Nevertheless, for our purpose the following weaker result is sufficient.

**Lemma 2.** *For every snark  $G$  there exists a snark  $G'$  of order not exceeding that of  $G$  such that  $\omega(G') \geq \omega(G)$  and the girth of  $G'$  is at least 5.*

*Proof.* For every 4-circuit in  $G$  there are two possibilities for applying the standard reduction which correspond to two pairs of opposite edges in a quadrilateral. It is not difficult to see that one of the choices always creates a 2-connected graph [20]. Furthermore, any 2-factor of the reduced graph can be easily extended to a 2-factor of the original graph without changing the parity of the lengths of the circuits and by adding at most one 4-circuit. Consequently, after reducing a 4-cycle in a snark we obtain a smaller snark with the same oddness. We repeatedly apply reductions of circuits of length 4 and, if necessary, we also reduce circuits of length 2 and 3 in the manner described in the previous proof. Eventually we obtain a snark  $G'$  which has the same oddness as  $G$ , girth at least 5, and is 2-connected.  $\square$

The remaining two lemmas deal with small edge-cuts in snarks.

**Lemma 3.** *For every snark  $G$  there exists a snark  $G'$  of order not exceeding that of  $G$  such that  $\omega(G') = \omega(G)$  and every 2-edge-cut in  $G'$  separates two uncolourable subgraphs of  $G'$ .*

*Proof.* Let  $G$  be a snark with a 2-edge-cut  $S$  that separates  $G$  into two components one of which is colourable. If both components were colourable, then so would be  $G$ , by the Parity Lemma. It follows that one of the components is colourable and the other is not. Let  $G_1$  be the uncolourable component, and let  $G_2$  be the colourable one. For  $i \in \{1, 2\}$ , let  $G'_i$  arise from  $G_i$  by joining its vertices of degree 2 by an edge  $e_i$ . By the Parity Lemma,  $G'_2$  is colourable, so there is an even 2-factor in  $G'_2$  passing through  $e_2$  and another even 2-factor of  $G'_2$  missing  $e_2$ . Consequently, every 2-factor  $F_1$  of  $G'_1$  can be extended to a 2-factor  $F$  of  $G$  such that no odd circuit of  $F$  is contained in  $G_2$ . Since  $G_2$  has even number of vertices, the oddness of  $G$  does not exceed that of  $G'_1$ . By repeating the procedure with  $G'_1$  we eventually obtain the desired graph  $G'$ .  $\square$

Similar arguments can be used to prove an analogous result for 3-edge-cuts.

**Lemma 4.** *For every snark  $G$  there exists a snark  $G'$  of order not exceeding that of  $G$  such that  $\omega(G') = \omega(G)$  and every 3-edge-cut in  $G'$  separates two uncolourable subgraphs of  $G'$ .*

## 4 Lower bounds on oddness ratio

Lemma 1 implies that the oddness ratio of every snark is at least 5. This bound is best possible, because the Petersen graph has ten vertices and oddness 2. However, the Petersen graph is the only snark for which equality holds, as it is the only 2-edge-connected

cubic graph having only 5-circuits in each 2-factor [2]. The purpose of this section is to improve this bound for snarks different from the Petersen graph.

We begin with the key observation that in a cubic graph one can always find a 2-factor that avoids most of the chosen 5-circuits.

**Proposition 5.** *Let  $\mathcal{C}$  be an arbitrary set of 5-circuits of a bridgeless cubic graph  $G$ . Then  $G$  has a 2-factor that contains at most  $1/6$  of the 5-circuits from  $\mathcal{C}$ .*

In the proof of this proposition we employ the concept of a perfect matching polytope introduced in [3]. Let  $G$  be a graph with  $E(G) = \{e_1, e_2, \dots, e_m\}$ . Each perfect matching  $M$  of  $G$  can be represented by its characteristic vector  $\mathbf{x} \in \mathbb{R}^m$  in which the  $i$ -th entry is 1 if  $e_i$  belongs to  $M$  and is 0 otherwise. The *perfect matching polytope*  $\mathcal{P}(G)$  of a graph  $G$  is the convex hull in  $\mathbb{R}^m$  of the set of characteristic vectors of all perfect matchings of  $G$ .

Let us denote the entry of a vector  $\mathbf{x} \in \mathbb{R}^m$  corresponding to an edge  $e \in E(G)$  by  $\mathbf{x}(e)$ , and let

$$\mathbf{x}(S) = \sum_{e \in S} \mathbf{x}(e)$$

whenever  $S$  is a subset of  $E(G)$ . For a set of vertices or a subgraph  $U$  of  $G$  let  $\delta(U)$  denote the set of all edges with precisely one end in  $U$ . With this notation we can equivalently describe  $\mathcal{P}(G)$  as the set of all vectors from  $\mathbb{R}^m$  that satisfy the following inequalities:

$$\begin{aligned} \mathbf{x}(e) &\geq 0 && \text{for each } e \in E(G), \\ \mathbf{x}(\delta(v)) &= 1 && \text{for each } v \in V(G), \\ \mathbf{x}(\delta(U)) &\geq 1 && \text{for each } U \subseteq V(G) \text{ with } |U| \text{ odd.} \end{aligned}$$

Note that if  $G$  is cubic and bridgeless, the vector  $(1/3, 1/3, \dots, 1/3)$  always belongs to  $\mathcal{P}(G)$ .

*Proof of Proposition 5.* Let  $\mathcal{P}$  be the perfect matching polytope of  $G$ ; since  $G$  is cubic and bridgeless,  $\mathcal{P}$  is nonempty. Consider the function

$$f(\mathbf{x}) = \sum_{C \in \mathcal{C}} \sum_{e \in \delta(C)} \mathbf{x}(e)$$

defined for each  $\mathbf{x} \in \mathcal{P}$ . The function  $f$  is linear, hence there is a vector  $\mathbf{x}_0$  such that  $f(\mathbf{x}_0)$  is minimal and  $\mathbf{x}_0$  is a vertex of  $\mathcal{P}$ . Since  $(1/3, 1/3, \dots, 1/3) \in \mathcal{P}$ , we have  $f(\mathbf{x}_0) \leq 5/3 \cdot |\mathcal{C}|$ .

