Covering Partial Cubes with Zones

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Abstract

A partial cube is a graph having an isometric embedding in a hypercube. Partial cubes are characterized by a natural equivalence relation on the edges, whose classes are called *zones*. The number of zones determines the minimal dimension of a hypercube in which the graph can be embedded. We consider the problem of covering the vertices of a partial cube with the minimum number of zones. The problem admits several special cases, among which are the following:

- cover the cells of a line arrangement with a minimum number of lines,
- select a smallest subset of edges in a graph such that for every acyclic orientation, there exists a selected edge that can be flipped without creating a cycle,
- find a smallest set of incomparable pairs of elements in a poset such that in every linear extension, at least one such pair is consecutive,
- find a minimum-size fibre in a bipartite poset.

We give upper and lower bounds on the worst-case minimum size of a covering by zones in several of those cases. We also consider the computational complexity of those problems, and establish some hardness results.

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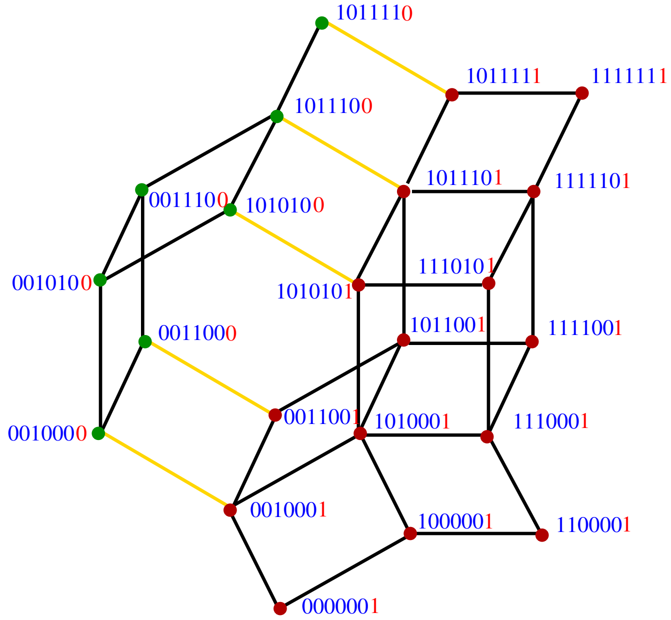

Figure 1: A partial cube with vertex labels representing an isometric embedding in Q_7 and a highlighted zone.

1 Introduction

As an introduction and motivation to the problems we consider, let us look at two puzzles.

Hitting a consecutive pair. Given a set of n elements, how many pairs of them must be chosen so that every permutation of the n elements has two consecutive elements forming a chosen pair?

Guarding cells of a line arrangement. Given an arrangement of n straight lines in the plane, how many lines must be chosen so that every cell of the arrangement is bounded by at least one of the chosen line?

While different, the two problems can be cast as special cases of a general problem involving *partial cubes*. The n-dimensional hypercube Q_n is the graph with the set $\{0,1\}^n$ of binary words of length n as vertex set, and an edge between every pair of vertices that differ on exactly one bit. A *partial cube* G is a subgraph of an n-dimensional hypercube with the property that the distance between two vertices in G is equal to their Hamming distance, that is, their distance in Q_n. In general, a graph G is said to have an *isometric embedding* in another graph H whenever G is a subgraph of H and the distance between any two vertices in G is equal to their distance in H. Hence partial cubes are the graphs admitting an isometric embedding in Q_n, for some n. The *dimension* of the partial cube is the minimum n for which such an embedding exists. The embedding can then be shown to be unique (see for instance Hammack et al. [14] for details). The edges of a partial cube can be partitioned into at most n equivalence classes called *zones*, each corresponding to one of the n directions of the edges of Q_n.

We consider the following

Minimum Zone Cover Problem: *Given a partial cube, find a smallest subset S of its zones such that every vertex is incident to an edge of one of the zones in S.*

In what follows, we use the expression *zone cover* for any subset of zones covering the vertices (not necessarily minimum). If we refer to the labeling of the vertices by words in $\{0,1\}^n$, the problem amounts to finding a smallest subset I of $[n]$ such that for every vertex v of the input partial cube, there is at least one $i \in I$ such that flipping bit i of the word of v yields another vertex.

2 Classes of Partial Cubes

The reader is referred to the books of Ovchinnikov [17] and Hammack, Imrich, and Klavžar [14] for known results on partial cubes. Let us also mention that some other structures previously defined in the literature are essentially equivalent to partial cubes. Among them are *well-graded families of sets* defined by Doignon and Falmagne [7], and *Media*, defined by Eppstein, Falmagne, and Ovchinnikov [10].

We are interested in giving bounds on the minimum number of zones required to cover the vertices of a partial cube, but also in the computational complexity of the problem of finding such a minimum cover. Regarding bounds, it would have been nice to have a general nontrivial result holding for every partial cube. Unfortunately, only trivial bounds hold in general.

In one extreme case, the partial cube $G = (V, E)$ is a star, consisting of one central vertex connected to $|V| - 1$ other vertices of degree one. This is indeed a partial cube in dimension $n = |V| - 1$, every zone of which consists of a single edge. Since there are n vertices of degree one, all n zones must be chosen to cover all vertices. In the other extreme case, the partial cube is such that there exists a single zone covering all vertices. This lower bound is attained by the hypercube Q_n.

