Thin Edges in Braces*

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Abstract

The *bicontraction* of a vertex v of degree two in a graph, with precisely two neighbours v_1 and v_2, consists of shrinking the set $\{v_1, v, v_2\}$ to a single vertex. The *retract* of a matching covered graph G, denoted by $\widehat{G}$, is the graph obtained from G by repeatedly bicontracting vertices of degree two. Up to isomorphism, the retract of a matching covered graph G is unique. If G is a brace on six or more vertices, an edge e of G is *thin* if $\widehat{G-e}$ is a brace. A thin edge e in a simple brace G is *strictly thin* if $\widehat{G-e}$ is a simple brace. Theorems concerning the existence of strictly thin edges have been used (implicitly by McCuaig (Pólya's Permanent Problem, *Electron. J. Combin., 11, 2004*) and explicitly by the authors (On the Number of Perfect Matchings in a Bipartite Graph, *SIAM J. Discrete Math.*, **27**, 940–958, 2013)) as inductive tools for establishing properties of braces.

Let G and J be two distinct braces, where G is of order six or more and J is a simple matching minor of G. It follows from a theorem of McCuaig (Brace Generation, *J. Graph Theory*, **38**, 124–169, 2001) that G has a thin edge e such that J is a matching minor of $G - e$. In Section 2, we give an alternative, and simpler proof, of this assertion. Our method of proof lends itself to proving stronger results concerning thin edges.

Let $\mathcal{G}^+$ denote the family of braces consisting of all prisms, all Möbius ladders, all biwheels, and all extended biwheels. Strengthening another result of McCuaig on brace generation, we show that every simple brace of order six or more which is not a member of $\mathcal{G}^+$ has at least two strictly thin edges. We also give examples to show that this result is best possible.

Keywords: Graph theory; perfect matchings; matching covered graphs; braces; bricks

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1 Matching Covered Graphs

Graphs considered here are loopless, but they may have multiple edges. For graph theoretical notation and terminology, we essentially follow Bondy and Murty [1]. One notable exception is that here we denote the subgraph of a graph G obtained by deleting an edge e from it by $G - e$, in [1] it is denoted by $G \setminus e$.

McCuaig's paper [11] provides an excellent introduction to the study of procedures for generating braces, and the motivation that led to his work. For the convenience of the reader, in the first part of this section we briefly review the relevant terminology, definitions and results from the theory of matching covered graphs. The later parts of this section include several useful basic results concerning tight cuts and removable edges in bipartite matching covered graphs.

A graph G is *matching covered* if it is connected, has at least two vertices and each edge lies in a perfect matching. Some authors refer to matching covered graphs as *1-extendable* graphs. Every 2-edge-connected cubic graph is matching covered. The treatise by Lovász and Plummer [10] contains the basic theory of matching covered graphs. One simple property stated in that book is the following result:

Proposition 1 ([10, 5.1 (3)])**.** *Every matching covered graph is 2-connected.* □

We shall denote a bipartite graph G with bipartition (A, B) by $G[A, B]$, and assume throughout that $|A| = |B| \geqslant 1$. The following result provides a characterization of bipartite matching covered graphs. It follows immediately from Theorem 4.1.1 in Lovász and Plummer's book [10].

Proposition 2. *Let $G := G[A, B]$ be a bipartite graph on four or more vertices. Then, G is matching covered if and only if for every partition (A', A'') of A into two nonempty sets, and every partition (B', B'') of B such that $|A'| = |B'|$, graph G has at least one edge that joins a vertex in B' to a vertex in A''.* □

1.1 Tight cuts

Let X be a subset of the vertex set of a graph G. We denote by $\partial(X)$ the set of all edges with one end in X and one end in $\overline{X} = V \setminus X$. Clearly, $\partial(X)$ is an edge cut of G; we shall simply refer to such sets of edges as *cuts*. If G is connected and $\partial(X) = \partial(Y)$, then either $Y = X$ or $Y = \overline{X}$; these two sets are then referred to as the *shores* of $\partial(X)$. A cut is *trivial* if it has a shore consisting of exactly one vertex.

Given any cut $C := \partial(X)$ in a connected graph G, where X is a nonempty proper subset of $V(G)$, one may obtain two other graphs, namely G/X and $G/\overline{X}$, by contracting the shores of C to single vertices. These two graphs are called the *C-contractions* of G. When it is necessary to name the *contraction vertices* (that is, the vertices resulting from the contractions of shores), we shall use an alternative notation to represent C-contractions. Thus, $G/(X \to x)$ and $G/(\overline{X} \to \overline{x})$ denote G/X and $G/\overline{X}$, respectively, where x and $\overline{x}$ are the corresponding contraction vertices.

Proposition 3. *Let $\partial(X)$ be a cut of a graph G. If both G/X and $G/\overline{X}$ are matching covered then G is also matching covered.* □

Now let G be a matching covered graph. A cut C of G is *tight* if $|C \cap M| = 1$, for every perfect matching M of G. Simplest examples of tight cuts are the trivial cuts. A basic fact concerning matching covered graphs is that, if C is a tight cut of G, then both C-contractions of G are also matching covered. By Proposition 1, we then have the following simple proposition.

Proposition 4. *The subgraphs of a matching covered graph induced by the shores of a tight cut are both connected.* □

Let $G := G[A, B]$ be a bipartite matching covered graph, and let X be a set of vertices of G such that $|X|$ is odd. Then $|X \cap A|$ and $|X \cap B|$ are clearly distinct; one with smaller cardinality is called the *minority part* and is denoted X_-, and the other, with larger cardinality, is called the *majority part* of X and is denoted X_+. The following property, which is easily proved, gives a description of tight cuts in bipartite matching covered graphs.

Proposition 5 (See Lemma 1.4 in [9]). *Let G be a bipartite matching covered graph, $C := \partial(X)$ a cut of G, $|X|$ odd. Then, C is tight if and only if (i) $|X_+| = |X_-| + 1$ and (ii) every edge of C is incident with a vertex of X_+.* □

1.1.1 Uncrossing tight cuts

Let G be a matching covered graph. Consider two cuts $C := \partial(X)$ and $D := \partial(Y)$ of G. The four sets $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$ and $\overline{X} \cap \overline{Y}$ are the *quadrants* defined by C and D. The two cuts C and D *cross* if each of the four quadrants is nonnull. A collection $\mathcal{C}$ of cuts of G is *laminar* if no two of its cuts cross. The following result, proved in [9], is a fundamental property of tight cuts in graphs.

Proposition 6 (Modularity). *Let G be a matching covered graph, $C := \partial(X)$ and $D := \partial(Y)$ two tight cuts of G. If $|X \cap Y|$ is odd then each of $\partial(X \cap Y)$ and $\partial(\overline{X} \cap \overline{Y})$ is tight and no edge of G joins a vertex of $X \cap \overline{Y}$ to a vertex of $\overline{X} \cap Y$.* □

The following corollary will play a useful role in this paper.

Corollary 7. *Let G be a matching covered graph, $C := \partial(X)$ and $D := \partial(Y)$ be tight cuts of G that cross. If $|X \cap Y|$ is odd then the graphs $G/X/(\overline{X} \cap \overline{Y})$ and $G/\overline{Y}/(X \cap Y)$ are isomorphic, up to multiple edges (Figure 1).*

Proof. The vertex set of each of the graphs $G/X/(\overline{X} \cap \overline{Y})$ and $G/\overline{Y}/(X \cap Y)$ consists of $\overline{X} \cap Y$ and two contraction vertices. Since there are no edges between $\overline{X} \cap Y$ and $X \cap \overline{Y}$ (by Proposition 6), to establish the required isomorphism it suffices to show that the contraction vertices are adjacent in both graphs. But this follows from the fact that, by Proposition 4, G has edges joining vertices of $X \cap \overline{Y}$ to vertices of $X \cap Y$ and also to vertices of $\overline{X} \cap \overline{Y}$. □

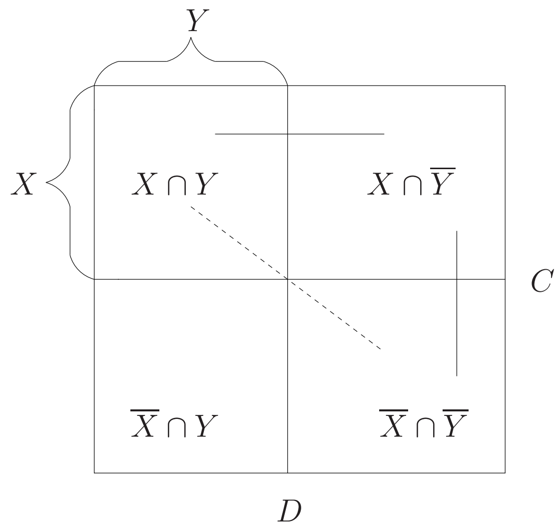


Figure 1: The edges that make the two contraction vertices adjacent in the graphs in Corollary 7 - dashed line indicates the possibility of one or more edges

Let $G := G[A, B]$ be a matching covered graph. *In our illustrations, we shall represent vertices in A by hollow, white discs and the vertices in B by black discs.* In view of Proposition 5, all edges in a tight cut $C := \partial(X)$ must emanate from vertices of the same colour in X.

1.1.2 Tight cut decompositions

A matching covered graph without nontrivial tight cuts is called a *brace* if it is bipartite, and a *brick* if it is nonbipartite. The complete bipartite graphs K_2, C_4 and $K_{3,3}$ are the unique simple braces on two, four and six vertices, respectively. Every brace on eight vertices contains the cube as a spanning subgraph.

Given any matching covered graph G, we may apply to it a procedure, called a *tight cut decomposition* of G, which produces a list of bricks and braces. If G itself is a brick or a brace then the list consists of just G. Otherwise, let C be any nontrivial tight cut of G. Then, both C-contractions of G are matching covered. One may recursively apply the tight cut decomposition procedure to each C-contraction of G, and then combine the resulting lists to produce a tight cut decomposition of G. We remark that, associated with a tight cut decomposition of G there is a maximal laminar collection $\mathcal{C}$ of nontrivial tight cuts of G.

Based on the modularity property (Proposition 6), Lovász [9] proved the following remarkable result on tight cut decompositions.

Theorem 8. *Any two applications of the tight cut decomposition procedure to a matching covered graph produce the same list of bricks and braces, up to multiple edges.* □

In particular, the numbers of bricks and braces are numerical invariants of matching covered graphs. The following result is a consequence of Proposition 5.

Corollary 9. *Every tight cut decomposition of a bipartite matching covered graph consists solely of braces.* □

1.1.3 Bicontractions and retracts

The *bicontraction* of a vertex v of degree two in a graph G, with precisely two neighbours v_1 and v_2, is the graph G/X, where $X := \{v_1, v, v_2\}$. The *retract* of a matching covered graph G, denoted by $\widehat{G}$, is the graph obtained from G by repeatedly bicontracting vertices of degree two. Up to isomorphism, the retract of a matching covered graph G is unique (see [4, Proposition 3.11]). If G is not a cycle, then, in its retract, each vertex has degree three or more.

1.2 Braces

Recall that a bipartite matching covered graph is a *brace* if it has no nontrivial tight cuts. The following characterization of braces will play an important role in this paper. For any graph G and any set X of vertices of G, $N_G(X)$ denotes the set of neighbours of X in G. We omit the subscript G if it is understood and write simply $N(X)$.

Theorem 10 ([9, 1.4], [10])**.** *Let G be a matching covered graph with bipartition (A, B). The following are equivalent:*

(a) *G is a brace;*

(b) *$G - a_1 - a_2 - b_1 - b_2$ has a perfect matching, for any two vertices a_1 and a_2 in A and any two vertices b_1 and b_2 in B;*

(c) *$|N(X)| > |X| + 1$, for every subset X of A such that $0 < |X| < |A| - 1$.* □

The above theorem implies that every vertex of a brace on at least six vertices has at least three distinct neighbours.

Lemma 11. *Let $G := G(A, B)$ be a brace, let S be a set of three vertices of G not all in the same part of G. Then, $G - S$ is connected.*

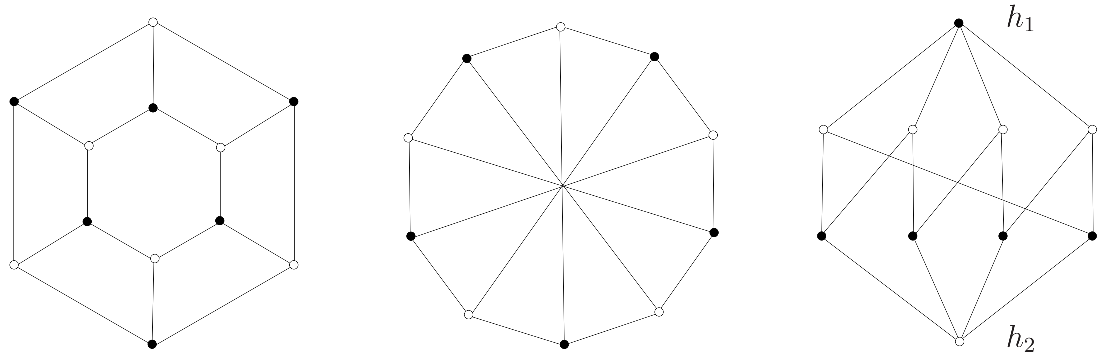
Proof. The assertion holds immediately if G has order four. Assume thus that G has order six or more. Adjust notation so that $S = \{a_1, a_2, b_1\}$, where $a_1, a_2 \in A$, and $b_1 \in B$. As $G - S$ has an odd number of vertices, it must have an odd component, say K. Suppose, contrary to the assertion, that $G - S$ is not connected, and let L be a component of $G - S$ different from K. Clearly L must have a vertex in B; otherwise, the vertex in $A \cap V(L)$ would have just one neighbour, namely b_1. Let b_2 be any vertex in $B \cap V(L)$. By Theorem 10, $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching, say M. The restriction of M to $E(K)$ would then be a perfect matching of K. This is impossible because K has an odd number of vertices. □

1.2.1 Prisms, Möbius ladders, and biwheels

We now describe the four families of braces mentioned in the abstract. Their relevance to the theory of braces was first established by McCuaig [11].