Let  $M$  be the perfect matching corresponding to  $\mathbf{x}_0$  and let  $F$  be the 2-factor complementary to  $M$ . Assume that  $F$  contains  $k$  circuits from  $\mathcal{C}$ . If a 5-circuit  $C \in \mathcal{C}$  belongs to  $F$ , it adds 5 to the sum in  $f(\mathbf{x}_0)$ . If  $C$  does not belong to  $F$ , it adds at least 1. Altogether  $f(\mathbf{x}_0) \geq 5k + (|\mathcal{C}| - k) = |\mathcal{C}| + 4k$ . Since  $f(\mathbf{x}_0) \leq 5/3 \cdot |\mathcal{C}|$ , we obtain  $k \leq |\mathcal{C}|/6$ .  $\square$

By inspecting all the vertices at distance at most 2 from a given vertex one can see that every vertex of a cubic graph is contained in at most six 5-circuits. In fact, the Petersen

graph is the only snark that has a vertex contained in precisely six 5-circuits. As the Petersen graph is vertex-transitive, it follows that it contains precisely twelve 5-circuits.

Consider the set  $\mathcal{C}$  of all 5-circuits of the Petersen graph. Since every 2-factor of the Petersen graph consists of two 5-circuits [2], the constant  $1/6$  in Proposition 5 is best possible.

**Corollary 6.** *Let  $G$  be a snark different from the Petersen graph. For every vertex  $v$  of  $G$  there exists a 2-factor  $F$  of  $G$  such that every 5-circuit of  $F$  misses  $v$ .*

*Proof.* Let  $\mathcal{C}$  be the set of all 5-circuits passing through a given vertex  $v$  of  $G$ . Since  $G$  is not the Petersen graph,  $|\mathcal{C}| \leq 5$ . By Proposition 5, there is a 2-factor  $F$  that contains at most  $5/6 < 1$  circuits from  $\mathcal{C}$ . Thus  $F$  contains no 5-circuit passing through  $v$ .  $\square$

To prove the main result of this section we need several lemmas.

**Lemma 7.** *Let  $G$  be a snark of order  $n$  different from the Petersen graph. Assume that  $G$  has  $n_i$  vertices contained in exactly  $i$  5-circuits. Then the following are true:*

- (i)  $n_6 = 0$
- (ii)  $n_5 \leq 2n/5$
- (iii) *If  $G$  is cyclically 3-connected, then  $n_5 = 0$ .*
- (iv) *If  $G$  is cyclically 5-connected, then  $n_4 = 0$ .*

*Proof.* We have already discussed the statement (i) before Corollary 6. We give a detailed proof only for (ii); the remaining statements follow by similar considerations.

Let  $v$  be a vertex of  $G$  contained in five 5-circuits. It follows that one of the edges incident with  $v$  belongs to four of the 5-circuits and the other two edges belong to three of them. In particular,  $v$  cannot be contained in a triangle.

Consider the subgraph  $H$  of  $G$  induced by the set of all vertices at distance 2 from  $v$ . Each 5-circuit passing through  $v$  contributes one edge to  $H$ , so  $H$  has five edges and at most six vertices. If  $v$  belonged to a 4-circuit, there would be at most 5 vertices in  $H$  and at least one of them would be of degree at most 1 in  $H$ . But then  $H$  could not have five edges. So  $v$  belongs to no 4-circuit and  $H$  has six vertices.

Let  $K$  be the subgraph of  $G$  induced by all vertices at distance at most 2 from  $v$ . Since  $H$  has five edges, there are two edges separating  $K$  from the rest of  $G$ . The subgraph  $K$  has ten vertices, and a short case analysis shows that  $K$  contains at most three other vertices contained in five 5-circuits in  $G$ . Hence, at most four of the ten vertices of  $K$  are contained in five 5-circuits. By repeating this argument for any vertex  $v$  contained in five 5-circuits which we have not counted yet we arrive at the conclusion that  $n_5 \leq 4n/10 = 2n/5$ . Note that no edge of  $K$  belongs to a 2-edge-cut, therefore all the subgraphs  $K$  arising in the described way are pairwise disjoint.  $\square$

**Lemma 8.** *Let  $G$  be a bridgeless cubic graph of order  $n$  that has  $n_i$  vertices contained in exactly  $i$  5-circuits. Let  $\mathcal{C}$  be the set of all 5-circuits of  $G$ . Then*

$$|\mathcal{C}| = \frac{1}{5} \sum_{i=0}^6 i \cdot n_i.$$

*Proof.* The desired equality immediately follows from counting the number of pairs  $(v, C)$ , where  $C \in \mathcal{C}$  and  $v$  is a vertex contained  $C$ , in two ways.  $\square$

**Lemma 9.** *Let  $G$  be a snark of order  $n$  with girth at least 4. If  $G$  has  $q$  circuits of length 5, then*

$$\omega(G) \leq \frac{3n + q}{21}.$$

*Proof.* Let  $\omega$  be the oddness of  $G$ . By Proposition 5, there is a 2-factor  $F$  of  $G$  which contains at most  $q/6$  circuits of length 5. Let  $p_i$  denote the number of  $i$ -circuits in  $F$ . Note that  $p_5 \leq q/6$ . We are minimising the value of  $n/\omega$  subject to the constraints

$$\begin{aligned} n &= 4p_4 + 5p_5 + 6p_6 + 7p_7 + \cdots + (n-1)p_{n-1}, \\ \omega &= p_5 + p_7 + p_9 + \cdots + p_{n-1}. \end{aligned}$$

Roughly speaking, the “worst case” occurs when  $F$  contains no circuits of even length, there is a maximal possible number of 5-circuits, and all the longer odd circuits have length 7. In other words,  $n \geq 5p_5 + 7(\omega - p_5)$ , and thus

$$7\omega \leq n + 2p_5 \leq n + q/3.$$

The resulting inequality is equivalent to the desired one.  $\square$

**Theorem 10.** *Let  $G$  be a snark of order  $n$  different from the Petersen graph. Then:*

- (i)  $n/\omega(G) \geq 525/97 > 5.41$ ;
- (ii)  $n/\omega(G) \geq 105/19 > 5.52$ , if  $G$  is cyclically 3-connected;
- (iii)  $n/\omega(G) \geq 35/6 > 5.83$ , if  $G$  is cyclically 5-connected.