Table 1 gives a summary of our results for the various families of partial cubes that we considered. For each family, we consider upper and lower bounds and complexity results. We now briefly describe the various families we studied. The proofs of the new results are in the following sections.

2.1 Hyperplane Arrangements

The dual graph of a simple[1] arrangement of n hyperplanes in $\mathbb{R}^d$ is a partial cube of dimension n. The zone cover problem becomes the following: given a simple hyperplane arrangement, find a smallest subset S of the hyperplanes such that every cell of the arrangement is bounded by at least one hyperplane in S.

[1] An arrangement is called *simple* if any $d + 1$ hyperplanes have empty intersection.

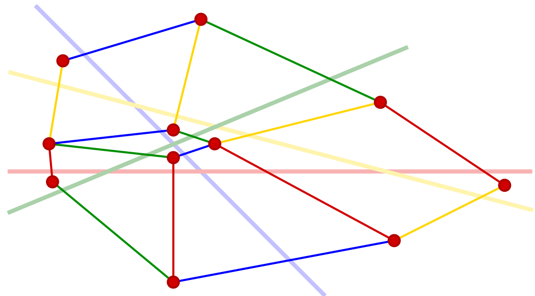

Figure 2: An arrangement of 4 lines superimposed with its dual.

The case of line arrangements was first considered in [3]. In fact it was this problem that motivated us to investigate the generalization to partial cubes. The best known lower and upper bounds for line arrangements on $|S|$ are of order $n/6$ and $n - O(\sqrt{n \log n})$ (see [1]). The complexity status of the optimization problem is unknown.

Instead of lines and hyperplanes it is also possible to consider Euclidean or spherical arrangements of pseudo-lines and pseudo-hyperplanes, their duals are still partial cubes. Actually, all the cited results apply to the case of arrangements of pseudo-lines.

Spherical arrangements of pseudo-hyperplanes are equivalent to oriented matroids, this is the Topological Representation Theorem of Folkman and Lawrence. The pseudo-hyperplanes correspond to the elements of the oriented matroid and the cells of the arrangement correspond to the topes of the oriented matroid. Hence the zone cover problem asks for a minimum size set C of elements such that for every tope T there is an element $c \in C$ such that $T \oplus c$ is another tope. For more on oriented matroids we refer to [2].

2.2 Acyclic Orientations

From a graph $G = (V, E)$, we can define a partial cube H in which every vertex is an acyclic orientation of G, and two orientations are adjacent whenever they differ by a single edge reversal (*flip*). This partial cube has dimension equal to the number of edges of G. It is also the dual graph of the arrangement of the $|E|$ hyperplanes of equation $x_i = x_j$ for $ij \in E$ in $\mathbb{R}^V$. Every cell of this arrangement corresponds to an acyclic orientation, and adjacent cells are exactly those that differ by a single edge. This observation, in particular, shows that H is connected. The graph H and the notion of edge flippability have been studied by Fukuda et al. [11] and more recently by Cordovil and Forge [6].

The zone cover problem now becomes the following: given a graph $G = (V, E)$, find a smallest subset $S \subseteq E$ such that for every acyclic orientation of G, there exists $e \in S$ such that flipping the orientation of e does not create a cycle.

Note that if G is the complete graph, H is the skeleton of the permutohedron. We can show that even in that case, deciding whether a given subset of edges is a feasible solution to the zone cover problem is NP-complete.

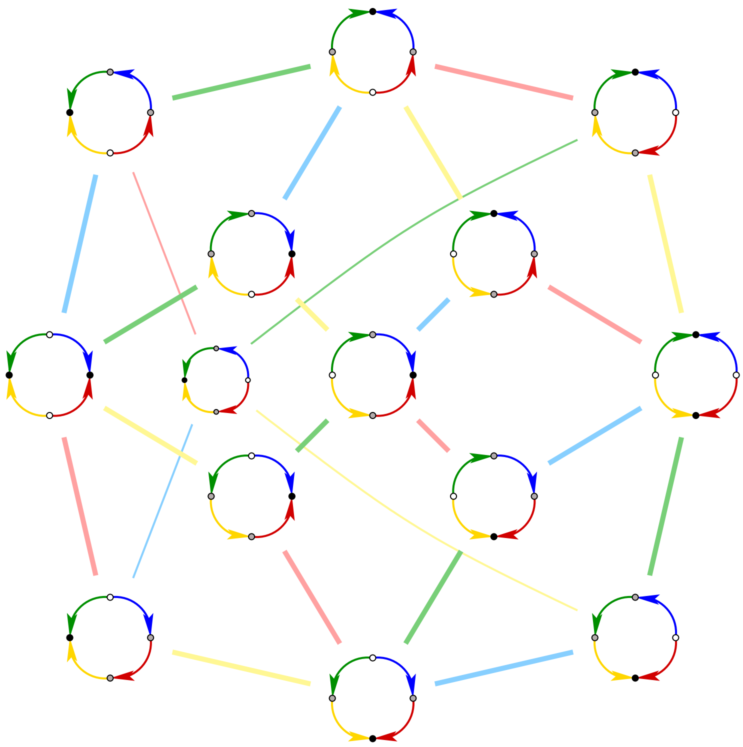

Figure 3: Flip graph of acyclic orientations of the 4-cycle.