A *prism* P_{4n}, $n \geqslant 2$, is the graph obtained from two disjoint cycles of length $2n$, $(u_1, u_2, \ldots, u_{2n}, u_1)$ and $(v_1, v_2, \ldots, v_{2n}, v_1)$ by the addition of the $2n$ edges u_iv_i, $i = 1, 2, \ldots, 2n$. The family of prisms is denoted $\mathcal{P}$. Figure 2a shows the prism P_{12}.

A *Möbius ladder* M_{4n+2}, $n \geqslant 1$, is the graph obtained from a cycle of length $4n+2$, $(v_1, v_2, \ldots, v_{4n+2}, v_1)$, by the addition of the $2n+1$ *chords* v_iv_{i+2n+1}, $1 \leqslant i \leqslant 2n+1$ of the cycle, where the addition in the suffixes is understood to be taken modulo $4n+2$. The family of Möbius ladders is denoted $\mathcal{M}$. Figure 2b shows the Möbius ladder M_{10}.



(*a*) (*b*) (*c*)

Figure 2: (a) prism P_{12}, (b) Möbius ladder M_{10}, (c) biwheel B_{10}

A *biwheel* B_{2n}, $n \geqslant 4$, is the graph obtained from a cycle $(v_1, v_2, \ldots, v_{2n-2}, v_1)$ of length $2n-2$, called the *rim* of B_{2n}, by the addition of two vertices, h_1 and h_2, called the *hubs* of B_{2n}, and by the addition of edges $h_1v_1, h_1v_3, \ldots, h_1v_{2n-3}$ and edges $h_2v_2, h_2v_4, \ldots, h_2v_{2n-2}$. The family of biwheels is denoted $\mathcal{B}$. Figure 2c shows the biwheel B_{10}.

Apart from the three families defined above, there is a fourth family of braces, related to biwheels, which appears in McCuaig's work. For $n \geqslant 4$, the *extended biwheel* B^+_{2n} is obtained from the biwheel B_{2n} by adding an edge joining the two hubs. In addition, we take $K_{3,3}$ to be the *extended biwheel* B^+_6. The family of extended biwheels is denoted $\mathcal{B}^+$. Note that $K_{3,3}$ is both an extended biwheel and a Möbius ladder, whereas the cube is both prism P_8 and biwheel B_8.

1.2.2 A lemma concerning crossing cuts

We shall now present a lemma which plays a crucial role in this paper. If S is any set and e is an element of S, we shall simply write $S - e$ for the set $S \setminus \{e\}$.

Lemma 12. *Let $G := G[A, B]$ be a brace. Let e and f be two (not necessarily distinct) edges of G such that each of the graphs $G - e$, $G - f$ and $G - e - f$ is matching covered. Let $C := \partial(X)$ and $D := \partial(Y)$ be two crossing cuts of G such that $C - e$ is tight in $G - e$*

and $D - f$ is tight in $G - f$. Assume that $X_- \cap Y_-$ is nonnull. Then, $|X \cap Y|$ is odd and the cuts $\partial(X \cap Y) - e - f$ and $\partial(\overline{X} \cap \overline{Y}) - e - f$ are both tight in $G - e - f$. Moreover, $\partial(X \cap Y)$ is nontrivial (Figure 3).

Proof. Let s denote a vertex in $X_- \cap Y_-$. Adjust notation so that s lies in A. Then, $X_- \subset A$ and $Y_- \subset A$.

Cut $D - f$ is tight in $G - f$, and hence $D - e - f$ is tight in $G - e - f$. Therefore, by Proposition 4, the subgraph of $G - e - f$ induced by $\overline{Y}$ is connected. This implies that some edge of $G - e - f$ joins a vertex b_1 in $X \cap \overline{Y}$ to a vertex a_1 in $\overline{X} \cap \overline{Y}$. Thus, b_1a_1 lies in C. As b_1a_1 is distinct from e, it follows that its end b_1 lies in X_+, which is a subset of B (Figure 3).

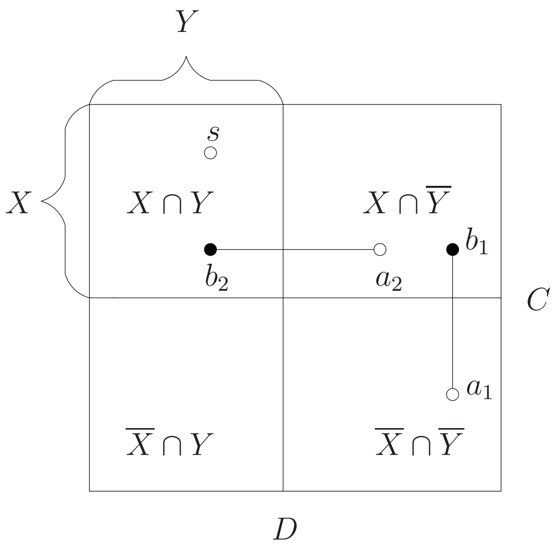


Figure 3: The four quadrants in Lemma 12.

Likewise, cut $C - e$ is tight in $G - e$, and hence $C - e - f$ is tight in $G - e - f$. Therefore, by Proposition 4, the subgraph of $G - e - f$ induced by X is connected. This implies that some edge of $G - e - f$ joins a vertex b_2 in $X \cap Y$ to a vertex a_2 in $X \cap \overline{Y}$. Thus, b_2a_2 lies in D. As b_2a_2 is distinct from f, it follows that b_2 lies in Y_+, which a subset of B. Consequently, b_2 lies in B, whence a_2 lies in A.

The cut $\partial(X \cap \overline{Y}) - e - f$ cannot be tight in $G - e - f$ because there are edges in it emanating from vertices b_1 and a_2 of different colours. Therefore, by Corollary 6, it follows that $|X \cap Y|$ is odd, and that $\partial(X \cap Y) - e - f$ and $\partial(\overline{X} \cap \overline{Y}) - e - f$ are tight cuts in $G - e - f$. The cut $\partial(X \cap Y)$ is nontrivial because both $X \cap Y$, and its complement, have more than two vertices each. □

By taking $e = f$ in the previous result we get the following consequence.

Corollary 13. *Let $G := G[A, B]$ be a brace. Let e be an edge of G such that $G - e$ is matching covered, and let $C := \partial(X)$ and $D := \partial(Y)$ be two nontrivial cuts of G such* *that $C - e$ and $D - e$ are both tight in $G - e$. If $X \cap Y$ contains an end of e, then the cuts $\partial(X \cap Y) - e$ and $\partial(\overline{X} \cap \overline{Y}) - e$ are both nontrivial and tight in $G - e$.*

Proof. Let u and v denote the ends of edge e. Adjust notation so that $X \cap Y$ contains the end u of e. As the cuts $C - e$ and $D - e$ are both nontrivial and tight in $G - e$, it follows that $u \in X_- \cap Y_-$ and v lies in $(\overline{X})_- \cap (\overline{Y})_-$. Now consider the quadrants $X \cap \overline{Y}$ and $\overline{X} \cap Y$. If either of them is empty, then one of X and Y is a subset of the other. In this case, the assertion holds immediately. We may thus assume that C and D cross. By Lemma 12, $\partial(X \cap Y) - e$ is nontrivial and tight in $G - e$. As the end v of e lies in $(\overline{X})_- \cap (\overline{Y})_-$, it also follows that $\partial(\overline{X} \cap \overline{Y}) - e$ is nontrivial and tight in $G - e$. □

1.3 Removable edges

An edge e in a matching covered graph G is *removable* if $G - e$ is also matching covered. Proposition 2 implies the following useful result concerning nonremovable edges in bipartite matching covered graphs.

Lemma 14. *Let $G[A, B]$ be a bipartite matching covered graph, and let e be a nonremovable edge of G. If G has two or more edges, then there exist partitions (A', A'') of A and (B', B'') of B such that $|A'| = |B'|$, and e is the only edge with one end in B' and one end in A''.* □

One may easily deduce the following theorem concerning braces from Lemma 14 and Theorem 10.

Theorem 15. *In a brace on six or more vertices, every edge is removable.* □

Corollary 16. *Let $G[A, B]$ be a bipartite matching covered graph on four or more vertices, and let x be a vertex of G. Then, either there is an edge of G, not incident with x, which is removable in G; or there is a subset Y of $V - x$, $|Y| = 3$, such that $\partial(Y)$ is a tight cut of G.*

Proof. If G has only four vertices then the assertion holds immediately. We may thus assume that G has order six or more. If G is a brace, the statement follows from Theorem 15. If not, let Y be a minimal subset of $V - x$ such that $\partial(Y)$ is a nontrivial tight cut of G. Then, $G/\overline{Y}$ is a brace, by the minimality of Y. If this brace has order at least six, one can again appeal to Theorem 15. Otherwise $|Y| = 3$, and the assertion follows. □

1.4 Graphs obtained by deleting an edge from a brace

Let G be a brace on six or more vertices, and let $e = uv$ be an edge of G. Then, by Theorem 15, $G - e$ is matching covered. We shall now establish two special properties of $G - e$ which play crucial roles in this paper. The first concerns tight cut decompositions of $G - e$, and the second concerns removable edges in $G - e$.

1.4.1 Tight cut decompositions of $G - e$

Suppose that $G-e$ is not a brace, then u and v belong to different shores of any nontrivial tight cut of $G-e$. Let $\mathcal{C}$ be the family of tight cuts corresponding to a tight cut decomposition of $G-e$. Then the shores of the cuts in $\mathcal{C}$ containing u form a nested family of subsets of V, and may be described in the following simple manner. Corollary 13 implies that there is a unique minimal subset of V containing u, say X_1, such that $\partial(X_1)$ is a nontrivial tight cut of $G-e$. By the minimality of X_1, it follows that $(G-e)/\overline{X_1}$ is a brace. The graph $(G-e)/X_1$ may or may not be a brace. If it is not, let X_2 denote a (not necessarily unique) minimal subset of V, properly containing X_1, such that $\partial(X_1)$ is a nontrivial tight cut of $G-e$. Then $(G-e)/X_1/\overline{X_2}$ is a brace (by the minimality of X_2). If $(G-e)/X_2$ is a brace, we have a tight cut decomposition of $G-e$. Otherwise $(G-e)/X_2/\overline{X_3}$ is a brace for a minimal subset X_3 of V, properly containing X_2, for which $\partial(X_3)$ is a nontrivial tight of $G-e$. Proceeding in this manner, we obtain a nested family of subsets of V as described in the following lemma:

Lemma 17. *Let G be a brace of order at least six, and let $e = uv$ be any edge of G such that $G-e$ is not a brace. Then there there exists a nested family $X_1 \subset X_2 \subset \cdots \subset X_k$ of proper subsets of V (see Figure 4), each containing u, such that*

$$\left.\begin{array}{l} (G-e)/\overline{X_1} \ \ \textit{is a brace}, \\ (G-e)/X_i/\overline{X_{i+1}} \ \ \textit{is a brace for } 2 \leqslant i < k, \\ (G-e)/X_k \ \ \textit{is a brace.} \end{array}\right\} \qquad (1)$$

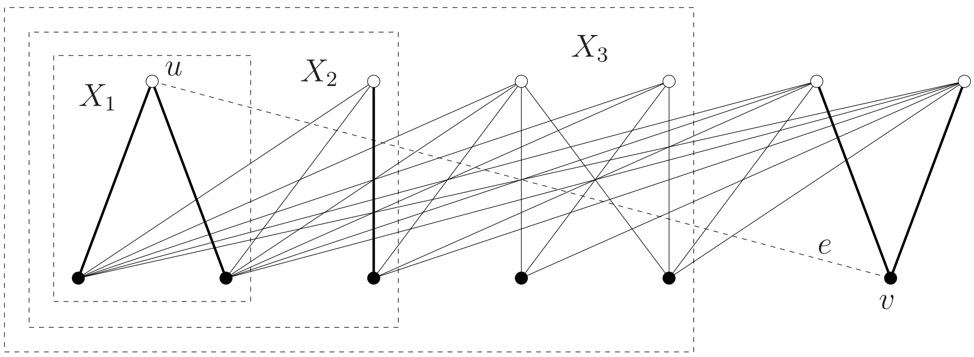


Figure 4: An illustration for Lemma 17, with $k = 3$

As mentioned above, the first member X_1 of the nested family is unique. Analogously, X_k is unique. This observation leads to the following corollary:

Corollary 18. *Let G be a brace of order at least six, and let e be an edge of G. If $G-e$ has at most three braces, then $G-e$ has a unique tight cut cut decomposition, that is, $G-e$ has a unique maximal laminar family of nontrivial tight cuts.* □

1.4.2 Removable edges in $G - e$

We shall now establish a useful general result concerning removable edges in a graph obtained from deleting an edge from a brace.