*Proof.* According to Lemma 1, it is enough to prove the theorem provided that  $G$  has girth at least 4. Lemma 7 asserts that each vertex of  $G$  is contained in at most five 5-circuits and that at most  $2n/5$  vertices are contained in five 5-circuits. Let  $\mathcal{C}$  be the set of all 5-circuits of  $G$ . By using Lemma 8 we have

$$|\mathcal{C}| \leq \frac{1}{5}(5n_5 + 4(n - n_5)) = \frac{4}{5}n + \frac{1}{5}n_5 \leq \frac{22}{25}n,$$

and Lemma 9 in turn yields that  $n/\omega(G) \geq 525/97 > 5.41$ . This proves (i).



To prove (ii), let  $G$  be cyclically 3-connected. Again, by Lemma 1 we may assume  $G$  to have girth at least 4. Let  $\mathcal{C}$  be the set of all 5-circuits of  $G$ . From Lemma 7 we infer that each vertex of  $G$  is contained in at most four 5-circuits. According to Lemma 8 we have  $|\mathcal{C}| \leq \frac{4}{5}n$  and from Lemma 9 we get  $n/\omega(G) \geq 105/19$ , as claimed.

Finally, if  $G$  is cyclically 5-connected, each vertex of  $G$  is contained in at most three 5-circuits, by Lemma 7. According to Lemma 8,  $|\mathcal{C}| \leq \frac{3}{5}n$ , and from Lemma 9 we get  $n/\omega(G) \geq 105/18 = 35/6$ , which proves (iii).  $\square$

## 5 Construction methods

In the next few sections we construct snarks with small size when compared to their oddness. Most of our constructions produce larger graphs by putting smaller parts together. For this purpose the following terminology will be useful: A *network* is a pair  $(G, T)$  consisting of a graph  $G$  and a distinguished set  $T$  of vertices of degree 1 called *terminals*. An edge incident with a terminal is called a *terminal edge*. A network with  $k$  terminals will be called a *k-pole*. In all networks considered below nonterminal vertices will always have degree 3.

Each terminal of a network serves as a place of connection with another terminal. Two terminal edges of either the same network or of two disjoint networks can be naturally joined to form a new nonterminal edge by identifying the corresponding terminal vertices and suppressing the resulting 2-valent vertex. This operation is called the *junction* of two terminals.

A standard way to create terminal vertices in a graph or network is by *splitting off* a vertex  $v$  from a graph  $G$ ; by this we mean the removal of  $v$  from  $G$  and attaching a terminal vertex to each dangling edge originally incident with  $v$ . The terminals resulting from splitting off the vertex  $v$  from  $G$  will be said to be *corresponding to  $v$* .

The basic building blocks for our constructions are the following six networks obtained from the Petersen graph. All of them are displayed in Figure 1 together with their simplified diagrams used in the rest of the paper.

- The network  $P_2$  is formed from the Petersen graph by subdividing an edge and splitting off the new vertex. This network has ten nonterminal vertices. By the Parity Lemma, it is not 3-edge-colourable.
- The network  $P_3$  is formed from  $P$  by splitting off a vertex. This network has nine nonterminal vertices and is not 3-edge-colourable for similar reasons.
- The network  $P_4^v$  is formed from  $P$  by deleting an edge and splitting off the two resulting vertices of degree 2. The terminal edges constitute two pairs, each pair corresponding to one vertex of degree 2. Every 3-edge-colouring of  $P_4^v$  must assign the same colour to both edges within the same pair, otherwise one could properly colour the edges of  $P$  with three colours. This network has eight nonterminal vertices.

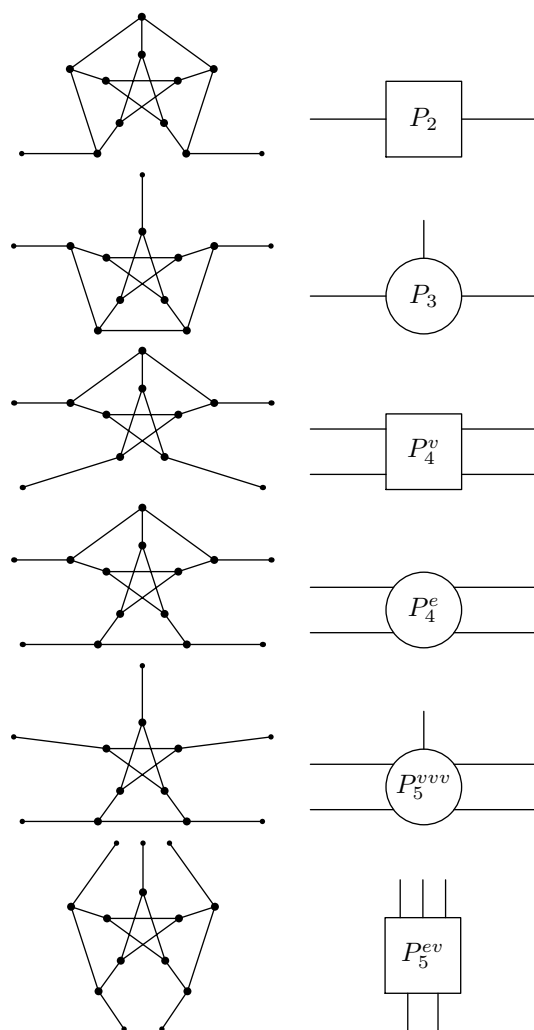


Figure 1: Networks  $P_2$ ,  $P_3$ ,  $P_4^v$ ,  $P_4^e$ ,  $P_5^{vvv}$ ,  $P_5^{ev}$  and their diagrams.