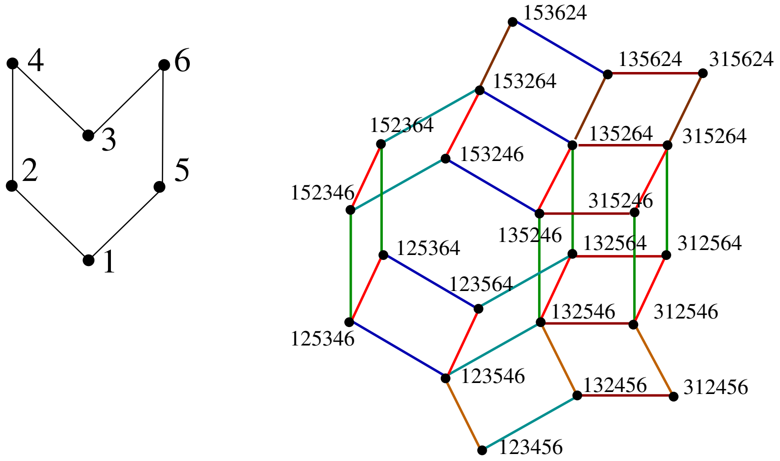

Figure 4: A poset (left) and its linear extension graph (right).

2.3 Linear Extension Graphs

The next class of partial cubes we consider are subgraphs of the permutohedron. Given a poset P, we define a partial cube, every vertex of which is a linear extension of P. Two such vertices are adjacent whenever the two corresponding linear extensions can be obtained from one another by swapping two consecutive elements. The linear extension graph is therefore the subgraph of the permutohedron induced by the linear extensions of P. An example is shown on Figure 4. It has been the subject of thorough investigations, in particular by Reuter [18], and Massow [16].

The zone cover problem can now be stated as follows: given a poset P, find the minimum number of incomparable pairs of elements such that in every linear extension of P, at least one such pair is consecutive.

In the special case where the order on P is empty (no pair is comparable), we indeed obtain the partial cube of acyclic orientations of the complete graph.

2.4 Median Graphs

A median graph is an undirected graph in which every three vertices x, y, and z have a unique *median*, that is, a vertex $\mu(x, y, z)$ that belongs to shortest paths between each pair of x, y, and z. Median graphs form structured subclass of partial cubes that has been studied extensively, see [5] and references therein. Since the star and the hypercube are median graph there are no non-trivial upper or lower bounds for the zone cover problem on this class. We use a construction of median graphs to prove hardness of the zone cover problem.

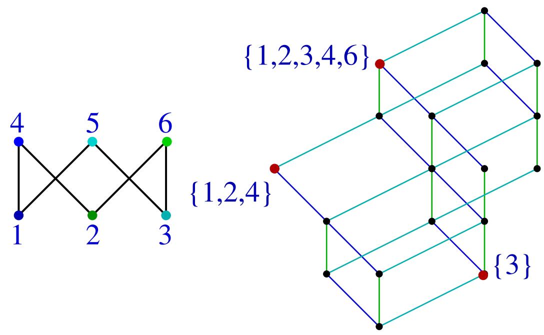

Figure 5: A poset (left) and the cover graph of its lattice of downsets (right).

2.5 Distributive Lattices

Cover graphs of distributive lattices[2] are partial cubes. From Birkhoff's representation theorem (a.k.a. Fundamental Theorem of Finite Distributive Lattices) we know that there is a poset P such that the vertices of the partial cube are the downsets of P, and the zones of the partial cube in turn correspond to the elements of P.

The problem becomes: given a poset P, find a smallest subset S of its elements such that for every downset D of P, there exists $v \in S$ such that either $D \cup \{v\}$ or $D \setminus \{v\}$ is a downset, distinct from D.

2.6 Trees

Trees with n edges are partial cubes of dimension exactly n. Since every zone contains exactly one edge, the zone cover problem on trees boils down to the edge cover problem on trees. There are instances attaining the upper bound of $n-1$, and there is a simple dynamic programming algorithm that computes an optimal cover in linear time.

3 Hyperplane Arrangements

We first consider the case of hyperplanes in $\mathbb{R}^2$, that is, line arrangements. Recall that we have defined a zone cover for an arrangement of lines as a subset of the lines so that every cell of the arrangement is bounded by at least one of the chosen lines. We first give a lower bound on the size of a zone cover for an arrangement of lines.

Theorem 1. *The minimum number of lines needed to cover the cells of any arrangement of n lines is $n - o(n)$.*

Proof. The proof is a direct consequence of known results on the following problem from Erdős: given a set of points in the plane with no four points on a line, find the largest subset in general position, that is, with no three points on a line. Let $\alpha(n)$ be the minimum

[2] For a very good introduction to terminology related to partial orders and lattices we refer to Chapter 3 of Stanley, *Enumerative Combinatorics Vol. I* [19].