Lemma 19. *Let $G[A, B]$ be a brace on six or more vertices, and let e be an edge of G. Then, every edge that lies in a nontrivial tight cut of $G - e$ is removable in $G - e$.*

Proof. Let $C := \partial(X)$ be a nontrivial cut of G such that $C - e$ is tight in $G - e$. Let f be any edge of $C - e$. Assume, to the contrary, that f is not removable in $G - e$. Then, f is not removable in some $(C - e)$-contraction of $G - e$. Adjust notation so that f is not removable in $H := (G - e)/(X \to x)$. Adjust notation so that $X_+ \subset A$. Then, edge e has an end, say v, in X_-, in turn a subset of B (Figure 5).

By Lemma 14, the sets $(A \cap V(H)) \cup \{x\}$ and $B \cap V(H)$ have partitions (A', A'') and (B', B'') such that $|A'| = |B'|$ and f is the only edge of H that joins a vertex of B' to a vertex of A''.

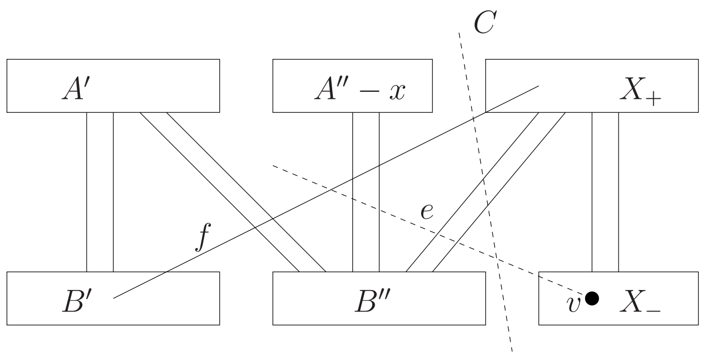


Figure 5:

Consider now the partition $(A', A - A')$ of A and $(B', B - B')$ of B. We have that $|A'| = |B'|$ and f is the only edge of G having one end in B', the other end in $A - A'$. Thus, f is not removable in G. This is a contradiction, as G is a brace on six or more vertices. □

Lemma 20. *Let $G[A, B]$ be a brace on six or more vertices, and let e be an edge of G. Let u be a vertex of G having degree three or more in $G - e$. Then, at most one edge of $\partial(u) - e$ is not removable in $G - e$.*

Proof. If every edge of $\partial(u) - e$ is removable then the assertion holds immediately. We may thus assume that $\partial(u) - e$ contains an edge, f, that is not removable in $G - e$. Let us prove that f is the only edge of $\partial(u) - e$ that is not removable in $G - e$.

Adjust notation so that vertex u lies in A. Edge f is not removable in $G - e$. Thus, there exists a partition (A', A'') of A and a partition (B', B'') such that $|A'| = |B'|$ and f is the only edge of $G - e$ that joins a vertex of B' to a vertex of A'' (which is u, see Figure 6).

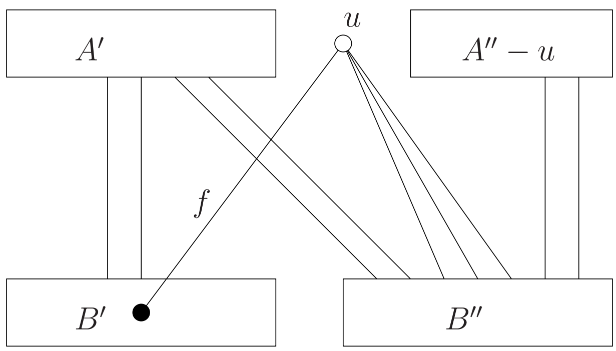


Figure 6:

If A'' has two or more vertices then $A' \cup B' \cup \{u\}$ is the shore of a nontrivial tight cut of $G-e$ that contains all the edges of $\partial(u)-e-f$. In that case, every edge of $\partial(u)-e-f$ is removable in $G-e$. We may thus assume that A'' is a singleton. In that case, B'' is also a singleton. As u has degree three or more in $G-e$, it follows that all the edges of $\partial(u)-e-f$ are multiple edges in $G-e$, whence removable. In both alternatives, we deduce that every edge of $\partial(u)-e-f$ is removable in $G-e$. $\square$

1.5 Thin edges and their indices

Let e be an edge of a graph G. The *index* of e in G is the number of ends of e having degree two in $G-e$. An edge e of a brace G is *thin* if $\widehat{G-e}$ (the retract of $G-e$) is a brace. Figure 7 illustrates thin edges and non-thin edges of a brace. The three edges e_0, e_1 and e_2 in Figure 7 are, respectively, thin edges of index zero, one and two in that brace. If an edge e of brace G is thin then $\widehat{G-e}$ is a *reduction* of G.

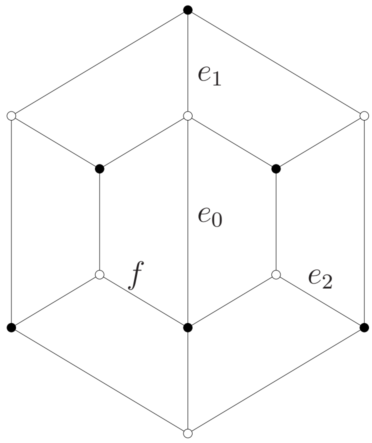


Figure 7: Edges e_0, e_1, and e_2 are thin, but f is not.

In the prism P_{4n}, as described in Section 1.2.1, each of the edges u_iv_i, $1 \leqslant i \leqslant 2n$, is a thin edge of index two; no other edge is thin. In the Möbius ladder M_{4n+2}, each of the chords $u_i u_{i+2n+1}$, $1 \leqslant i \leqslant 2n+1$, is a thin edge of index two; no other edge is thin. And, in the biwheel, each of the edges incident with a hub is a thin edge of index one; no other edge is thin.

We shall show in Section 2 that every brace on six or more vertices has a thin edge.

1.6 Matching minors

We now proceed to describe the notion of a matching minor which is central to this paper. Although the idea of a matching minor appears in McCuaig's work [11], the term itself was introduced by Norine and Thomas [13].

A matching covered subgraph H of a matching covered graph G is *well-fitted* if the graph $G-V(H)$ has a perfect matching. (Well-fitted subgraphs are referred to as *nice* subgraphs by Lovász and Plummer [10], and as *central* subgraphs by Norine and Thomas [13].) According to Norine and Thomas [13], a matching covered graph J is a *matching minor* of another matching covered graph G if there exists a well-fitted subgraph H of G such that a graph isomorphic to J is obtainable from H by means of a sequence of bicontractions of vertices of degree two.

Clearly K_2 is a matching minor of every matching covered graph. According to a theorem of Little, any two edges of a matching covered graph are contained together in a well-fitted cycle in that graph [8] (see also [10, Theorem 5.4.4]). It follows that C_4 is a matching minor of any matching covered graph of order at least four.

Using the theory of ear decompositions (see [10] or [2]), it can be shown that a matching covered subgraph H of a bipartite matching covered graph G is well-fitted if and only if H can be obtained from G by bicontractions of vertices of degree two, and deletions of removable edges. We thus have the following characterization of matching minors of bipartite matching covered graphs.

Lemma 21. *Let G and J be two bipartite matching covered graphs. Then J is a matching minor of G if and only if there exists a sequence $G_1, G_2, \ldots, G_r$ of graphs such that (i) $G_1 = G$, and $G_r \cong J$, and (ii) for $1 \leqslant i < r$, G_{i+1} is obtained from G_i either by the deletion of a removable edge of G_i, or by the bicontraction of a vertex of degree two in G_i. (An analogous result also holds for non-bipartite matching covered graphs but, in that case, the deletions of removable doubletons will also have to be permitted.)* □

As an immediate consequence, we have:

Corollary 22. *Let G be a matching covered graph, and let J be a matching minor of G. Any matching minor of J is also a matching minor of G.* □

Another useful result is the following:

Lemma 23. *Let G be a bipartite matching covered graph, and let $C := \partial(X)$ be a tight cut of G. Then, both C-contractions of G are matching minors of G.*

Proof. By induction on the number of edges of G. Let us first show that G/X is a matching minor of G.

If there is an edge e in $G/\overline{X} \to \overline{x}$, not incident with $\overline{x}$, which is removable, then $G/X = (G-e)/X$. By induction, $(G-e)/X$ is a matching minor of $G-e$, and hence G/X is a matching minor of G. So, we may assume that no such edge exists. If X is a singleton then G/X and G are isomorphic, the assertion holds immediately. We may thus assume that $|X| \geqslant 3$. By Corollary 16, there is a subset Y of X, with $|Y| = 3$, such that $\partial(Y)$ is a tight cut, where the minority vertex in Y has degree two in G. In this case, $G/Y \to y$ is a matching minor of G because it can be obtained from G by the bicontraction of the minority vertex in Y. However, C is tight in G/Y and $G/X = (G/Y)/(X-Y+y)$. Thus, G/X is a C-contraction of G/Y. Since G/Y has fewer edges than G, we may use induction, and deduce that G/X is a matching minor of G.

A similar argument shows that $G/\overline{X}$ is a matching minor of G. □

Corollary 24. *Every brace of a bipartite matching covered graph G is a matching minor of G.* □

We conclude this section with a crucial result concerning matching minors of bipartite matching covered graphs.

Lemma 25. *Let G be a bipartite matching covered graph. A simple brace is a matching minor of G if and only if it is a matching minor of some brace of G.*

Proof. Let J be a simple brace. If J is a matching minor of some brace of G then J is a matching minor of G, by Corollary 24.

We prove the converse by induction on the number of edges of G. Assume that J is a matching minor of G. If G is a brace then J, a matching minor of G, is a matching minor of a brace of G. We may thus assume that G is not a brace. By hypothesis, J is a brace. Thus, J and G are distinct. As J is a matching minor of G, then either J is a matching minor of a bicontraction of G or J is a matching minor of $G-e$, where e is a removable edge of G.

Consider first the case in which G has a vertex v such that J is a matching minor of the graph H, obtained from G by the bicontraction of v. Then, v has degree two and has two distinct neighbours, v_1 and v_2, where $H = G/X_0$ and $X_0 = \{v, v_1, v_2\}$. By induction, J is a matching minor of a brace of H. Cut $\partial(X_0)$ is tight in G. Every brace of H is a brace of G. Thus, J is a matching minor of a brace of G. The assertion holds.

We may thus assume that G has a removable edge e such that J is a matching minor of $G-e$. By induction, J is a matching minor of a brace of $G-e$, say K. Let K_0 denote the underlying simple graph of K. As J is simple, J is a matching minor of K_0. We have assumed that G is not a brace. Thus, G has a nontrivial tight cut, $C := \partial(X)$. Every perfect matching of $G-e$ is a perfect matching of G. Thus, $C-e$ is a tight cut of $G-e$. By the uniqueness of the tight cut decomposition, $G-e$ has a $(C-e)$-contraction that has a brace, K', which is isomorphic to K, up to multiple edges.

Adjust notation so that K' is a brace of $(G-e)/X$. The simple brace K_0 is isomorphic to the underlying simple graph of K'. Thus, J, a matching minor of K_0, is a matching minor of K'. By Corollary 24, J is a matching minor of $(G-e)/X$. However, either e is not an edge of G/X or e is a removable edge of G/X. In the former alternative,

$G/X = (G - e)/X$, whence J is a matching minor of G/X. In the latter alternative, $(G - e)/X$ is a matching minor of G/X, whence J is a matching minor of G/X, by Corollary 22. In both alternatives, J is a matching minor of G/X. By induction, J is a matching minor of a brace of G/X. Every brace of G/X is a brace of G. Then J is a matching minor of a brace of G. □

The brace C_4 is a matching minor of every brace of order four or more. There is a polynomial-time algorithm for deciding whether or not $K_{3,3}$ is a matching minor of a given input brace G. (This is due to McCuaig [12] and Robertson, Seymour and Thomas [14], and is related to their theory of Pfaffian orientations.) We are not aware of any work related to the complexity status of the problem of deciding whether or not a simple brace J of order eight or more is a matching minor of a given input brace G.

We conclude this section by noting that if G is member of the family $\mathcal{G}^+$, and a simple brace J of order six or more is a matching minor of G, then J is also a member of $\mathcal{G}^+$.

2 Existence of Thin Edges

Throughout this section, G and J denote two distinct braces where G has order at least six, J is simple and is a matching minor of G. We shall first establish the main result of this paper, which asserts that there exists a thin edge e of G such that J is a matching minor of $G - e$. As noted in the abstract, this result may be derived from McCuaig's work [11]. But our approach is quite different and makes it possible for us to deduce that any brace of order six or more has at least two thin edges.