- The network  $P_4^e$  is formed from  $P$  by subdividing two edges at distance 1 and splitting off the resulting vertices of degree 2. The terminal edges again constitute two pairs, each pair corresponding to one vertex of degree 2. Every 3-edge-colouring of  $P_4^e$  must assign different colours to the edges in the same pair of terminal edges, otherwise one could properly colour the edges of  $P$  with three colours. This network has ten nonterminal vertices.
- The network  $P_5^{vvv}$  is formed from  $P$  by deleting two adjacent edges and splitting off the two resulting vertices of degree 2. The terminal edges constitute two pairs, each pair corresponding to one vertex of degree 2, and one single terminal edge. Since there are five terminal edges, the Parity Lemma implies that in every 3-edge-colouring of  $P_5^{vvv}$  one colour must be used exactly three times and the other two colours must be used once. As we cannot colour  $P$ , every 3-edge-colouring of  $P_5^{vvv}$  must assign the same colours to the terminal edges of one of the two pairs. The other pair of terminal edges must have different colours assigned. This network has seven nonterminal vertices.
- The network  $P_5^{ev}$  is formed from  $P$  by splitting of a vertex  $v$  and subdividing an edge at distance two from  $v$  and by subsequently splitting off the new vertex of degree 2. This creates one pair and one triple of terminal edges, respectively. Since  $P$  is not 3-edge colourable, the Parity Lemma implies that the terminal edges contained in the pair must have different colours. This network has nine nonterminal vertices.

Another construction method which we employ is superposition [12]. Given a cubic graph  $G$ , take a collection  $\{X_v; v \in V(G)\}$  of disjoint networks called *supervertices* and a collection  $\{Y_e; e \in E(G)\}$  of disjoint networks called *superedges*. Each supervertex is a network whose terminals are partitioned into three subsets and each superedge is a network whose terminals are partitioned into two subsets; the partition sets are called *connectors*. For each vertex  $v$  of  $G$ , associate each connector of the supervertex  $X_v$  with the end of an edge of  $G$  incident with  $v$  in such a way that no two connectors are associated with the same end. For each edge  $e$  of  $G$ , associate each connector of the superedge  $Y_e$  with an end of the edge  $e$  in such a way that the connectors corresponding to an incidence between a vertex and an edge in  $G$  have the same size; again, the connectors of  $Y_e$  are associated with different ends of  $e$ . Now, substitute each vertex  $v$  of  $G$  with the supervertex  $X_v$  and each edge  $e$  of  $G$  with the superedge  $Y_e$ , and perform all the junctions between supervertices and superedges that correspond to the incidences between vertices and edges of  $G$ . Let  $\tilde{G}$  be the resulting graph. Note that the graph  $\tilde{G}$  is cubic since all the nonterminal vertices in supervertices and superedges have degree 3. We call  $\tilde{G}$  a *superposition* of  $G$ .

There is a natural incidence-preserving surjective mapping  $p: \tilde{G} \rightarrow G$ , called a *projection*, such that for each vertex  $\tilde{v}$  of  $\tilde{G}$  the image  $p(\tilde{v})$  is a vertex of  $G$  and for each edge  $\tilde{e}$  of  $\tilde{G}$  the image  $p(\tilde{e})$  is either an edge of  $G$  or a vertex resulting from the contraction of  $\tilde{e}$ . This mapping takes every nonterminal vertex from a supervertex  $X_v$  to  $v$  itself, and every nonterminal vertex from a superedge  $Y_e$  that replaces an edge  $e = uv$  of  $G$  to either  $u$  or  $v$ . It should be remarked that  $p$  is not uniquely determined, but the difference between any two such mappings is insubstantial.

Note that in a superposition  $\tilde{G}$  of  $G$  a vertex  $v$  of  $G$  may be substituted with a *trivial supervertex*  $X_v$ , one which consists of a single nonterminal vertex  $\tilde{v}$  incident with three pendant edges whose pendant vertices are terminals. Similarly, an edge  $e$  may be substituted with a *trivial superedge*  $Y_e$  which consists of a single edge whose endvertices are both terminals. If all the substitutions are trivial, then  $p: \tilde{G} \rightarrow G$  is clearly an isomorphism.

A superedge  $Y$  is *proper* if for every 3-edge-colouring of  $Y$  that uses nonzero elements of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  as colours the sum of colours on the terminal edges incident with the terminals in either connector is nonzero. In particular, a trivial superedge is proper. It follows from the Parity Lemma that the two sums must be the same element of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

There is a standard method of constructing proper superedges from a snark. Take a snark  $H$ , and choose a vertex  $v$  which can be either a vertex of  $H$  or a 2-valent vertex arising from the subdivision of an arbitrary edge of  $H$ . Let us create a connector by splitting off  $v$  from  $H$  and repeat the operation with another such vertex. Clearly, splitting off a vertex of  $H$  creates a connector of size 3 whereas splitting off a subdivision vertex creates a connector of size 2. It is an easy consequence of the Parity Lemma that any superedge arising in the just described way is always proper.

A superposition  $\tilde{G}$  of  $G$  is *proper* if  $G$  is a snark and every superedge is proper. We claim that in this case  $\tilde{G}$  cannot be 3-edge-colourable. Suppose it is, and consider a 3-edge-colouring of  $\tilde{G}$  that uses nonzero elements of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  as colours. We show that the projection  $p: \tilde{G} \rightarrow G$  induces a 3-edge-colouring of  $G$ . We can colour each edge  $e$  of  $G$  with the element of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  equal to the sum of colours occurring on the terminal edges of either connector of  $Y_e$ . This colouring is easily seen to be a proper 3-edge-colouring of  $G$ , which is impossible since  $G$  is a snark. It follows that  $\tilde{G}$  is a snark.

The following more general result is useful for constructing graphs with large resistance or oddness.

**Proposition 11.** *Let  $\tilde{G}$  be a snark resulting from a proper superposition of a snark  $G$ . Then  $\rho(\tilde{G}) \geq \rho(G)$ .*

*Proof.* Let  $W$  be a set of vertices of  $\tilde{G}$  such that  $\tilde{G} - W$  is a 3-edge-colourable graph and  $|W| = \rho(\tilde{G})$ . Consider the image  $p(W)$  under a projection  $p: \tilde{G} \rightarrow G$ . The restriction of  $p$  to  $\tilde{G} - W$  maps  $\tilde{G} - W$  to  $G - p(W)$  and induces a 3-edge-colouring of  $G - p(W)$  from any 3-edge-colouring of  $\tilde{G} - W$  in the manner described above. Therefore

$$\rho(\tilde{G}) = |W| \geq |p(W)| \geq \rho(G),$$

as claimed. □

It is not known whether the statement about oddness analogous to Proposition 11 is true.