Partial Cubes	Lower bound	Upper bound	Complexity
Arrangements of n lines in $\mathbb{R}^2$	$n - o(n)$ (Thm. 1)	$n - \Omega(\sqrt{n \log n})$ [1]	–
Arrangements of n hyperplanes in $\mathbb{R}^d$	–	$n - \Omega(n^{1/d})$ (Thm. 2)	–
Acyclic orientations	–	Minimum edge cut (Thm. 6)	Recognition is coNP-complete, even for complete graphs (Thm. 3)
Linear Extension Graphs	$2n$ (Cor.13)		Recognition is coNP-complete, even for empty orders (Cor. 9)
Median graphs	n		NP-complete (Cor. 15), APX-hard (Cor. 16)
Distributive lattices with representative poset of n elements	–	$2n/3$ (Cor. 18)	Recognition is coNP-complete (Cor. 20) Σ_2^P-complete (Cor. 21)
Trees with n edges	$n - 1$		P (min edge cover)

Table 1: Worst-case bounds and complexities for the special cases of the zone cover problem. When used, n denotes the dimension of the partial cube.

size of such a set over all arrangements of n points. Füredi observed [12] that $\alpha(n) = o(n)$ follows from the density version of the Hales-Jewett theorem [13]. But this directly proves that we need at least $n - o(n)$ lines to cover all cells of an arrangement. The reduction is the one observed by Ackerman et al. [1]: consider the line arrangement that is dual to the point set, and slightly perturb it so that each triple of concurrent lines forms a cell of size three. Now the complement of any zone cover is in general position in the primal point set. □

We now show that for arrangements of hyperplanes in $\mathbb{R}^d$, with $d = O(1)$, there always exists a zone cover of size at most $n - \Omega(n^{1/d})$. The proof is along the lines of the proof given in [3] for the $d = 2$ case.

Theorem 2. *In every arrangement of n hyperplanes in general position in $\mathbb{R}^d$ (with $d = O(1)$), there exists a subset of the hyperplanes of size at most $n - \Omega(n^{1/d})$ such that every cell is bounded by at least one hyperplane in the subset.*

Proof. We will prove that every such arrangement has an *independent set* of size $\Omega(n^{1/d})$, where an independent set is defined as the complement of a zone cover, that is, a subset of the hyperplanes such that no cell is bounded by hyperplanes of the subset only.

Let H be a set of n hyperplanes, and consider an arbitrary, inclusionwise maximal independent set I. For each hyperplane $h \in H \setminus I$, there must be a cell c_h of the arrangement that is bounded by a set of hyperplanes $C \cup \{h\}$ with $C \subseteq I$, since otherwise we could add h to I, and I would not be maximal. If c_h is bounded by at least $d + 1$

hyperplanes, then there must be a vertex of the arrangement induced by I that is also a vertex of c_h. We charge h to this vertex. Note that each vertex can only be charged $2^d = O(1)$ times. If c_h is bounded by d hyperplanes, then we charge h to the remaining $(d-1)$-subset of hyperplanes of I bounding c_h. This subset can also be charged at most $O(1)$ times. The total charge is therefore at most $O(\binom{|I|}{d}) + O(\binom{|I|}{d-1}) = O(|I|^d)$, and this must be an upper bound on $|H \setminus I|$. Hence

$$\begin{aligned} |H \setminus I| = n - |I| &\leqslant O(|I|^d) \\ |I| &= \Omega(n^{1/d}). \end{aligned}$$

□

4 Acyclic Orientations

Given a graph $G = (V, E)$ we wish to find a subset $S \subseteq E$ such that for every acyclic orientation of G, there exists a *flippable* edge $e \in S$, that is, an edge $e \in S$ such that changing the orientation of e does not create any cycle. Let us call such a set an *AO-zone cover* for G. Note that an oriented edge $e = uv$ in an acyclic orientation is flippable if and only if it is not *transitive*, that is, if and only if uv is the only oriented path from u to v.

4.1 Complexity

Theorem 3. *Given a graph $G = (V, E)$ and a subset $S \subseteq E$ of its edges, the problem of deciding whether S is an AO-zone cover is coNP-complete, even if G is a complete graph.*

Proof. The set S is not an AO-zone cover if and only if there exists an acyclic orientation of G in which all edges $e \in S$ are transitive.

Consider a simple graph H on the vertex set V, and define G as the complete graph on V, and S as the set of non-edges of H. We claim that S is an AO-zone cover for G if and only if H does not have a Hamiltonian path. Since deciding the existence of a Hamiltonian path is NP-complete, this proves the result.

To prove the claim, first suppose that H has a Hamiltonian path, and consider the acyclic orientation of G that corresponds to the order of the vertices in the path. Then by definition, no edge of S is in the path, hence all of them are transitive, and S is not an AO-zone cover. Conversely, suppose that S is not an AO-zone cover. Then there exists an acyclic orientation of G in which all edges of S are transitive. The corresponding ordering of the vertices in V yields a Hamiltonian path in H. □

4.2 Special cases and an upper bound

Lemma 4. *The minimum size of an AO-zone cover in a complete graph on n vertices is $n - 1$.*

Proof. First note that there always exists an AO-zone cover of size $n - 1$ that consists of all edges incident to one vertex.

Now we need to show that any other edge subset of smaller size is not an AO-zone cover. This amounts to stating that every graph with at least $\binom{n}{2} - (n-2)$ edges has a Hamiltonian path. To see this, proceed by induction on n. Suppose this holds for graphs with strictly less than n vertices. Consider a set S of at most $n-2$ edges of the complete graph. Let u, v be two vertices with $uv \in S$. One of the two vertices, say v, is incident to at most $1 + \lfloor (n-3)/2 \rfloor = \lfloor (n-1)/2 \rfloor$ edges of S. Hence v is incident to at least $\lceil (n-1)/2 \rceil$ edges that are not in S.