Our proof technique is constructive and is based on the notion of rank of an edge. Given an edge e of G which is not thin, we shall show that there exists an edge of higher rank than e. This leads us to the conclusion that an edge of G of maximum rank is a thin edge with the desired property. (We used a similar technique in [5] for showing that every brick different from K_4, $\overline{C_6}$, and the Petersen graph has a 'thin' edge.)

2.1 The rank of an edge

Since J is a matching minor of G, by Lemma 21, a graph isomorphic to J is obtainable from G by a sequence of bicontractions and deletions of removable edges. As G and J are distinct, any such sequence has at least two members, and its second member is obtained from G either by the bicontraction of a vertex of degree two, or by the deletion of an edge of G. However, as G is a brace of order six or more, it has no vertices of degree two. Therefore there must exist an edge e of G such that J is a matching minor of $G - e$.

Let $\mathcal{R}$ denote the set of edges e of G such that J is a matching minor of $G - e$. Let e be an edge in $\mathcal{R}$. Then, by Lemma 25, J is a matching minor of one of the braces of $G - e$. Our objective is to define the notion of the rank of an edge and then show that an edge of maximum rank in $\mathcal{R}$ is a thin edge. But, first, we introduce a closely related function which we shall refer to as 'pre-rank'. For an edge e in $\mathcal{R}$, the *pre-rank* of e, denoted by $r_0(e)$, is defined to be the maximum of the orders of all braces of $G - e$ that have J as a matching minor. As an example, consider the brace G shown in Figure 8. If

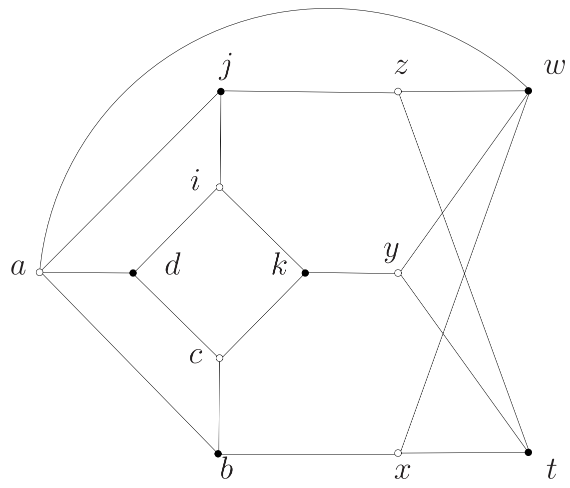


Figure 8: The pre-rank of an edge, example: $J = K_{3,3}$, $r_0(aw) = 6$, $r_0(ab) = 10$

J is $K_{3,3}$, then it can be verified that $\mathcal{R} = \{ab, ad, aj, aw, bc, cd, ck, id, ij, ik, xw, yw, yk\}$. The graph $G - aw$ has two braces, one being the cube and the other being $K_{3,3}$. The cube, being planar, does not contain $K_{3,3}$ as a matching minor. Thus $r_0(aw) = 6$. The graph $G - ab$ has two braces, one has four vertices and the other has ten vertices and has $K_{3,3}$ as a matching minor. Thus $r_0(ab) = 10$.

In the brace shown in Figure 8, an edge in $\mathcal{R}$ with the largest possible pre-rank is a thin edge. However, this is not in general true. It turns out that, in defining the rank of an edge, in addition to considering the number of vertices in a largest brace of $G - e$, we need also to consider the number of contraction vertices that brace has. To illustrate this point, consider the brace G in Figure 9, where J is brace $K_{3,3}$.

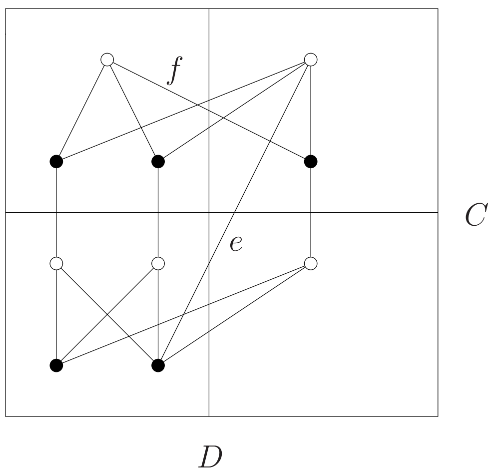


Figure 9: With $J = K_{3,3}$: $r_0(f) = r_0(e) = 3$, but $r(f) = 4$ and $r(e) = 3$

Both graphs $G - e$ and $G - f$ have $K_{3,3}$ as a brace, up to multiple edges. The graph $G - e$ has two braces isomorphic to $K_{3,3}$, up to multiple edges (consider the tight cut

$C - e$). Edge f is thin and the graph $G - f$ has only one brace isomorphic to $K_{3,3}$, up to multiple edges; it has two other braces, both of order four. Thus, both e and f lie in $\mathcal{R}$ and have pre-rank six. But edge e is not thin, whereas f is thin. Note that the braces of $G - e$ have only one contraction vertex, whereas the brace of $G - f$ on six vertices has two contraction vertices. So, in defining the rank of an edge e, we take into account the number of contraction vertices in a largest brace of $G - e$, giving preference to braces having two contraction vertices.

We thus define the *rank* $r(e)$ of an edge e in $\mathcal{R}$ to be:

$$r(e) := \begin{cases} r_0(e) + 1, & \text{if } G - e \text{ has a brace of order } r_0(e) \\ & \text{with two contraction vertices} \\ r_0(e), & \text{otherwise.} \end{cases}$$

We remark that the same edge e may provide different choices for the brace of $G - e$ having J as a matching minor. For example, in Figure 10, $G-e$ has two braces isomorphic to $K_{3,3}$ up to multiple edges. One brace contains the vertices in $\{1, 2, 7, 8, 9\}$, plus a contraction vertex. The other brace contains vertices in $\{3, 4, a, b\}$, plus two contraction vertices. Thus, the latter is responsible for the value seven for $r(e)$.

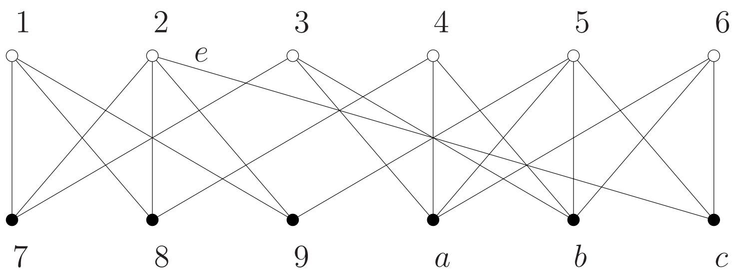


Figure 10: The graph $G - e$ has two braces of order six

Theorem 26 (The Rank Augmentation Theorem)**.** *Let $e = uv$ be an edge in $\mathcal{R}$ and suppose that e is <u>not</u> thin. Then there exist two edges f and g in $\mathcal{R}$ such that:*

(i) *f and g are adjacent to each other, but not to e;*

(ii) *$r(f) \geqslant r(e)$, $r(g) \geqslant r(e)$; and*

(iii) *either $r(f) > r(e)$ or $r(g) > r(e)$.*

Proof. By the definition of $r(e)$, there is a brace of $G - e$ of order $r_0(e)$ which has J as a matching minor. As e is not thin by hypothesis, $G - e$ has at least two braces, and $r(e) < |V(G)|$. For every tight cut decomposition of $G - e$, every brace has at most two contraction vertices, by Lemma 17. Consider all the tight cut decompositions of $G - e$. Let $\mathcal{G}^\star$ denote the set of those braces $G^\star$ of $G - e$ that satisfy the following properties:

(i) $G^\star$ has J as a matching minor;

(ii) $G^\star$ has order $r_0(e)$; and

(iii) if $r(e)$ is odd then $G^\star$ has two contraction vertices.

Let $\mathcal{G}$ be a tight cut decomposition of $G-e$ that contains braces in $\mathcal{G}^\star$. We must now choose a brace $G^\star$ of $\mathcal{G}$ that lies in $\mathcal{G}^\star$.

As e is not thin, every brace in $\mathcal{G}$ has a contraction vertex that is the result of the contraction of a set having five or more vertices. In particular, every brace in $\mathcal{G}$ that lies in $\mathcal{G}^\star$ has a contraction vertex that is the result of the contraction of a set having five or more vertices. Let X be a maximal such set, let $G^\star$ be the corresponding brace.

We note that $r_0(e)$ is at least four (all braces of any bipartite matching covered graph of order at least four have order at least four). Furthermore, clearly,

$$r(e) \leqslant 1 + |\overline{X}|, \tag{2}$$

with equality only if $G^\star$ has precisely one contraction vertex.

Note that edge e has its ends in X_- and $\overline{X}_-$. Let u denote the end of e in X_-, v its other end. Recall that (A, B) denotes the bipartition of G. Adjust notation so that u lies in A. (See Figure 11).

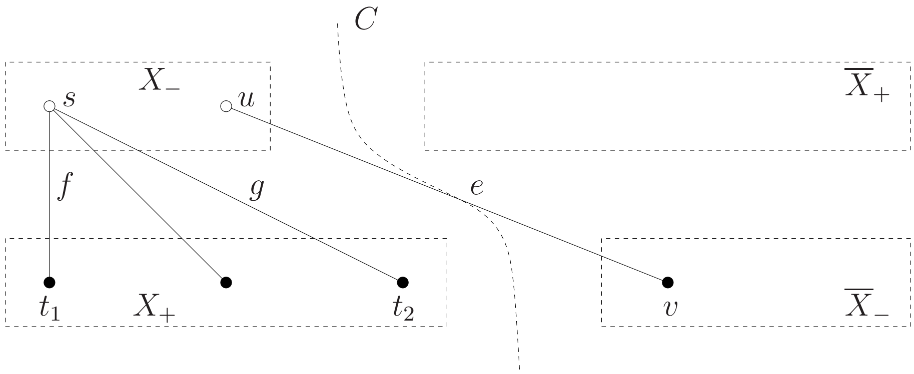


Figure 11: Edges e, f and g.

As $|X| \geqslant 5$, it follows that X_- contains two or more vertices. Let $s \in X_-$ be a vertex distinct from u. Then, s has degree at least three in $G-e$. By Lemma 20, there are edges f and g, both incident with s and removable in $G-e$. Then $G-e-f$ and $G-e-g$ are matching covered subgraphs of $G-e$.

Let t_1 and t_2 denote the ends of f and g in B. The vertices of G adjacent to s in G lie all in X_+. Thus, both t_1 and t_2 lie in X_+ (Figure 11). This establishes that f and g are adjacent to each other, but not to e. We shall now prove that f lies in $\mathcal{R}$ and $r(f) \geqslant r(e)$, and characterize under what conditions equality holds. For this, we consider some cases. In all cases we show that f lies in $\mathcal{R}$. In all cases except the last, we show that $r(f) > r(e)$. In the last case we conclude that $r(f) = r(e)$. A similar reasoning holds for g. We then finally show that equalities $r(f) = r(e)$ and $r(g) = r(e)$ cannot both hold.

Case 1. *No tight cut in $G - f$ crosses C.*

In this case, either X or $\overline{X}$ is contained in a shore of every tight cut in $G - f$. For each nontrivial tight cut D in $G - f$, the ends of f lie in the minority parts of the shores of D. As f has no end in $\overline{X}$, it follows that $\overline{X}$ is contained in a shore of every nontrivial tight cut in $G - f$. Then $\overline{X}$ is contained in the vertex set of a brace F of $G - f$. As C is not tight in $G - f$, shore $\overline{X}$ is, in fact, strictly contained in a shore of every nontrivial tight cut D in $G - f$. It follows that F has at least $|\overline{X}| + 3$ vertices.

Moreover, as $(F - e)/X = (G - e)/X$, the brace J is a matching minor of F. Thus, $r(f) \geqslant |V(F)| \geqslant |\overline{X}| + 3 > r(e)$, establishing that f lies in $\mathcal{R}$ and $r(f) > r(e)$.

Case 2. *There exist tight cuts in $G - f$ that cross C.*

For each nontrivial tight cut of $G - f$, edge f is incident with vertices in the minority parts of both shores of that cut. Among the (nontrivial) tight cuts of $G - f$ that cross cut C, choose one so that its shore that contains the end s of f in X_- is minimal. Let D denote the cut in G, let Y denote the shore that contains vertex s. See Figure 12.