## 6 Connectivity 2

In this section we identify the smallest snark with oddness 4 and construct a family of snarks with oddness  $2q$  having less than  $15q$  vertices.

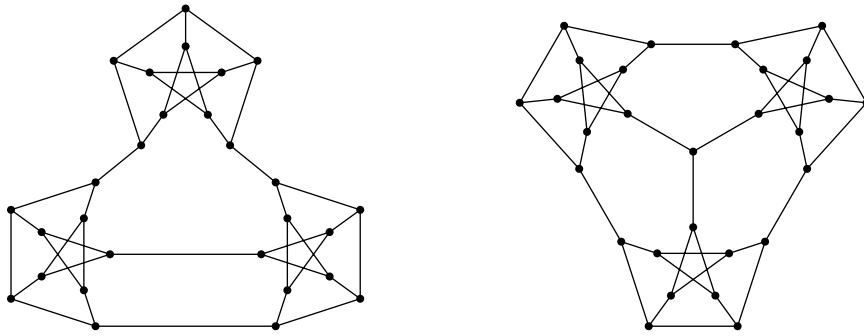


Figure 2: The smallest snarks of oddness 4; both have order 28.

By virtue of Lemma 2, we only need to consider snarks of girth at least 5. Since cyclically 4-connected snarks are catalogued up to order 36 and have oddness two up to this order [1], we may restrict to graphs that either have a 2-edge-cut or a 3-edge-cut.

**Theorem 12.** *The smallest snark with oddness 4 has 28 vertices. There is one such snark with cyclic connectivity 2 and one with cyclic connectivity 3.*

*Proof.* Figure 2 shows two snarks of order 28,  $H_1$  and  $H_2$ , with cyclic connectivity 2 and 3, respectively. Each of them contains three disjoint copies of  $P_3$ , which is uncolourable, therefore  $\rho(H_i) \geq 3$ , and hence  $\omega(H_i) \geq 4$ . It is not difficult to show that in fact  $\omega(H_1) = \omega(H_2) = 4$ . In the rest of the proof we show that there exists no snark with oddness 4 on fewer than 28 vertices.

Let  $G$  be a snark with oddness 4 of minimum order, and suppose that its order is at most 26. By Lemmas 1 and 2 its girth is at least 5. It is known [1] that all cyclically 4-connected snarks with girth 5 or more and order not exceeding 36 have oddness 2. Hence the cyclic connectivity of  $G$  equals either 2 or 3.

First, let us suppose that  $G$  is cyclically 3-connected. Since  $G$  is not cyclically 4-connected, it contains a cycle-separating 3-edge-cut  $S$ . By Lemma 4, the edge-cut  $S$  separates two uncolourable subgraphs  $G_1$  and  $G_2$ . Let  $G'_i$  be the snark obtained from  $G_i$  by joining a new vertex  $v_i$  to the three vertices of degree 2 in  $G_i$ . The snark  $G'_i$  is cyclically 3-connected because  $G$  was cyclically 3-connected. In other words,  $G$  arises from two smaller cyclically 3-connected snarks  $G'_1$  and  $G'_2$  by the reverse process, which splits off a vertex from both  $G'_1$  and  $G'_2$ , and joins the terminals from different snarks. Since  $G$  has girth at least 5, none of the two smaller snarks contains two vertex-disjoint cycles of length at most 4. The last condition significantly reduces the number of candidates for  $G'_1$  and  $G'_2$ . Moreover, it is computationally efficient because short cycles can be quickly discovered by repeating breadth-first search from every vertex of a graph. With the help of a computer, we have tried all pairs of candidates and for each pair checked all the possible combinations of feasible choices of  $v_1$  and  $v_2$ . We have found no snark  $G$  with oddness 4 on at most 26 vertices that arises in this way.

It follows that  $G$  has a 2-edge-cut. Let  $S$  be a 2-edge-cut in  $G$  such that one of the components separated by  $S$  is as small as possible. Again, we may assume that  $G$  has

girth at least 5. Further, by Lemma 3, the cut  $S$  separates two uncolourable subgraphs  $G_1$  and  $G_2$ , both with even number of vertices. Both subgraphs must have at least ten vertices, for otherwise we would obtain a snark of order at most eight by connecting the two vertices of either  $G_1$  or  $G_2$  with an edge, but there are no such snarks. Thus the larger of the two subgraphs has at most 16 vertices. Let  $G'_i$  be the snark obtained from  $G_i$  by joining the vertices of degree 2 by an edge  $e_i$ . We may assume that  $10 \leq |V(G'_1)| \leq |V(G'_2)| \leq 16$ , hence  $G'_1$  has 10 or 12 vertices.

If both  $G'_1$  and  $G'_2$  have a 2-factor such that the added edge  $e_i$  belongs to an odd circuit of this 2-factor, then there  $G$  has a 2-factor  $F$  containing an even circuit  $C$  that passes through  $S$ . Let  $F_i$  be the 2-factor of  $G'_i$  obtained by adding  $e_i$  to  $F \cap G_i$ . The circuit  $C$  has length at least 6 because  $G$  has girth at least 5. Since  $G$  contains at least four odd circuits in  $F$ , we have at least  $4 \cdot 5 + 6 = 26$  vertices in  $G$ . Thus  $G$  has exactly 26 vertices,  $C$  has length 6, and  $F$  contains four circuits of length 5 plus the circuit  $C$ . One of the odd circuits of  $F$  is in  $F_1$  while three of them are in  $F_2$ . Both  $G'_1$  and  $G'_2$  contain at least two vertices of  $C$ , for otherwise  $G$  would have a bridge. But then  $G'_2$  contains at least 17 vertices (three odd 5-circuits plus at least two vertices of  $C$ ), which is a contradiction.