Consider a Hamiltonian path on the $n-1$ vertices other than v, which exists by induction. If v is adjacent to the first or last vertex in this path, and the corresponding edge is not in S, then we can extend this path to a Hamiltonian path containing v. Otherwise, since v is incident to at least $\lceil (n-1)/2 \rceil$ edges that are not in S, there must be two consecutive vertices of the path that are adjacent to v, and again we can include v in the path. □

Since acyclic orientations of the complete graph K_n correspond to the permutations of S_n, this is in fact the solution to the puzzle "hitting a consecutive pair" in the introduction. The dual graph of the arrangements of hyperplanes corresponding to the complete graph (graphic hyperplane arrangement of K_n) is the skeleton graph of the permutohedron. Hence the above result can also be stated in the following form.

Corollary 5. *The minimum size of a zone cover of the n-dimensional permutohedron is $n-1$.*

We now give a simple, polynomial-time computable upper bound on the size of an AO-zone cover. A set $C \subseteq E$ is an *edge cut* whenever the graph $(V, E \setminus C)$ is not connected.

Theorem 6. *Every edge cut of G is an AO-zone cover of G.*

Proof. Consider an edge cut $C \subseteq E$ and an acyclic orientation A_G of G. This acyclic orientation can be used to define a partial order on V. Let us consider a total ordering of V that extends this partial order, and pick an edge $e = uv \in C$ that minimizes the rank difference between u and v. We claim that e is flippable. Suppose for the sake of contradiction that e is not flippable. Then e must be transitive and there exists a directed path P in A_G between u and v that does not use e. Since C is a cut, u and v belong to distinct connected components of $(V, E \setminus C)$, and P must use another edge $e' \in C$. By definition, the endpoints of e' have a rank difference that is smaller than that of u and v, contradicting the choice of e. □

An even shorter proof of the above can be obtained by reusing the following observation from Cordovil and Forge [6]: for every acyclic orientation of G, the set of flippable edges is a spanning set of edges. Therefore, every such set must intersect every edge cut.

While AO-zone covers are hard to recognize, even for complete graphs, we show that a minimum-size AO-zone cover can be found in polynomial time whenever the input graph is chordal, i.e., if every cycle of length at least 4 has a chord. In that case, the upper bound given by the minimum edge cut is tight, and the result generalizes Lemma 4.

Theorem 7. *The minimum size of an AO-zone cover in a chordal graph is the size of a minimum edge cut.*

Proof. We need to show that whenever a set S of edges of a chordal graph G has size strictly less than the edge connectivity of G, there exists an acyclic orientation of G in which all edges of S are transitive. Let us denote by k the edge connectivity of G, and n its number of vertices.

We proceed by induction on n. For the base case we consider complete graphs. From Lemma 4, the minimum size of an AO-zone cover in K_{k+1} is k. Now suppose the statement holds for every chordal graph with $n-1$ vertices, and that there exists a k edge connected chordal graph G on n vertices with an AO-zone cover S of size $k-1$. Since G is chordal, it has at least two nonadjacent simplicial vertices u and v, i.e., vertices whose neighborhood induces a clique. The degree of both u and v is at least k. Hence one of them, say v, is incident to at most $\lfloor (k-1)/2 \rfloor \leqslant \lfloor (d(v)-1)/2 \rfloor$ edges of S. Now remove v and consider, using the induction hypothesis, a suitable acyclic orientation of the remaining graph. This orientation induces a total order, i.e., a path p with $d(v)$ vertices, on the neighbors $N(v)$ of v. Then vertex v must have one or two incident edges that do not belong to S and connect v to the first, or the last, or two consecutive vertices of path p. Hence we can integrate v in the path such that all the edges of S that are incident to v are transitive. This yields a suitable acyclic orientation for G and completes the induction step. □

When the graph is not chordal, it may happen that the minimum size of an AO-zone cover is arbitrarily small compared to the edge connectivity. We can in fact construct a large family of such examples.

Theorem 8. *For every natural number t and odd natural number g such that $3 \leqslant g \leqslant t$, there exists a graph G with edge connectivity t and an AO-zone cover of size g.*

Proof. The graph G is constructed by considering a wheel with $g+1$ vertices and center c, and replacing every edge incident to the center c by a copy of the complete graph K_{t-1} such that each vertex of the complete graph is connected to both endpoints of the original wheel edge. Figure 6 shows an example. Let us call C_g the cycle induced by the non-center vertices.

Let us first look at the edge connectivity of G. Vertices belonging to one of the K_{t-1} have degree t. On the other hand we easily construct t edge-disjoint paths between every pair of vertices. Hence, the edge connectivity of G equals t.

We now show that the set of edges of C_g is an AO-zone cover. Suppose the opposite: there exists an acyclic orientation for which no edge of the cycle is flippable, hence all of them are transitive. Since C_g is odd, there must exist two consecutive edges of C_g with the same orientation, say uv and vw, with uv oriented from u to v, and vw from v to w.

The only way to make vw transitive is to construct an oriented path from v to w going through the center c. Hence there must exist an oriented path of the form vPc, where P is a path in the complete graph K_{t-1} attached to v. To make uv transitive we need a directed path $cP'v$. The oriented cycle $vPcP'v$ is in contradiction to the acyclicity. Therefore, some edge of C_g is always flippable. □

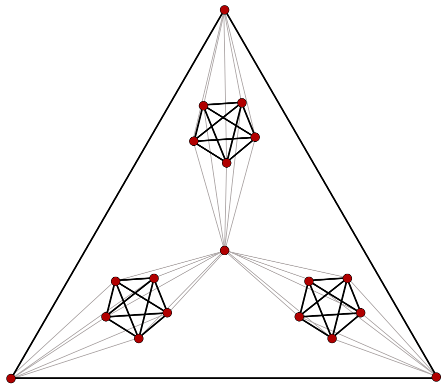

Figure 6: A graph G with edge connectivity 6 and an AO-zone cover of size 3.