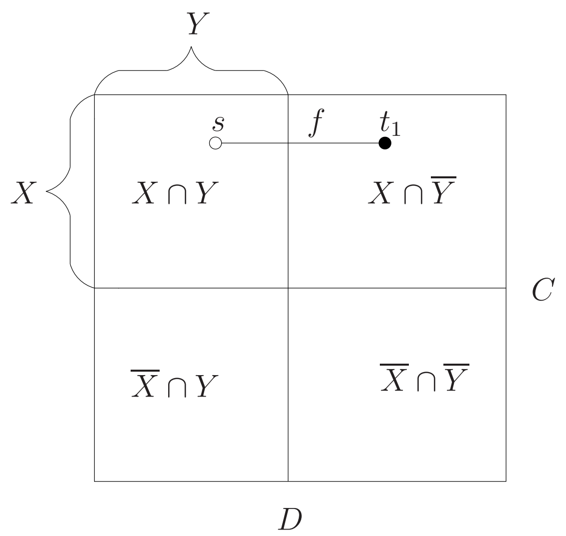


Figure 12: The four quadrants in Case 2

By definition, the end s of f belongs to X_-, and also to Y; more specifically to Y_-, by Proposition 5. Thus, s belongs to $X_- \cap Y_-$. By using Lemma 12, we deduce the following facts:

- $\partial(X \cap Y) - e - f$ is a nontrivial tight cut of $G - e - f$, and
- $\partial(\overline{X} \cap \overline{Y}) - e - f$ is a tight cut of $G - e - f$.

Since the edge f has both its ends in X, it does not belong to $\partial(\overline{X} \cap \overline{Y}) - e$, implying that this cut is tight in $G - e$.

With the view to finding a lower bound for the rank of f, we now consider nontrivial tight cuts of $(G - f)/(\overline{Y} \to \overline{y})$.

Lemma 27. *Let D' be nontrivial tight cut of $(G-f)/(\overline{Y} \to \overline{y})$. The shore of D' that contains vertex s is a subset of $X \cap Y$.*

Proof. Let Y' be the shore of D' in $(G-f)/(\overline{Y} \to \overline{y})$ that contains vertex s. Note that D' is a nontrivial tight cut of $G-f$. If Y' contains the contraction vertex $\overline{y}$ then D' would be a nontrivial tight cut of G itself, a contradiction. Thus, Y' is a proper subset of Y. By the minimality of Y, it follows that D' and C do not cross. Thus, either $Y' \subseteq X \cap Y$, or $Y' \subseteq \overline{X} \cap Y$. However, Y' contains s. We conclude that $Y' \subseteq X \cap Y$. ☐

Corollary 28. *Let S denote the maximal subset of $X \cap Y$ which contains s and is the shore of a (possibly trivial) tight cut of $(G-f)/\overline{Y}$. Then $H := (G-f)/\overline{Y}/S$ is a brace of $G-f$. Furthermore, $|V(H)| \geqslant |\overline{X} \cap Y| + 2$, with equality only if $\partial(X \cap Y) - f$ is a (nontrivial) tight cut of $(G-f)/\overline{Y}$.* ☐

We now consider two subcases depending on whether or not $\partial(\overline{X} \cap \overline{Y})$ is trivial.

Case 2.1. *The cut $\partial(\overline{X} \cap \overline{Y})$ is trivial.*

In this case, $|\overline{X} \cap \overline{Y}| = 1$, and we have:

$$\begin{aligned}
(G-e)/X &= (G-e-f)/X, \text{ because } f \text{ has both ends in } X \\
&= (G-e-f)/X/(\overline{X} \cap \overline{Y}), \text{ because } |\overline{X} \cap \overline{Y}| = 1 \\
&\cong (G-e-f)/\overline{Y}/(X \cap Y), \text{ up to multiple edges, by Corollary 7.}
\end{aligned}$$

By definition, the graph $(G-e)/X$ has $G^\star$ as a brace. It follows then that the graph $(G-e-f)/\overline{Y}/(X \cap Y)$ has a brace isomorphic to $G^\star$, up to multiple edges. Let S denote the subset of $X \cap Y$, and $H := (G-f)/\overline{Y}/S$ denote the brace as defined in Corollary 28. Then $\partial(X \cap Y) - e - f$ is a tight cut of $H - e$, implying that $(G-e-f)/\overline{Y}/X \cap Y$ is a matching minor of $H - e$. Thus, the underlying simple graph of $G^\star$ is a matching minor of H. As J is simple, it follows that J is a matching minor of H. Thus, f lies in $\mathcal{R}$. Now, as H is a brace of $G-f$, we have:

$$\begin{aligned}
r_0(f) &\geqslant |V(H)|, \text{ by the definition of } r_0(f) \\
&\geqslant |\overline{X} \cap Y| + 2, \text{ by Corollary 28} \\
&= |\overline{X}| + 1, \text{ because, by assumption, } |\overline{X} \cap \overline{Y}| = 1 \\
&\geqslant r(e), \text{ by (2).}
\end{aligned}$$

If equality does not hold all the way through, then, $r(f) > r(e)$. Alternatively, if equality holds throughout then, by (2), $(G-e)/X$ is a brace of $G-e$ of order $r(e)$ with one contraction vertex, whereas $H = (G-f)/\overline{Y}/(X \cap Y)$ is a brace of $G-f$ of order $r(e)$ with two contraction vertices. By definition of the rank function, we deduce that $r(f) > r(e)$. In both alternatives, $r(f) > r(e)$. In this case we also conclude that f lies $\mathcal{R}$ and $r(f) > r(e)$.

Case 2.2. *The cut $\partial(\overline{X} \cap \overline{Y}) - e$ is nontrivial and tight in $G-e$.*

It follows that, in this case, $(G-e)/X$ is not a brace, whence $G^\star$ has two contraction vertices. Let X' be the subset of $\overline{X}$ such that $G^\star = (G-e)/X/X'$.

We shall prove that in this case $X' = \overline{X} \cap \overline{Y}$. In other words, $G^\star$ is precisely equal to $(G-e)/X/(\overline{X} \cap \overline{Y})$. We shall also prove that $\partial(X \cap Y) - f$ is tight in $G - f$ and $G^\star$ is isomorphic, up to multiple edges, to $(G-f)/\overline{Y}/(X \cap Y)$.

Lemma 29. *The cut $\partial(X \cap Y) - f$ is tight in $G - f$. Furthermore, the graphs $(G - f)/\overline{Y}/(X \cap Y)$ and $(G-e)/X/(\overline{X} \cap \overline{Y})$ are braces, which are isomorphic, up to multiple edges.*

Proof. Since $\partial(\overline{X} \cap \overline{Y}) - e$ is nontrivial and tight in $G - e$, the end v of e in $\overline{X}$ belongs to $\overline{Y}$. But, as $v \in B$, it follows that v belongs, in fact, to $\overline{Y}_-$. This implies that all neighbours of v in $G - f$ belong to $\overline{Y}_+$ because $\partial(Y) - f$ is a tight cut of $G - f$. In particular, vertex u, which is joined to v by e, belongs to $\overline{Y}_+$. Thus, both ends of e lie in $\overline{Y}$, whence e does not lie in $\partial(X \cap Y)$.

Now, since $\partial(X \cap Y) - e - f$ is a tight cut in $G - e - f$, and $e \notin \partial(X \cap Y)$, it follows that $\partial(X \cap Y) - f$ is a tight cut of $G - f$. By Corollary 28, $(G-f)/\overline{Y}/(X \cap Y)$ is a brace.

Observe now the following implications:

$$
\begin{aligned}
(G-f)/\overline{Y}/(X \cap Y) &= (G-e-f)/\overline{Y}/(X \cap Y), \text{ because } e \text{ has both ends in } \overline{Y} \\
&\cong (G-e-f)/X/(\overline{X} \cap \overline{Y}), \text{ up to multiple edges,} \\
&\quad\ \text{by Corollary 7} \\
&= (G-e)/X/\overline{X} \cap \overline{Y}, \text{ because } f \text{ has both its ends in } X
\end{aligned}
$$

As noted above, $(G-f)/\overline{Y}/(X \cap Y)$ is a brace. Thus, $(G-e)/X/(\overline{X} \cap \overline{Y})$ is also a brace. □

Lemma 30. $X' = \overline{X} \cap \overline{Y}$.

Proof. Let $C' := \partial(X')$. The cuts $\partial(\overline{X} \cap \overline{Y}) - e$ and $C' - e$ are nontrivial and tight in $G - e$. The end v of e lies in $\overline{X}$, a superset of both X' and $\overline{X} \cap \overline{Y}$. Thus, the end v of e lies in $X' \cap \overline{X} \cap \overline{Y}$.

Suppose that $\partial(\overline{X} \cap \overline{Y})$ and C' cross. By Corollary 13, $\partial(X' \cup (\overline{X} \cap \overline{Y})) - e$ and $\partial(X' \cap \overline{X} \cap \overline{Y}) - e$ are both nontrivial tight cuts of $G - e$.

If $X' \cup (\overline{X} \cap \overline{Y})$ is a proper subset of $\overline{X}$, then $\partial(X' \cup (\overline{X} \cap \overline{Y})) - e$ would be a nontrivial tight cut of the brace $G^\star$. Thus $X' \cup (\overline{X} \cap \overline{Y}) = \overline{X}$ (Figure 13).

Let $L := (G-e)/(X \cup Y)/(X' \cap \overline{X} \cap \overline{Y})$. The cut $\partial(X' \cap \overline{X} \cap \overline{Y})$ is nontrivial. Thus, L has two contraction vertices. Let $X_0 := X \cup Y$ and $Y_0 := \overline{X'}$. The cuts $\partial(X_0)$ and $\partial(Y_0)$ cross. Moreover,

$$
\begin{aligned}
L &= (G-e)/X_0/(\overline{X_0} \cap \overline{Y_0}) \cong (G-e)/\overline{Y_0}/(X_0 \cap Y_0) \\
&= (G-e)/X'/((X \cup Y) \cap \overline{X'}) = (G-e)/X'/X = G^\star,
\end{aligned}
$$

where the congruence, up to multiple edges, follows by Corollary 7. Thus, $G^\star$ is isomorphic to L, up to multiple edges. This is a contradiction to the choice of X, as X is a proper subset of $X \cup Y$.

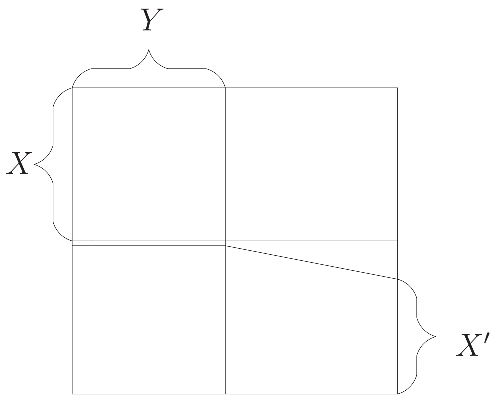


Figure 13: Shore X is not maximal

Thus, $C' = \partial(X')$ and $\partial(\overline{X} \cap \overline{Y})$ do not cross. As $v \in X' \cap \overline{X} \cap \overline{Y}$, one of X' and $\overline{X} \cap \overline{Y}$ is a subset of the other. If X' is a proper subset of $\overline{X} \cap \overline{Y}$, then $\partial(\overline{X} \cap \overline{Y}) - e$ would be a nontrivial tight cut of brace $G^\star$. On the other hand, note that $(G - e)/X/(\overline{X} \cap \overline{Y})$ is a brace, by Lemma 29. If X' were a strict super set of $\overline{X} \cap \overline{Y}$, then $\partial(X') - e$ would be a nontrivial tight cut of this brace. We deduce that $X' = \overline{X} \cap \overline{Y}$.

Finally, by Lemma 29, $G^\star$ is isomorphic to $(G - f)/\overline{Y}/(X \cap Y)$, up to multiple edges. As J is simple, it follows that f lies in $\mathcal{R}$. Moreover, neither $\overline{Y}$ nor $X \cap Y$ is a singleton. Thus, $r(f) \geqslant |V(G^\star)| + 1 = r(e)$, whence $r(f) \geqslant r(e)$. □

In sum, in all cases we have shown that edge f lies in $\mathcal{R}$. We have also proved that $r(f) \geqslant r(e)$, with equality only if G has a cut $D := \partial(Y)$ such that $D - f$ is tight in $G - f$, D crosses C, the end s of f lies in X and $X' = \overline{X} \cap \overline{Y}$. Similar conclusions hold for edge g. It now remains to be shown that the last part of the statement holds. For this, assume, to the contrary, that $r(f) = r(e) = r(g)$.

We then have a shore Y of a nontrivial tight cut of $G - f$ and a shore Z of a nontrivial tight cut of $G - g$ such that the common end s of f and g lies in $Y \cap Z$, the contraction set X' of $G - e$ is nontrivial, and $\overline{X} \cap \overline{Y} = X' = \overline{X} \cap \overline{Z}$.

Let a_1 and a_2 denote two neighbours of v in $G - e$. The end v of e lies in $B \cap X'$, therefore a_1 and a_2 lie all in $A \cap X'$. As $\overline{X} \cap \overline{Y} = X' = \overline{X} \cap \overline{Z}$, it follows that a_1 and a_2 lie both in $A \cap \overline{Y} \cap \overline{Z}$.

The brace $G^\star$ contains at least four vertices, two of which are contraction vertices lying in distinct parts of the bipartition of $G^\star$. As $X' = \overline{X} \cap \overline{Y}$, the set of vertices of $G^\star$ that are internal, distinct from the contraction vertices, is $\overline{X} \cap Y$. This implies that $\overline{X} \cap Y$ contains as many vertices in A as it contains in B. Let b_1 denote a vertex of $B \cap \overline{X} \cap Y$. As $\overline{X} \cap \overline{Y} = \overline{X} \cap \overline{Z}$, it follows that $\overline{X} \cap Y = \overline{X} \cap Z$. Thus b_1 lies in $B \cap Y \cap Z$.