We are thus left with the case where  $G'_1$  or  $G'_2$  has a *special* edge  $e$ , that is, an edge that belongs to no odd circuit of any 2-factor. The existence of a special edge can be checked by a straightforward backtracking algorithm enumerating all perfect matchings and their complementary 2-factors. By an exhaustive computer search we have verified that no snark of order at most 16 has a special edge. (It is enough to look at snarks without parallel edges and with no two disjoint cycles of length at most 4. There is a unique snark on 18 vertices with a special edge; it is created by joining the terminals of two copies of  $P_3$  in such a way that the graph is bridgeless.) This completes the proof.  $\square$

The computer searches referred to in the proof of Theorem 12 can be extended to show that there are exactly two snarks with oddness 4 on 28 vertices—those displayed in Figure 2.

The following theorem provides an upper bound for the oddness ratio of a snark without restriction on connectivity.

**Theorem 13.** *For every even integer  $\omega \geq 2$  there exists a snark with oddness  $\omega$  having fewer than  $7.5\omega$  vertices.*

*Proof.* Let  $G$  be a snark and let  $v$  be an arbitrary vertex of  $G$ . Construct the graph  $G^{(v)}$  by inserting a copy of  $P_2$  into each of the three edges incident with the vertex  $v$  of  $G$ . This construction increases the number of vertices of  $G$  by 30 and the oddness by 4. Indeed, let  $F$  be a 2-factor of  $G^{(v)}$ ; this 2-factor has a corresponding 2-factor  $F'$  in  $G$  obtained by discarding the circuits of  $F$  contained in the inserted copies of  $P_2$  and replacing each path formed by the traversal of  $F$  through a copy of  $P_2$  by the appropriate edge of  $G$ . The vertex  $v$  has degree 2 in  $F$ , hence  $F$  passes through two edges incident with  $v$  in  $G^{(v)}$ . Each of the copies of  $P_2$  inserted into those edges contains exactly one 5-circuit of  $F$ . Moreover, the remaining copy of  $P_2$  contains two 5-circuits of  $F$ . The circuit of  $F$  passing through  $v$  has length increased by 10 compared to the corresponding circuit of  $F'$ . Put together,  $F$  has four more 5-circuits than the corresponding 2-factor  $F'$  of  $G$ .

Let  $R_0$  be the Petersen graph and for each even  $i \geq 1$  set  $R_{i+2} = R_i^{(v)}$ , where  $v$  is an arbitrary vertex of  $R_i$ . As explained above, for each integer  $t \geq 0$  the graph  $R_{2t}$  has order  $30t + 10$  and oddness  $4t + 2$ . Thus  $|V(R_{2t})| = 7.5 \cdot \omega(R_{2t}) - 5$ .

Further, let  $R_1$  be one of the two snarks of order 28 with oddness 4 described in the proof of Theorem 12, say  $H_1$ , and for each odd  $i \geq 3$  set  $R_{i+2} = R_i^{(v)}$ , where  $v$  is an arbitrary vertex of  $R_i$ . It follows that for each integer  $t \geq 0$  the graph  $R_{2t+1}$  has order  $30t - 2$  and oddness  $4t$ . Therefore  $|V(R_{2t+1})| = 7.5 \cdot \omega(R_{2t+1}) - 2$ .

Summing up, for each even integer  $\omega \geq 2$  we have constructed a snark with oddness  $\omega$  and order smaller than  $7.5\omega$ .  $\square$

We can construct many different sequences  $(G_i)$  of graphs with oddness ratio converging to 7.5 from below. However, no construction has led to any constant better than 7.5. This fact and examples like the family  $(R_n)_{n \geq 0}$  from the previous theorem lead us to conjecture the following.

**Conjecture 14.** *Every snark  $G$  with oddness  $\omega$  has at least  $7.5\omega - 5$  vertices.*

The smallest snark with oddness 6 we are aware of is the graph  $R_2$  of order 40 from the proof of Theorem 13; it is determined uniquely up to isomorphism. Somewhat surprisingly, the oddness ratio of  $R_2$  is  $20/3$ , which is better than the ratio 7 reached by  $R_1$ , one of two smallest graphs with oddness 4.

A bit more support for Conjecture 14 is given by the following argument. To generalise the construction given in Theorem 13 we attempt to insert a copy of  $P_2$  into several edges of an appropriate cubic graph  $H$ . Let  $S$  be the set of edges of  $H$  which are replaced by copies of  $P_2$ . We have added  $10|S|$  vertices and have guaranteed one or two odd circuits in each copy of  $P_2$ , depending on whether a 2-factor of the resulting graph  $H'$  passes through the copy or not. We show that there is a 2-factor  $F$  passing through at least  $2/3$  of the copies of  $P_2$  we have inserted into  $H$ . This result is, in fact, due to Fan [4, Equation 2] where it was proved by the technique of light 1-factors. Our proof uses the matching polytope instead.

**Lemma 15** (Fan [4]). *For any set  $S$  of edges of a bridgeless cubic graph  $G$  there exists a 2-factor in  $G$  which contains at least  $2/3 \cdot |S|$  edges from  $S$ .*

*Proof.* Let  $\mathbf{x}$  be a point of the perfect matching polytope  $\mathcal{P}(G)$  such that the sum

$$s(\mathbf{x}) = \sum_{e \in S} \mathbf{x}(e)$$

is minimal. The minimum of  $s(\mathbf{x})$  is attained on at least one vertex of  $\mathcal{P}(G)$ , hence we may assume that  $\mathbf{x}$  is a vertex. Since  $(1/3, 1/3, \dots, 1/3) \in \mathcal{P}(G)$ , the perfect matching corresponding to  $\mathbf{x}$  has at most  $1/3 \cdot |S|$  edges in  $S$ . The complementary 2-factor therefore contains at least  $2/3 \cdot |S|$  edges from  $S$ .  $\square$

From Lemma 15 it follows that the number of added odd circuits is at most  $4/3 \cdot |S|$ . Since we have added  $10 \cdot |S|$  vertices, the oddness ratio of  $H$  cannot be improved to be

less than 7.5, unless we have created many odd circuits by inserting copies of  $P_2$  into even circuits of the 2-factor of  $H$  corresponding to  $F$ —and many even circuits in a 2-factor of  $H$  mean that  $H$  has a large oddness ratio.