5 Linear Extension Graphs

We now turn to the class of linear extension graphs, which are restriction of the permutohedron to linear extensions of a partially ordered set. A zone cover of a linear extension graph is a set of incomparable pairs of the poset such that in any linear extension, there is at least one such pair that is consecutive. In this section, we give a tight worst-case bound on the minimum size of such a zone cover.

The special case of the empty order (no two elements are comparable) is tackled in Theorem 3. We therefore inherit the following complexity result from the previous section.

Corollary 9. *Given a partially ordered set and a subset S of its incomparable pairs, the problem of deciding whether S is a zone cover of the linear extension graph of P is coNP-complete.*

In order to prove a tight worst-case bound on the size of a minimum zone cover, we need the following technical lemma. This can be seen as a directed version of Lemma 4.

Lemma 10. *Consider the complete directed graph on $k \geqslant 2$ vertices having two directed edges between every pair of vertices. Every subgraph obtained by deleting at most $2k-3$ edges contains a directed Hamiltonian path.*

Proof. We prove this by induction on k. The base case $k = 2$ is easy. Suppose the results holds for $k-1$ and let us prove it for k. Let us delete a set S of at most $2k-3$ edges.

First suppose there exist two vertices u and v such that both edges uv and vu belong to S. One of the two vertices, say v, must be incident to at most $2+\lfloor(2k-5)/2\rfloor = k-1$ edges of S. Hence v has at least $2(k-1)-(k-1) = k-1$ incident edges that are not deleted. The graph on $n-1$ vertices obtained by removing v has at most $2k-3-2 = 2(k-1)-3$ deleted edges and therefore, by induction, contains a Hamiltonian path.

Then we can insert v in the path and obtain a directed Hamiltonian path for the whole graph. Suppose by contradiction that it is impossible. Then for each pair of successive vertices in the path, say xy, one of the edge xv or vy must be missing, for otherwise we

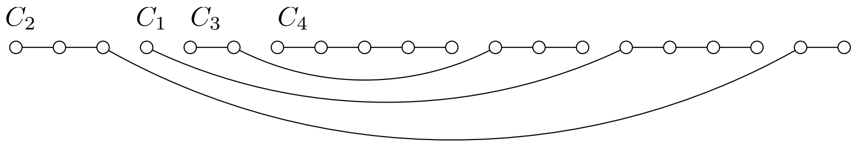

Figure 7: A nested linear extension of a poset consisting of $k = 4$ chains of length $\ell = 5$.

have a path including v. Also, if the path starts at a and ends at b, we cannot have edges va nor bv. Hence at least $(k-2)+2 = k$ edges must be missing, among all possible $2(k-1)$ edges incident to v. But then only at most $2(k-1)-k = k-2$ edges can remain, and we know we have $k-1$, a contradiction.

Now suppose there does not exist two vertices u and v such that both edges uv and vu belong to S. Then, since S has size greater than $k-1$, the graph induced by S must contain an undirected cycle. Hence there exists two consecutive vertices u and v on this cycle, both of which are incident to at least two deleted edges. Letting v be the one incident with the smallest number of edges of S, we can apply the same reasoning as above. $\square$

We now prove a lower bound for a special family of posets. The case of the permutohedron is that of the empty order, for which $k = n, \ell = 1$.

Lemma 11. *Consider a disjoint union of k chains of length ℓ, for $k, \ell \geqslant 1$. Every zone cover of the linear extension graph of such an order must be of size at least* $(2k-2)\ell-k+1$.

Proof. We consider a specific family of linear extensions of such a partial order, each of which is defined by two parameters:

- a linear order on the k chains,
- an index $j_a \in [\ell]$ for each chain C_a.

The corresponding linear extension is composed of blocks of the first j_a minimum elements of each chain C_a, with the chains taken in the prescribed linear order, followed by the $\ell - j_a$ remaining elements of each chain C_a, where the chains are now taken in the reversed linear order. We call this family of linear extensions *nested linear extensions*. An example is given in Figure 7. We give a lower bound on the number of zones required to cover all *nested* linear extensions.

Consider such a nested linear extension. Some incomparable pairs that are consecutive in this extension are of the type $(x_{j_a}, \min_b)$, for two chains C_a and C_b, $a \neq b \in [k]$, where $x_{j_a} \in C_a$, and $\min_b$ is the minimum of chain C_b. Another type of consecutive pairs ar those of the form $(\max_b, x_{j_a+1})$, where $\max_b$ is the maximum in C_b.

We now construct a complete directed graph with k vertices, where every vertex corresponds to one of the k chains. Every set of incomparable pairs of the form $(x_{j_a}, \min_b)$ will be associated with the directed edge ab in this graph. We now say that such an edge is *covered* when all the zones of the form $(x_{j_a}, \min_b)$, for $j_a \in [\ell]$ and $x_{j_a} \in C_a$, are taken in a zone cover. Note that a zone of the form $(\max_b, x_{j_a+1})$ can be traded for one of the form $(x_{j_a}, \min_b)$. The linear order on the k chains defining a nested linear extension induces a directed Hamiltonian path in the graph. In order to find a set of zones covering all such linear extensions, we have to make sure that for each directed Hamiltonian path, at least one of the used edges is covered. From Lemma 10, we know that at least $2k-2$ edges of the graph must be chosen.