Let b_2 denote a vertex adjacent to s and distinct from t_1 and t_2. As s lies in X_-, in turn a subset of A, and since $\partial(Y) - f$ is tight in $G - f$ and $\partial(Z) - g$ is tight in $G - g$, it follows that b_2 lies in $B \cap Y \cap Z$.

In sum, a_1 and a_2 are two vertices in $A \cap \overline{Y} \cap \overline{Z}$, whereas b_1 and b_2 are two vertices in $B \cap Y \cap Z$. Graph G is a brace. By Theorem 10(b), $G - a_1 - a_2 - b_1 - b_2$ has a perfect matching, M.

Vertices b_1 and b_2 lie in $B \cap Y = Y_+$, whereas vertices a_1 and a_2 lie in $\overline{Y}$. As $\partial(Y) - f$ is tight in $G - f$, it follows that f lies in M.

Likewise, vertices b_1 and b_2 lie in $B \cap Z = Z_+$, vertices a_1 and a_2 lie in $\overline{Z}$, and $\partial(Z) - g$ is a tight cut of $G - g$. Thus, edge g lies in M. We conclude that M contains both edges f and g. This is a contradiction, as f and g are adjacent. We have thus established the validity of the last part of Theorem 26. □

The above theorem implies that an edge of maximum rank in $\mathcal{R}$ is thin. We thus have the main result we set out to prove.

Theorem 31 (The Thin Edge Theorem for Braces)**.** *Let G be a brace of order at least six, and let J be a simple brace distinct from G. If J is a matching minor of G, then G has a thin edge e such that J is a matching minor of $G - e$.* □

2.2 Multiple thin edges in braces

We now turn our attention to proving that every brace G on six or more vertices has at least two thin edges. We first note that if a brace J is a matching minor of G, then it is, in general, not true that G has two thin edges e and e' such that J is a matching minor of both $G-e$ and $G-e'$. For instance, take J to be any cubic brace on at least eight vertices and let G be obtained from J by adding an edge e joining two nonadjacent vertices. Then e is the only thin edge of G such that J is a matching minor of $G - e$, because any edge $e' \neq e$ is incident with a vertex of degree three, implying that the retract of $G - e'$ has at most $|V(G)| - 2 = |V(J)| - 2$ vertices. Thus, J cannot be a matching minor of $G - e'$, for any edge $e' \neq e$. Nevertheless, it would be of interest to establish properties of braces without reference to matching minors of large orders. The following result is a strengthening of Theorem 31 in the case where $J = C_4$. Every bipartite matching covered graph of order six or more has C_4 as a matching minor. Thus, when $J = C_4$, every edge of the brace belongs to the set $\mathcal{R}$ as defined in the beginning of Section 2.1.

Theorem 32. *Every brace of order six or more has at least two thin edges.*

Proof. Let G be a brace of order six or more, and take J to be C_4. By applying Theorem 31, we immediately deduce that G has a thin edge. Let e_0 be a thin edge of G. As noted above, all edges in E belong to $\mathcal{R}$. Let e_1 be an edge in $E - e_0$ of maximum possible rank. If e_1 is thin, then there is nothing more to prove. So, assume that e_1 is not thin. By Theorem 26, with e_1 playing the role of e, there exist two edges f_1 and g_1 such that (i) f_1 and g_1 are adjacent to each other, but are not adjacent to e_1, (ii) $r(f_1) \geqslant r(e_1)$, and $r(g_1) \geqslant r(e_1)$, and (iii) either $r(f_1) > r(e_1)$ or $r(g_1) > r(e_1)$ (or both). Assume without loss of generality that $r(f_1) > r(e_1)$. If $f_1 \neq e_0$, the maximality of the rank of e_1 would be violated. So, suppose that $f_1 = e_0$. If $r(g_1) > r(e_1)$, the maximality of the rank of e_1 would again be violated. So, suppose that $r(g_1) = r(e_1)$. If g_1 is thin, there is nothing more to prove. Assume that it is not. Now, by Theorem 26, with g_1 playing the role of e, there exist two edges f_2 and g_2 such that (i) f_2 and g_2 are adjacent to each other, but are not adjacent to g_1, (ii) $r(f_2) \geqslant r(g_1)$, and $r(g_2) \geqslant r(g_1)$, and (iii) either $r(f_2) > r(g_1)$ or $r(g_2) > r(g_1)$. Since $f_1 = e_0$ is adjacent to g_1, neither f_2 nor g_2 can be equal to e_0. But then we have a contradiction as to the maximality of the rank of e_1 among the edges in $E - e_0$. Hence g_1 is thin, and the desired assertion follows. $\square$

Most known families of braces have many thin edges. For instance, biwheels on $2n$ vertices have $2(n-1)$ thin edges. Möbius ladders and prisms on $2n$ vertices have n thin edges. This leads us to surmise the following:

Conjecture 33. There exists a positive constant c such that every brace on n vertices has cn thin edges.

New ideas seem to be necessary to approach even weaker conjectures. For example, we do not know if every brace of order six or more has two nonadjacent thin edges. Our hope is that, by gaining further knowledge about the existence of multiple thin edges, one might be able to settle questions on braces such as our biwheel conjecture. (It states that there exists an integer N such that, for all $n \geqslant N$, a brace of order $2n$ has at least $(n-1)^2$ perfect matchings, see [7]).

3 Strictly Thin Edges

McCuaig [11] implicitly used the notion of thin edges to devise recursive procedure for generating braces. In order to establish such procedures for generating simple braces, where all the intermediate graphs are also simple, one needs the notion of a strictly thin edge. Let G be a simple brace on six or more vertices. An edge e of G is *strictly thin* if e is thin and the retract of $G - e$ is simple.

We saw in the last section that every brace of order six or more has a thin edge (Theorem 32). However, not every brace has strictly thin edges. For example, Möbius ladders, prisms, and biwheels do not have any strictly thin edges. McCuaig [11] showed that, among other things, a simple brace which does not belong to any one of the above mentioned families has a strictly thin edge.

We shall now proceed to show how Theorems 31 and 32 may be used to deduce the following strenghtening of the above mentioned statement.

Theorem 34 (The Main Theorem)**.** *Let G and J be distinct simple braces, where G is not in $\mathcal{G}^+$ and has more than four vertices, and J is a matching minor of G. Then*

(i) *G has a strictly thin edge e such that J is a matching minor of $G - e$.*

(ii) *G has two strictly thin edges.*

Part (i) of the above theorem can be deduced from the main theorem in McCuaig [11]. Note that, as is the case with thin edges, one cannot claim that there are two strictly thin edges e and f of G such that J is a matching minor of both $G - e$ and $G - f$.

3.1 Multiple edges in retracts

Suppose that G is a simple brace, and e is a thin edge of G. We begin with a brief review of the conditions under which bicontractions of $G-e$ result in a graph with multiple edges. The simplest case arises when e is a thin edge of index one. (If the index of e is zero, then the retract of $G-e$ is itself, and it has no multiple edges.) Let $e = x_0y_0$ be such an edge. Let vertex x_0 be the end of e of degree three in G, and y_1 and y_2 be its neighbours distinct from y_0. If $e_1 = y_1w$ and $e_2 = y_2w$ are two edges incident with a common vertex w which belongs to $V(G) \setminus \{x_0, y_1, y_2\}$, then e_1 and e_2 are multiple edges in the retract of $G-e$. Note that the degree of w is at least four in G. See Figure 14. (The rectangle with rounded corners includes all the non-contraction vertices in the retracts.)

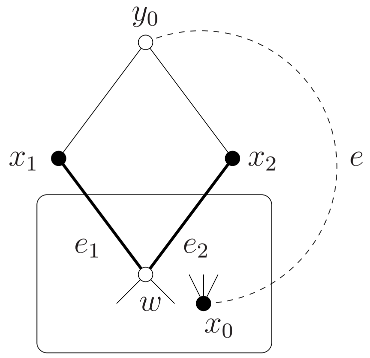


Figure 14: Multiple edges in the retract of $G-e$, index$(e) = 1$.

Now consider the case in which $e = x_0y_0$ is a thin edge of index two. There are essentially three possible situations under which two edges e_1 and e_2 of G become multiple edges in the retract of $G-e$. These three situations are illustrated in Figure 15. (The rectangles with rounded corners include all the non-contraction vertices in the retracts.)

3.2 An exchange property of thin edges

In this section, we shall investigate implications of a thin edge being not strictly thin. We shall adopt throughout the notation introduced below.

Notation 35. Let G, J be two distinct simple braces, where G is not in $\mathcal{G}^+$ and has at least six vertices, and J is a matching minor of G. There are two subsets of $E(G)$, namely $\mathcal{T}$ and $\mathcal{T}_-$, defined below, which are of special interest to us.

$\mathcal{T}$: the set of thin edges e of G such that J is a matching minor of $G-e$, and

$\mathcal{T}^*$: the subset of those edges in $\mathcal{T}$ which are also strictly thin.

By Theorem 31, the set $\mathcal{T}$ is nonempty. If G is a prism, or a Möbius ladder, or a biwheel, then $\mathcal{T}^* = \varnothing$. If G is an extended biwheel, then $\mathcal{T}^*$ has just one member, namely the edge that joins the two hubs.

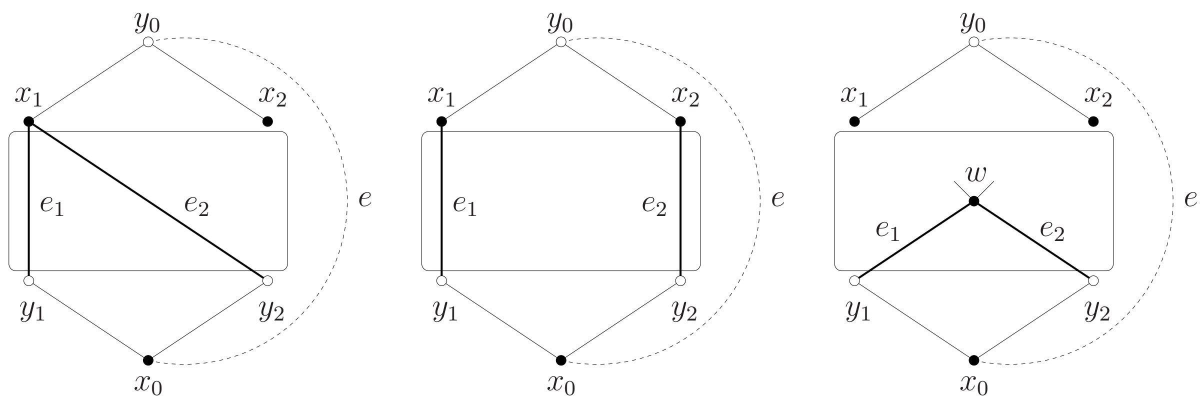


Figure 15: Multiple edges in the retract of $G - e$, index$(e) = 2$.

Any edge of index zero in $\mathcal{T}$ is also in $\mathcal{T}^*$. Thus, the index of any edge $e := x_0y_0$ in $\mathcal{T} \setminus \mathcal{T}^*$ will have to be either one or two. Consequently, at least one end of e has degree three. If the index of e is one, we shall adjust notation and assume that y_0 has degree three, and denote its two neighbours in $G - e$ by x_1 and x_2. If the index of e is two, then x_0 also has degree three, and we shall denote its two neighbours in $G - e$ by y_1 and y_2. In both cases, we shall let e_1 and e_2 denote two parallel edges in the retract of $G - e$. When e is of index two then two subsets of $V(G)$, described below, will play a special role in Lemma 40:

$X := \{x_0, x_1, x_2\}$, and $Y := \{y_0, y_1, y_2\}$.

Finally, for brevity, we shall use the following notation for the retract of $G - e$ and its underlying simple graph:

H: the retract $\widehat{G - e}$ of $G - e$, and

H_0: the underlying simple graph of H.

Having established the requisite notation, let us first note that G must have at least eight vertices because it is simple and, not being a member of $\mathcal{G}^+$, is different from $K_{3,3}$. Consequently, H_0 has at least four vertices. By Lemma 25, J is a matching minor of the underlying simple graph of a brace of G. Thus:

Proposition 36. *The brace J is a matching minor of H_0.* □

As our aim is to find a thin edge which is also strictly thin, we should look for thin edges other than e. The two edges e_1 and e_2, which are thin in H, are obvious candidates for being thin in G as well. As a first step, we establish that these two edges are removable in $G - e$.

Lemma 37. *For $i = 1, 2$, edge e_i is removable in both G and $G - e$. Moreover, J is a matching minor of both $G - e - e_i$ and $G - e_i$.*

Proof. By hypothesis, G is a brace and has more than four vertices. Thus, every edge of G is removable. In particular, e_i is removable in G.

Let us now prove that e_i is removable in $G - e$. Edge e_i is removable in H. If e_i is not incident with a contraction vertex of H then e_i is certainly removable in $G - e$. Alternatively, if e_i is incident with a contraction vertex of H then e_i is removable in $G-e$, by Lemma 19. In both alternatives, e_i is removable in $G - e$.