We do not have any better bounds on resistance than those established by Steffen [19]:  $8 \leq A_\rho^2 \leq 9$ . Our construction proves that  $A_\omega^2 \leq 7.5$ . This shows that  $A_\omega^2$  is strictly smaller than  $A_\rho^2$ .

## 7 Connectivity 4

The case of cyclic connectivity 4 is perhaps most interesting of all, because snarks with cyclic connectivity smaller than 4, or girth smaller than 5, are usually considered to be trivial. In a recent paper [5], Hägglund constructed an infinite family of cyclically 4-connected snarks whose oddness ratio is 15. The aim of this section is to present a family of snarks that improves the ratio to 13. We start with the 4-poles  $P_4^v$  and  $P_4^e$  defined in Section 5 and construct two specific uncolourable 4-poles  $N_1$  and  $N_2$ . Recall that the terminals of both  $P_4^v$  and  $P_4^e$  are partitioned into two pairs. The colouring properties of  $P_4^v$  and  $P_4^e$  described in Section 5 imply that if we join one pair of terminal edges from  $P_4^e$  to a pair of terminal edges from  $P_4^v$  we get an uncolourable 4-pole. This 4-pole has 18 nonterminal vertices and will be denoted by  $N_1$  (see Figure 3 left). The 4-pole  $N_2$  arises from  $P_4^e$  and two distinct copies of  $P_4^v$  by joining each pair of terminal edges of  $P_4^e$  to a pair of terminal edges in a different copy of  $P_4^v$  (see Figure 3 right). Thus  $N_2$  has 26 nonterminal vertices. The important property of  $N_2$  is that it is not only uncolourable, but it remains so even after removing an arbitrary nonterminal vertex. This is obviously true if the removed vertex  $w$  lies in a copy of  $P_4^v$  in  $N_2$ , because the removal of such a vertex leaves a copy of  $N_1$  in  $N_2 - w$  intact, and  $N_1$  is uncolourable. Assume that we have removed a vertex  $w$  from a copy of  $P_4^e$  in  $N_2$ . Then in every 3-edge-colouring of  $P_4^e - w$  at least one of the pairs of terminal edges of  $P_4^e$  will receive distinct colours, for otherwise a 3-edge-colouring of  $P_4^e - w$  would induce a 3-edge-colouring of  $P - w$ , which is, however, uncolourable by the Parity Lemma. On the other hand, the same pair of edges is forced to have identical colours from the adjacent copy of  $P_4^v$ . This again implies that  $N_2 - w$  is uncolourable.

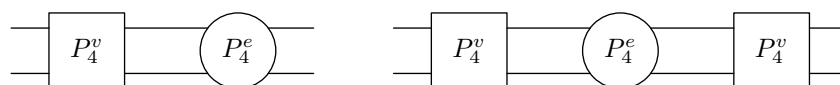


Figure 3: Networks  $N_1$  (left) and  $N_2$  (right)

To construct a cyclically 4-connected snark with arbitrarily high oddness we take a number of copies of  $N_1$  and a number of  $N_2$ , arrange them into a circuit, and join one pair of terminal edges from each copy to a pair of terminal edges of its predecessor and another pair of terminal edges to a pair of terminal edges of its successor. The way in which copies of  $N_1$  and  $N_2$  are arranged is not unique, therefore we may get several non-isomorphic graphs even if we only use copies of one of  $N_1$  and  $N_2$ .



In this construction, each copy of  $N_1$  adds 1 and each copy of  $N_2$  adds 2 to the resistance of the resulting graph. Thus if we take  $r$  copies of  $N_2$ , we get a cyclically 4-connected snark of order  $26r$  with resistance  $2r$  and oddness at least  $2r$ . If we take  $r$  copies of  $N_2$  and one copy of  $N_1$  we get a cyclically 4-connected snark of order  $26r + 18$  with resistance  $2r + 1$  and oddness at least  $2r + 2$ . In particular, this shows that  $A_\omega^4 \leq A_\rho^4 \leq 13$ .

For  $r = 1$  our construction produces a cyclically 4-connected snark of order 44 with resistance 3 and oddness 4 (see Figure 4). This is currently the smallest known non-trivial snark of oddness at least 4. It improves the best previously known values of 50 and 46 (see [1, 5]).

We have tried various other approaches to construct a cyclically 4-connected snark with oddness 4 on fewer vertices (mostly computer-assisted), but all of them eventually led to snarks of order 44, not necessarily isomorphic to the one from Figure 4. We therefore believe that the following is true.

**Conjecture 16.** *The smallest cyclically 4-connected snark with oddness 4 has 44 vertices.*

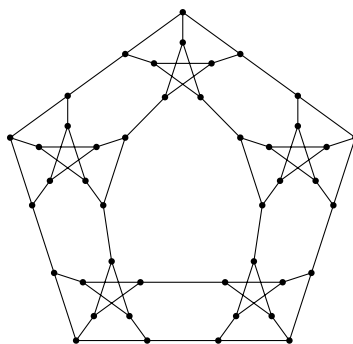


Figure 4: A cyclically 4-connected snark of order 44 with oddness 4.

## 8 Connectivity 5

Steffen in [19, Theorem 2.3] constructed a cyclically 5-connected snark of order  $608r$  with oddness at least  $8r$  and resistance  $6r$  for every positive integer  $r$ . His construction shows that  $A_\rho^5 \leq 101 + 1/3$  and  $A_\omega^5 \leq 76$ . Here we construct, for each integer  $r \geq 2$ , a cyclically 5-connected snark of order  $25r$  with resistance  $r$ . Our construction thus yields  $A_\omega^5 \leq A_\rho^5 \leq 25$ .