Covering an edge of the graph requires ℓ zones. However, we may double count the zones of the form $(\min_a, \min_b)$. To prevent double counting, we have to remove at most $k-1$ zones from our cover. The total number of zones is therefore at least $(2k-2)\ell-k+1$, as claimed. $\square$

Letting $n = k\ell$, this bound can be rewritten as $\frac{2(k-1)}{k}n - k + 1$. We now prove that it is tight.

Theorem 12. *A minimum zone cover of the linear extension graph of an n-element poset P with k minimal elements has size at most*

$$\frac{2(k-1)}{k}n - k + 1.$$

Proof. We introduce a few notations. Let MIN be the set of minimal elements of P, and for $x \in P$, let $\mathrm{INC}(x)$ be the set of elements that are incomparable to x, $\downarrow(x)$ the set of elements smaller than x, and $\mathrm{PRIV}(x) = \{y \in P - x : \downarrow(y) \cap \mathrm{MIN} = \{x\}\}$. Note that by definition, the sets $\mathrm{PRIV}(x)$ for the different values of $x \in \mathrm{MIN}$ are pairwise disjoint.

We now consider one arbitrary element $x \in \mathrm{MIN}$, and define the following set of incomparable pairs:

$$\{(x,y) : y \in \mathrm{INC}(x)\} \cup \{(m,y) : m \in \mathrm{MIN} \wedge y \in \mathrm{PRIV}(x)\}.$$

We claim that this set is a zone cover of P. Indeed, suppose for contradiction that a linear extension is not covered. Then the first set in the union forces x to be the minimum in this linear extension, for otherwise x must be preceded by an element that is incomparable to it. Now in all such extensions, the first occurence of another minimal element of P must be preceded either by x or by an element that is in $\mathrm{PRIV}(x)$. In the first case, the extension is covered by a pair in the first set in the union, and in the second case, it is covered by a pair of the second set.

In order to bound the size of such a zone cover, we proceed by computing the average size of the cover with respect to the choice of x:

$$\begin{aligned}
& \frac{1}{k}\sum_{x\in\mathrm{MIN}} \left(|\mathrm{INC}(x)| + (k-1)|\mathrm{PRIV}(x)|\right) \\
\leqslant\ & \frac{1}{k}\left(n(k-1) + (k-1)\sum_{x\in\mathrm{MIN}} |\mathrm{PRIV}(x)|\right) \\
\leqslant\ & \frac{1}{k}\left(n(k-1) + (k-1)(n-k)\right) \\
=\ & \frac{2(k-1)}{k}n - k + 1.
\end{aligned}$$

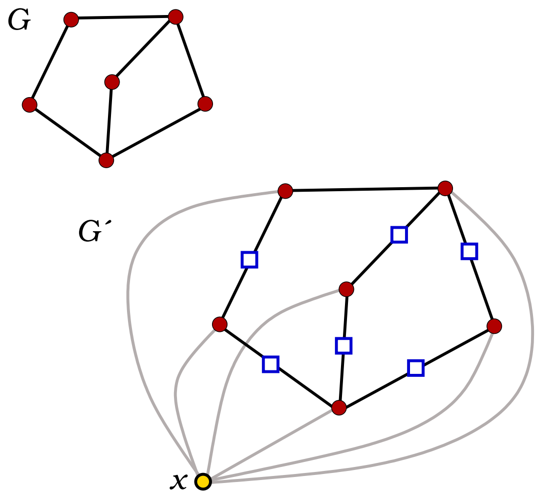

Figure 8: A graph G and the G' obtained by the construction.

Hence there must exist an $x \in \mathrm{MIN}$ such that the zone cover has this size at most. $\square$

Corollary 13. *The worst-case bound on the size of a minimum zone cover of the linear extension graph of a poset P on n elements is $2n - o(n)$. Hence in any poset, there exists a set of at most $2n$ incomparable pairs, such that in any linear extension of P, at least one such pair is consecutive.*

6 Median Graphs

A median graph is an undirected graph in which every three vertices x, y, and z have a unique *median*, i.e., a vertex $\mu(x, y, z)$ that belongs to shortest paths between each pair of x, y, and z. Imrich, Klavžar, and Mulder [15] proposed the following construction of median graphs: start with any triangle-free graph $G = (V, E)$. First add an apex vertex x adjacent to all vertices of V, then subdivide every edge of E once, and let G' be the resulting graph. Figure 8 illustrates the construction.

We only need that G' is a partial cube. This can be shown with the explicit construction of an isometric embedding into Q_n. Let $V = \{v_1, \dots, v_n\}$ and map these vertices bijectively to the standard basis, i.e., $v_i \to e_i$. Apex x is mapped to 0 and the subdivision vertex w_{ij} of an edge v_iv_j is mapped to $e_i + e_j$. The embedding shows that the zones are in one-to-one correspondence with the vertices of G, where the zone of v_i consists of the edge xv_i together with all edges $w_{ij}v_j$. Hence, a zone cover of G' corresponds to a subset of $V(G)$. Let us call w_e the vertex of G' subdividing the edge $e \in E(G)$. The construction of the embedding into Q_n shows that the transformation $G \to G'$ yields a partial cube even if G contains triangles, only membership in the smaller class of median graphs is lost in this case.