In order to obtain H from $G - e$, index(e) bicontractions are performed. Applying to $G - e - e_i$ the same bicontractions, we obtain $H - e_i$. Thus, $H - e_i$ is a matching minor of $G - e - e_i$. Consequently, H_0 is a matching minor of $G - e - e_i$. By Proposition 36, J is a matching minor of H_0. It follows that J is a matching minor of both $G - e - e_i$ and $G - e_i$. $\square$

Corollary 38. *For $i = 1, 2$, if e_i is thin, then $e_i \in \mathcal{T}$.* $\square$

Lemma 39. *Suppose that e belongs to $\mathcal{T} \setminus \mathcal{T}^*$ and that its index is one. Then both e_1 and e_2 are thin edges of index one and belong to $\mathcal{T}$.*

Proof. Since e is a thin edge of index one, then $G - e$ has just two braces, one of then is H, and the other has order four. Clearly, for $i = 1, 2$, $G - e - e_i$ also has only two braces, one of them being $H - e_i$, and the other being of order four. Graph $G - e_i$, which is obtained by adding e to $G - e - e_i$, is either a brace or has only two braces, one of order four, the other of order $|V(H)|$. It follows that e_i is a thin edge of index one. The assertion follows from Corollary 38. $\square$

If the index of the edge e is two, and e_1 and e_2 are parallel edges in the retract of $G-e$, it is not necessary for both e_1 and e_2 to be thin in G. (For example, consider the brace shown in Figure 16, where $e = x_0y_0$ is a thin edge of index two. The edges $e_1 = y_1x_3$ and $e_2 = y_2x_3$ are parallel edges in the retract H of $G - e$. The cut $\partial(\{y_0, y_1, x_0, x_1, x_2\})$ is a tight cut of $G - e_1$ both of whose shores have at least five vertices, implying that e_1 is not a thin edge of G.) Among other things, Lemma 40 asserts that at least one of the two edges e_1 and e_2 is thin in G.

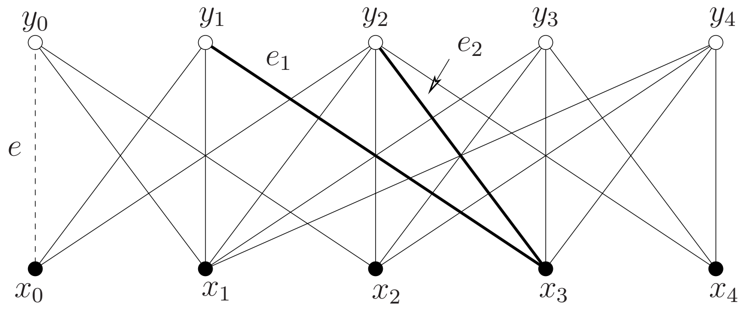


Figure 16: Index of e is two, e_1 is not thin

Lemma 40. *Suppose that e belongs to $\mathcal{T} \setminus \mathcal{T}^*$ and that its index is two. If e_1 is not thin, then the following properties hold:*

(i) *edges* e_1 *and* e_2 *share an end* w *that is not adjacent to any end of* e*, as in Figure 15(c);*

(ii) *edge* e_2 *belongs to* $\mathcal{T}$ *and has index zero or one;*

(iii) *brace* G *has an edge* f *distinct from* e_2 *that belongs to* $\mathcal{T}$ *and such that* e_2 *is not a multiple edge in the retract of* $G - f$*.*

Proof. Since e is a thin edge of G of index two, the graph $G-e$ has precisely three braces, one is H, the other two have order four. Moreover, H has two contraction vertices. Thus, $G - e - e_1$ also has three braces, one is $H - e_1$, the other two have order four. Brace H has order $|V(G)| - 4$ and $H - e_1$ is a brace of $G - e - e_1$. By hypothesis, edge e_1 is not thin. Thus G has a cut $C := \partial(Z)$ such that $C - e_1$ is tight in $G - e_1$ and both Z and $\overline{Z}$ have five or more vertices. Consequently, G has 10 or more vertices, and H has six or more vertices. Moreover, one of the $(C - e - e_1)$-contractions of $G - e - e_1$ is isomorphic to $H - e_1$, the other $(C - e - e_1)$-contraction has two braces, both of order four. Adjust notation so that $(G - e - e_1)/(Z \to z) \simeq H - e_1$, whereupon $|Z| = 5$. Because $H - e_1$ is a brace of order at least six, no vertex of $\overline{Z}$ has degree two in $G - e - e_1$. Thus, edge e has both ends in Z. Likewise, the end w of e_1 in $\overline{Z}$ has degree four or more in G. As e has both ends in Z, $(G - e_1)/(Z \to z) \simeq H - e_1$ (See Figure 17).

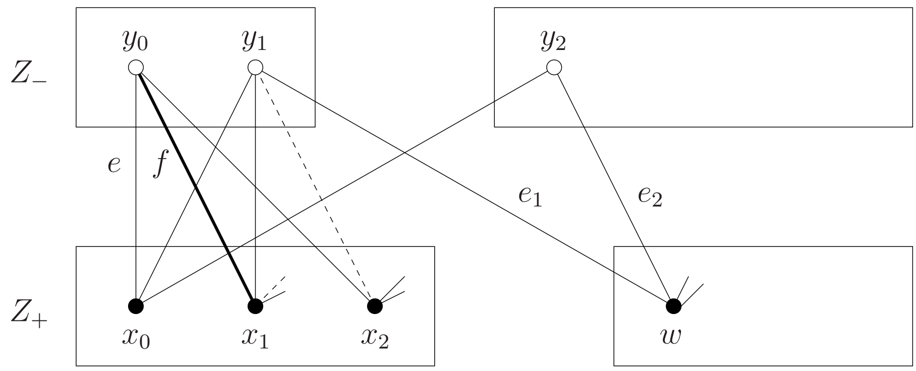


Figure 17: The cut C

Let X denote the set $\{x_0, x_1, x_2\}$ and Y the set $\{y_0, y_1, y_2\}$. Edge e has both ends in Z. Thus, either $X \subseteq Z_+$ or $Y \subseteq Z_+$. Edge e_1 has one end in Z_-, the other in $\overline{Z}$, in the same part of G containing the vertices of Z_+. Thus, the end w of e_1 in $\overline{Z}$ does not lie in $X \cup Y$. Edge e_1 is incident with y_1. Thus, y_1 is the end of e_1 in Z_-. Edge e has both ends in X. Thus, $Z_- = \{y_0, y_1\}$ and $Z_+ = X$. Vertex y_1 is adjacent to at least two vertices of Z_+. It certainly is adjacent to x_0, by definition. Adjust notation so that y_1 is adjacent to x_1. It is possible that y_1 is also adjacent to x_2.

Edges e_1 and e_2 are parallel in H. Moreover, e_2 is incident with y_2, whence $e_2 = wy_2$. Assume, to the contrary, that e_2 is not thin. Repeating the reasoning done with e_1, we deduce that $N(y_2) \subseteq X \cup \{w\}$. Then, $N(Y) = X \cup \{w\}$. But G is a brace, $|Y| = 3$ and $|X \cup \{w\}| = 4$, therefore G has only eight vertices. This is a contradiction, as

$|\overline{Z}| \geqslant 5 = |Z|$. We deduce that e_2 is thin in G. We also know that J is a matching minor of $G - e_2$. Thus, e_2 lies in $\mathcal{T}$. The end w of e_2 has degree four or more. Thus, e_2 has index zero or one.

Now let us turn to the proof of the third part of the assertion.

40.1. *Edge $f := x_1 y_0$ belongs to $\mathcal{T}$.*

Proof. Let $G' := G - f$, let $G'' := G - f - e_1 = G' - e_1$. Consider the cut $C - e_1$ of G''. One of the $(C - e_1)$-contractions of G'' is $H - e_1$, a brace. The other $(C - e_1)$-contraction of G'' is a bipartite graph whose underlying simple graph is a 6-cycle with one or two chords. It follows that G'' is the splicing of two matching covered graphs, whence it is matching covered. Moreover, $C - e_1$ is tight in G''.

Let us consider a tight cut decomposition of G'' in which we use $C - e_1$ as one of the tight cuts of G''. Then, G'' has precisely three braces, two of order four, the third is the brace $H - e_1$, up to multiple edges. Brace J is a matching minor of $H - e_1$, whence it is also a matching minor of G''. We deduce that J is a matching minor of G'.

To complete the proof, we must show that f is thin in G. The graph G'' has precisely three braces, two of which of order four, the third of order $|V(G)| - 4$. For every cut D of G, if $D - f$ is tight in G' then $D - f - e_1$ is tight in G''. We may thus obtain a tight cut decomposition of G'' by starting with a tight cut decomposition of G' and then proceed by removing e_1 from each brace obtained and continue with the tight cut decomposition procedure. By doing this, we obtain two braces of order four plus a brace of order $|V(G)| - 4$.

Let G_1 be the graph obtained from G' by bicontracting vertex y_0. This operation corresponds to a (possibly partial) tight cut decomposition of $G - f$, where the two graphs are a brace of order four and G_1. Moreover, $G_1 - e_1$ has precisely two braces, one of order four, the other of order $|V(G)| - 4$. Thus, either G_1 is a brace or G_1 has precisely two braces, one of order four.

If G_1 is a brace then certainly f is thin, of index one. Assume thus that G_1 has precisely two braces, one of order four. Graph G_1 has order $|V(G)| - 2$, whence it has order eight or more. We deduce that G_1 has a vertex which is adjacent only to two vertices. Every vertex of $H - e_1$, a brace on six or more vertices, is adjacent to three or more vertices. Thus, every vertex of $\overline{Z}$ is adjacent to three or more vertices in G_1. Vertex y_1 is adjacent to three vertices in G_1, namely, w, x_1 and the contraction vertex of G_1. The contraction vertex of G_1 is adjacent to three or more vertices, otherwise $N(x_0, x_2) = Y$, a contradiction to the fact that G is a brace on ten or more vertices. We deduce that x_1 is the vertex of G_1 that is adjacent only to two vertices. As G is simple, it follows that x_1 has degree two in G_1. Thus, the retract of G_1 is a brace of order $|V(G)| - 4$. Indeed, f is thin in G. □

To complete the proof, let us now show that e_2 is not a multiple edge of $\widehat{G - f}$. For this, assume the contrary. Then, an end of e_2 is adjacent to an end of f of degree three. The end w of e_2 is not adjacent to the end y_0 of f. Thus, y_2 is adjacent to x_1, and x_1 must have degree three. In that case, $N(x_0, x_1) = Y$, whence brace G has only six vertices, a contradiction. □

4 Proof of the Main Theorem

Our proof of the Main Theorem relies on the following crucial result.

Lemma 41 (Key Lemma)**.** *Let G and J be distinct simple braces, where G is not in $\mathcal{G}^+$ and has more than four vertices, and J is a matching minor of G. Suppose that G has a thin edge e such that J is a matching minor of $G - e$. If e is not strictly thin then G has two strictly thin edges f and g such that J is a matching minor of both $G - f$ and $G - g$.*

Proof. By hypothesis, G is a simple brace of order six or more that is not a member of $\mathcal{G}^+$. The only simple brace on six vertices is $K_{3,3}$, a Möbius ladder. Thus, G has order eight or more. In what follows, we adopt the notation introduced at the beginning of Section 3.2.

Case 1. *There are edges of index one in $\mathcal{T} \setminus \mathcal{T}^*$.*

Let $e = x_0y_0$ be an edge of index one in $\mathcal{T} \setminus \mathcal{T}^*$, where y_0 has degree three, and let e_1 and e_2, incident with x_1 and x_2, respectively, be parallel edges in the retract H. Since e_1 and e_2 are parallel edges in the retract H of $G - e$, they must have a common end, say w, which has degree four or more in G (see Figure 14).

By Lemma 39, both e_1 and e_2 belong to $\mathcal{T}$. If they are both strictly thin, then there would be nothing more to prove. On the other hand, if either of them, say e_2, is not strictly thin, then it would be possible to apply Lemma 39 with e_2 playing the role of e and assert the existence of a configuration similar to the one in Figure 14, which is based on e_1. If that does not yield two strictly thin edges in $\mathcal{T}$, the procedure can be repeated with a new thin in $\mathcal{T}$ that is not strictly thin. In this manner, as we argue below, we would be able to show that either there exist two strictly thin edges of the required type, or we would be able to obtain a contradiction to the hypothesis that $G \notin \mathcal{G}^+$ by showing that G is either a biwheel or an extended biwheel.

Let h_1 and h_2 denote two distinct vertices of G, let $P := (v_1, v_2, \ldots, v_k)$, $k \geqslant 3$, be a maximal path in $G - \{h_1, h_2\}$ such that the following properties are satisfied:

(i) Each vertex of P is adjacent either to h_1 or to h_2 (but not both).

(ii) Each internal vertex of P has degree three, and the edge that joins it to h_1 or h_2 lies in $\mathcal{T} \setminus \mathcal{T}^*$ and has index one.