We take two disjoint copies of  $P_5^{ev}$  and one copy of  $P_5^{vvv}$ . Recall that  $P_5^{vvv}$  has two pairs of terminal edges and one single terminal edge. We join one pair of terminals to the pair of terminals in the first copy of  $P_5^{ev}$  and the other pair to the pair in the second copy of  $P_5^{ev}$  (see Figure 5). The resulting 7-pole  $Z$  has 25 vertices; it coincides with the one constructed by Steffen [19, Figure 1].

In every 3-edge-colouring of  $P_5^{vvv}$  the terminal edges of one of the two pairs of terminals must have the same colour. On the other hand, in every 3-edge-colouring of  $P_5^{ev}$  the edges

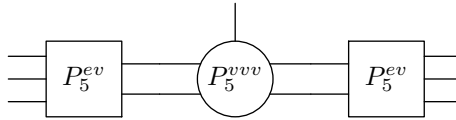


Figure 5: A non-colourable cyclically 5-connected 7-pole.

in the pair of terminals have different colours. Therefore the 7-pole  $Z$  is uncolourable. The required graphs can be easily constructed by joining terminals from  $r$  disjoint copies of  $Z$  in such a way that the graph becomes cyclically 5-connected. Since  $Z$  is uncolourable, we need to remove a vertex from each copy of  $Z$  in order to get a 3-edge-colourable subgraph. The resistance of the resulting graph is therefore at least  $r$ .

## 9 Connectivity 6

The first infinite class of cyclically 6-connected snarks with arbitrarily large oddness was described by Kochol [13]. For each positive integer  $n$  he constructed a cyclically 6-connected snark of order  $118r$  with resistance at least  $r$ , thereby establishing the bound  $A_\omega^6 \leq A_\rho^6 \leq 118$ . We improve this upper bound to 99 by constructing, for each integer  $r \geq 2$ , a snark with resistance at least  $r$  and with order  $99r$  or  $99r + 1$ , depending on whether  $r$  is even or odd, respectively.

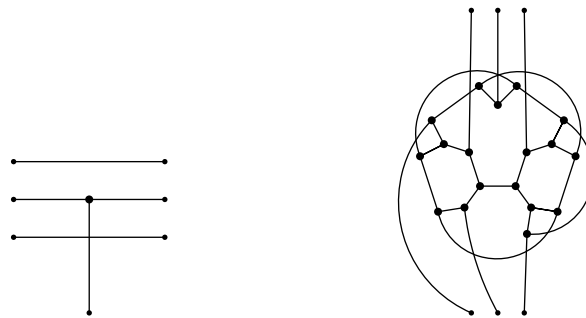


Figure 6: The supervertex  $X$  and the superedge  $Y$ .

Take  $r$  copies of  $P_3$ , say  $Q_1, Q_2, \dots, Q_r$ , where  $r \geq 2$ . Arrange them into a circuit and, for each  $i \in \{1, 2, \dots, r\}$ , join one of the terminals of  $Q_i$  to a terminal of  $Q_{i-1}$  and another terminal to a terminal of  $Q_{i+1}$ , with indices reduced modulo  $r$ . This leaves one terminal of each  $Q_i$  unmatched and produces an  $r$ -pole. Partition the terminals of this  $r$ -pole into pairs and, if  $r$  is odd, one triple. Identify the terminals of each partition set, and suppress the resulting 2-valent vertices to obtain a cubic graph  $L_r$ . Its order is  $9r$  or  $9r + 1$ , depending on whether  $r$  is even or odd, respectively. The identification process can clearly be performed in such a way that  $L_r$  is 3-connected.

To obtain a snark with cyclic connectivity 6 we apply superposition to  $L_r$ . Nontrivial supervertices will be copies of the 7-pole  $X$  with a single nonterminal vertex shown in

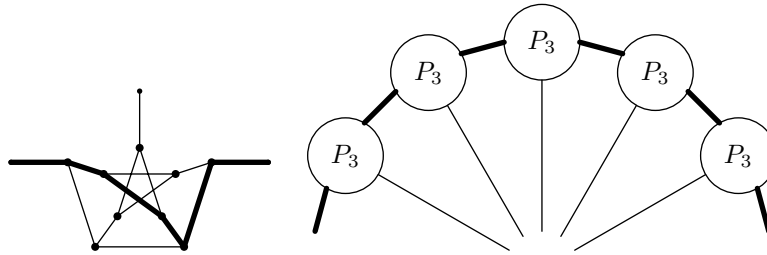


Figure 7: The superposed circuit passing through a copy of  $P_3$  and the arrangement of the copies of  $P_3$  in  $L_r$ .

Figure 6 (left), while nontrivial superedges will be copies of the 6-pole  $Y$  with 18 nonterminal vertices created from the flower snark  $J_5$  by splitting off two nonadjacent vertices, one from the single 5-circuit of  $J_5$ ; see Figure 6 (right). We choose a circuit  $C$  in  $L_r$  which intersects each copy of  $P_3$  in the edges displayed bold in Figure 7 (left) and finish the superposition by replacing each vertex on  $C$  with a copy of  $X$  and each edge on  $C$  with a copy of  $Y$ ; we use trivial supervertices and superedges everywhere outside  $C$ . The superposition is indicated in Figure 7.

Let  $M_r$  be the resulting graph. The construction replaces the  $5r$  vertices of  $C$  with the same number of new vertices, one for each nontrivial supervertex, and adds  $18 \cdot 5r = 90r$  vertices, 18 new vertices for each nontrivial superedge. It follows that  $M_r$  has  $99r$  or  $99r + 1$  vertices depending on whether  $n$  is even or odd, respectively. Due to the choice of supervertices and superedges and the fact that the removal of the edges of  $C$  from each copy of  $P_3$  in  $L_r$  leaves an acyclic subgraph, the graph  $M_r$  is indeed cyclically 6-connected.

We prove that  $\rho(M_r) \geq r$ . Observe that in order to get a colourable graph from  $L_r$ , one has to delete at least one vertex from each copy of  $P_3$ , that is, at least  $r$  vertices in total. By Proposition 11,  $\rho(M_r) \geq \rho(L_r) \geq n$ , and therefore  $\omega(M_r) \geq r$ . This yields the bounds  $A_\omega^6 \leq A_\rho^6 \leq 99$ .

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