Lemma 14. *If G has no isolated vertices, then the minimum size of a zone cover of G' equals the minimum size of a vertex cover of G.*

Proof. We first show that a vertex cover S of G corresponds to a zone cover of G' of the same size. For a vertex w_{ij} at least one of v_i and v_j is in S, hence w_{ij} is covered. If v_i belongs to S then it is covered by its own zone. Otherwise there is an edge v_iv_j and necessarily $v_j \in S$, therefore in this case v_i is covered by the zone of v_j. The apex x is covered by every zone.

The other direction is straightforward. A zone cover of G' must in particular cover all subdivision vertices $w_e \in V(G')$. Hence the zone corresponding to at least one of the endpoint of e must be selected, yielding a vertex cover in G. □

Given a connected triangle-free graph G, one can construct the median graph G' in polynomial time. Since deciding whether G has a vertex cover of size at most k is NP-complete, even on triangle-free graphs, we obtain:

Corollary 15. *Given a median graph G' and a positive integer k, deciding whether there exists a zone cover of size at most k of G' is NP-complete.*

The minimum vertex cover problem is hard to approximate, even on triangle-free graphs. This directly yields the following corollary.

Corollary 16. *Given a median graph G', finding a minimum size zone cover of G' is APX-hard.*

7 Distributive Lattices

In this special case of the partial cube covering problem, we are given a poset P, and we wish to find a subset S of its elements such that the following holds: for every downset D of P, there exists $x \in S$ such that either $D \cup \{x\}$ or $D \setminus \{x\}$ is a downset, distinct from D. Given a poset P, we refer to a suitable set S as a *DS-zone cover* for P. Figure 5 gives an example of poset and the corresponding partial cube.

7.1 Relation to poset fibres

We first establish a connection between DS-zone covers and fibres. A *fibre* of a poset is a subset of its elements that meets every nontrivial maximal antichain. Let $f(P)$ be the size of a smallest fibre of P.

Lemma 17. *Every fibre is a DS-zone cover. In particular, the minimum size of a DS-zone cover is at most $f(P)$.*

Proof. Consider a poset $P = (V, \leqslant)$ and one of its downset $D \subseteq V$. Let $F := V \setminus D$, $A := \max(D)$, and $B := \min(F)$, i.e., A is the antichain of maximal elements in the order induced by D and B is the antichain of minimal elements in the order induced by F. By definition, a DS-zone cover is a hitting set for the collection of subsets $A \cup B$ constructed in this way.

Let us now consider the subset $A \cup B' \subseteq A \cup B$, where $B' := \{b \in B : b \text{ is incomparable to } a, \forall a \in A\}$. This set is easily shown to be a maximal antichain. Hence, it must be hit by any fibre. □

Duffus, Kierstead, and Trotter [8] have shown that every poset on n elements has a fibre of size at most $2n/3$. This directly yields the following.

Corollary 18. *For every n-element poset P, there exists a DS-zone cover of size at most $2n/3$.*

We now consider a special case for which the notions of DS-zone cover and fibre coincide. A poset is bipartite if $P = \min(P) \cup \max(P)$, i.e., if the height of P is at most 2.

Lemma 19. *For a bipartite poset P, a set S is a DS-zone cover for P if and only if it is a fibre of P.*

Proof. We know from Lemma 17 that every fibre is a DS-zone cover. The other direction is as follows. Consider a DS-zone cover S and let A be any maximal antichain of P. Let $T := A \cap \max(P)$, and let D be the downset generated by T. As a downset D is guarded, hence either an element of T is hit by S, or an element of $\min(P \setminus D)$ is hit, but since A is maximal $A = T \cup \min(P \setminus D)$, hence A is hit. Therefore, S is a fibre. □

7.2 Complexity

Lemma 19 yields two interesting corollaries on the complexity of recognizing and finding DS-zone covers.

Corollary 20. *Given a poset P and a subset S of its elements, the problem of deciding whether S is a DS-zone cover is coNP-complete. This holds even if P is bipartite.*

Proof. Recognition of fibres in bipartite posets has been proved coNP-complete by Duffus et al. [8]. From Lemma 19, this is the same problem as recognizing DS-zone covers. □

Corollary 21. *Given a poset P and an integer k, the problem of deciding whether there exists a DS-zone cover of size at most k is Σ_2^P-complete. This holds even if P is bipartite.*

Proof. Again, this is a consequence of Lemma 19 and a recent result of Cardinal and Joret [4] showing that the corresponding problem for fibres in bipartite posets is Σ_2^P-complete. □

Duffus et al. [9] mention that it is possible to construct posets on $15n + 2$ elements, every fibre of which must contain at least $8n+1$ elements, which gives a lower bound with a factor 8/15. The DS-zone covers for these examples do not seem to require as many elements.

Open Problems

We left a number of problems open. For instance, we do not know the complexity of deciding whether there exists an AO-zone cover of size at most k in a given graph. A natural candidate class would be Σ_2^P. For the same problem, we do not have any nontrivial lower bound on the minimum size of an AO-zone cover. It would also be interesting to give tighter lower and upper bounds on the minimum size of a DS-zone cover in a poset and of a zone cover in a line arrangement. Finally, questions involving partial cubes induced by antimatroids, such as elimination orderings in chordal graphs, could be investigated.

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