We denote the subgraph of G consisting of the path P, vertices h_1 and h_2, together with the edges joining h_1 and h_2 to vertices of P by F. See Figure 18.
(Such a maximal configuration must exist because, with appropriate relabelling, the subgraph of G shown in Figure 14 yields a configuration with $k = 3$.) Adjust notation so that v_1 is adjacent to h_1, whereupon each vertex v_i of P, with i odd, is adjacent to h_1 and each vertex of v_i, with i even, is adjacent to h_2. For $i = 1, \ldots, k$, denote by f_i the edge that joins v_i to h_1 or to h_2.

By definition, edge f_2 lies in $\mathcal{T} \setminus \mathcal{T}^*$ and has index one. The edges f_1 andf_3 are multiple edges in the retract of $G - f_2$. By Lemma 39, with f_2 playing the role of e, f_1 playing the

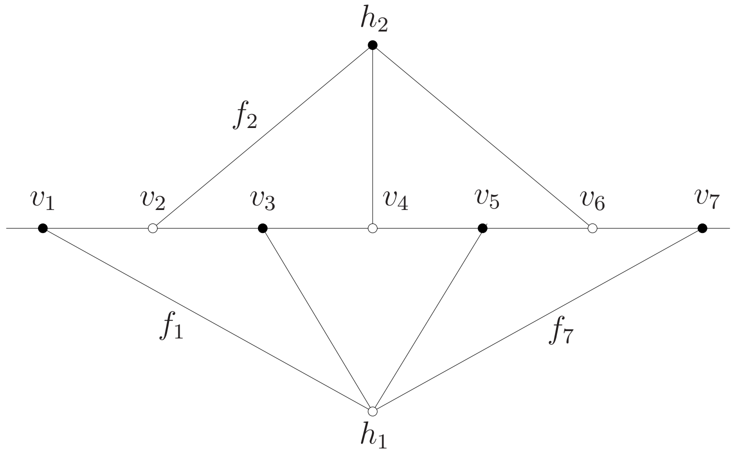


Figure 18: Subgraph F ($k = 7$)

role of e_1, and f_3 the role of e_2, we conclude that edge f_1 lies in $\mathcal{T}$. Likewise, f_k lies in $\mathcal{T}$. If f_1 and f_k are both strictly thin then the assertion holds in this case.

We may thus assume that one of f_1 and f_k is not strictly thin. Adjust notation so that f_1 is not strictly thin. Brace G has eight or more vertices, therefore the retract of $G - f_2$ has six or more vertices. Thus, the common end h_1 of f_1 and f_3 has three or more neighbours in the retract of $G - f_2$, whence h_1 has degree four or more in G. Thus, f_1 has index zero or one. We have assumed that f_1 is not strictly thin. Thus, it has index one, whence v_1 has degree three.

Let v denote the vertex of G distinct from h_1 and v_2 that is adjacent to v_1. As G is bipartite and every internal vertex of P has degree three, either $v = v_k$ or v does not lie in $V(F)$.

Assume that $v = v_k$, this implies that k is even. The graph $F - \{h_1, h_2, v_k\}$ is a connected component of $G - \{h_1, h_2, v_k\}$. By Lemma 11, $V(G) = V(F)$. If h_1 and h_2 are not adjacent then G is a biwheel, otherwise it is an extended biwheel. In any case, it is a contradiction, since by hypothesis G is not in $\mathcal{G}^+$.

Thus, v does not lie in $V(F)$. Edge f_1 is thin of index one, but not strictly thin. Moreover, v_1 has degree three. Thus, v is adjacent to v_3 or to h_2. By the maximality of P, vertex v is not adjacent to h_2. Thus, v is adjacent to v_3. Every internal vertex of P has degree three. Thus, $k = 3$ (see Figure 19).
If v_3 has degree three then P is a connected component of $G - \{h_1, h_2, v\}$; in that case, by Lemma 11, G has only six vertices, a contradiction. Thus, v_3 has degree four or more. We have seen that f_k lies in $\mathcal{T}$. Also, h_1 has degree four or more. But $k = 3$. Thus, $f_k = h_1v_3$. We deduce that f_3 has index zero, whence it is strictly thin.

Note that in $\widehat{G - f_2}$, the edges vv_1 and vv_3 are parallel. Moreover, as G has eight or more vertices, $\widehat{G - f_2}$ has six or more vertices, whence vertex v must be adjacent to three or more vertices in $\widehat{G - f_2}$. Thus, v has degree four or more in G. By Lemma 39, with f_2 playing the role of e, edge vv_1 playing the role of e_1 and edge vv_3 playing the role of e_2, we deduce that vv_3 lies in $\mathcal{T}$. In sum, vv_3 is an edge in $\mathcal{T}$ whose ends both have degree

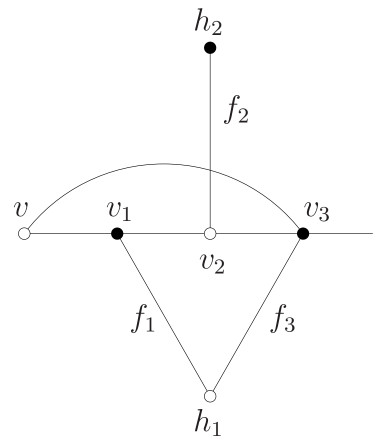


Figure 19: Edge f_1 is not strictly thin (and $k = 3$)

four or more. Thus, vv_3 is strictly thin. We conclude that vv_3 and f_3 are both strictly thin edges in $\mathcal{T}^{\star}$. The assertion holds.

Case 2. *There are no edges of index one in $\mathcal{T} \setminus \mathcal{T}^*$.*

Let $e = x_0y_0$ be an edge of index two in $\mathcal{T} \setminus \mathcal{T}^*$, and let e_1 and e_2 be two multiple edges in the retract H. Assume without loss of generality that e_1 is incident with y_1 and e_2 is incident with y_2. We shall divide the analysis of this case into three subcases depending on where the ends of e_1 and e_2 different from y_1 and y_2 are situated.

Case 2.1. *Edges e_1 and e_2 share a common end in $X \cup Y$.*

Adjust notation so that $e_i = x_1y_i$, for $i = 1, 2$ (Figure 15(a)). By Lemma 40, edges e_1 and e_2 are both thin, whence, by Corollary 38, they belong to $\mathcal{T}$.

Note that vertex x_1 has degree four or more, otherwise $N(\{x_0, x_1\}) = Y$, and brace G would have only six vertices, a contradiction. Thus, e_1 and e_2 both belong to $\mathcal{T}$ and have index less than two. If e_1 is not strictly thin then it has index one, contrary to the hypothesis of the case under consideration. Thus, e_1 is strictly thin. Likewise, e_2 is strictly thin. The assertion holds in this case.

Case 2.2. *Edges e_1 and e_2 are not adjacent (Figure 15(b)).*

Adjust notation so that $e_i = x_iy_i$, for $i = 1, 2$. Note that G has a subgraph formed by the union of two disjoint paths (x_1, y_0, x_2) and (y_1, x_0, y_2), and the addition of an edge joining x_i y_i, for $i = 0, 1, 2$. Moreover, the edge x_0y_0 lies in $\mathcal{T} \setminus \mathcal{T}^*$. Let F be a maximal subgraph of G formed by the union of two disjoint paths $P := (u_1, u_2, \ldots, u_k)$ and $Q := (v_1, v_2, \ldots, v_k)$ ($k \geqslant 3$) and the addition of the edges u_iv_i for $i = 1, \ldots, k$, and such that the edges u_iv_i lie in $\mathcal{T} \setminus \mathcal{T}^*$, for $i = 2, \ldots, k-1$ (Figure 20).

Note that every internal vertex of P and Q has degree three, as each edge u_iv_i, for $i = 2, \ldots, k-1$, lies in $\mathcal{T} \setminus \mathcal{T}^*$, and so it is a thin edge of index two.

By definition, edge u_2v_2 lies in $\mathcal{T} \setminus \mathcal{T}^*$. The retract of $G - u_2v_2$ has parallel edges u_1v_1 and u_3v_3. By Lemma 40, edge u_1v_1 lies in $\mathcal{T}$. Likewise, u_kv_k lies in $\mathcal{T}$.

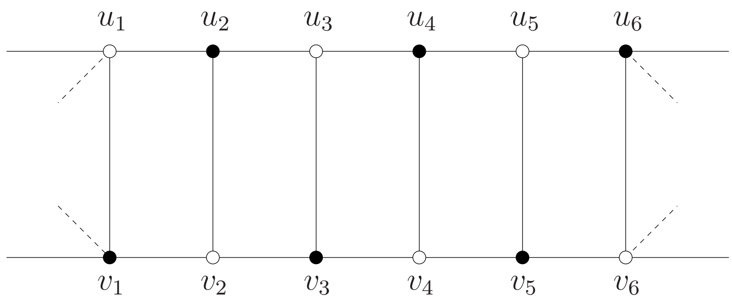


Figure 20: Subgraph F of G ($k = 6$)

If u_1v_1 and u_kv_k are both strictly thin then the assertion holds. We may thus assume that at least one of them lies in $\mathcal{T} \setminus \mathcal{T}^*$. Adjust notation so that u_1v_1 lies in $\mathcal{T} \setminus \mathcal{T}^*$. Then, u_1v_1 has index two, whence u_1 and v_1 both have degree three. Let u be the vertex of $V(G) - \{v_1, u_2\}$ adjacent to u_1. Let v be the vertex of $V(G) - \{u_1, v_2\}$ adjacent to v_1. Then, u is not an internal vertex of P, nor of Q. Likewise, v is not an internal vertex of P, nor of Q.

Proposition 42. *Vertices u and v do not belong to $V(F)$.*

Proof. Suppose that at least one of u and v is in $V(F)$. Adjust notation so that u is in $V(F)$. By definition, $u \neq v_1$. We have seen that u is not an internal vertex of P, nor of Q. Thus, $u \in \{u_k, v_k\}$. Then $F - \{v_1, u_k, v_k\}$ is a connected component of $G - \{v_1, u_k, v_k\}$. By Lemma 11, $V(G) = V(F)$.

If $u = u_k$ then k is even, and $v = v_k$, whence G is a prism. Alternatively, if $u = v_k$ then k is odd, and $v = u_k$, whence G is a Möbius ladder. In both alternatives we get a contradiction to the hypothesis that G is not in $\mathcal{G}^+$. □

We may thus assume that neither u nor v is in $V(F)$. We have assumed that u_1v_1 belongs to $\mathcal{T} \setminus \mathcal{T}^*$. Thus, the retract of $G - u_1v_1$ has multiple edges. Thus, either u is adjacent to a vertex in $\{v, v_2, u_3\}$ or v is adjacent to a vertex in $\{u, u_2, v_3\}$. Vertices u and v_2 are not adjacent. Likewise, vertices v and u_2 are not adjacent. By the maximality of F, vertices u and v are not adjacent. We deduce that either u is adjacent to u_3 or v is adjacent to v_3. Adjust notation so that u is adjacent to u_3. Then, $k = 3$ (Figure 21).

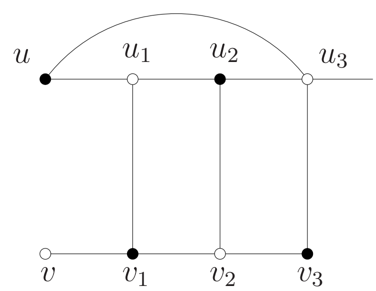


Figure 21: Vertices u and u_3 are adjacent

42.1. *Vertex u_3 has degree four or more in G and edge u_2u_3 is not thin in G.*

Proof. Assume, to the contrary, that $N(u_3) = \{u, u_2, v_3\}$. Then, $N(\{u_1, v_2, u_3\}) = \{u, v_1, u_2, v_3\}$. This implies that G has only eight vertices. As v is not adjacent to u_2, it follows that $N(v) = \{u, v_1, v_3\}$. In particular, u and v are adjacent, a contradiction. We conclude that u_3 has degree four or more in G.

This conclusion implies that u_2u_3 has index one. Thus, $\widehat{G - u_2u_3}$ has six or more vertices. But in $\widehat{G - u_2u_3}$, vertex v_1 is adjacent only to v and to the contraction vertex of $\widehat{G - u_2u_3}$. Thus, $\widehat{G - u_2u_3}$ is not a brace. We deduce that u_2u_3 is not thin in G. ☐

By Lemma 40, with u_1v_1 playing the role of e, u_2u_3 the role of e_1 and uu_3 the role of e_2, we have that uu_3 lies in $\mathcal{T}$. Edge u_3v_3 also lies in $\mathcal{T}$. Vertex u_3 has degree four or more, therefore both uu_3 and u_3v_3 have index less than two. By the hypothesis of the case, they are both strictly thin. We conclude that u_3u and u_3v_3 are strictly thin edges in $\mathcal{T}^\star$.

Case 2.3. *Edges e_1 and e_2 share a common w end not in $X \cup Y$ (Figure 15(c)).*

In this case, we shall also prove that at least one of e_1 and e_2 lies in $\mathcal{T}$ and is strictly thin.

42.2. *Vertex w has degree four or more in G.*

Proof. Assume that w has degree three. In the retract H of $G - e$, vertex w is adjacent only to two vertices. Then, H has only four vertices. We conclude that G has only eight vertices (Figure 22).

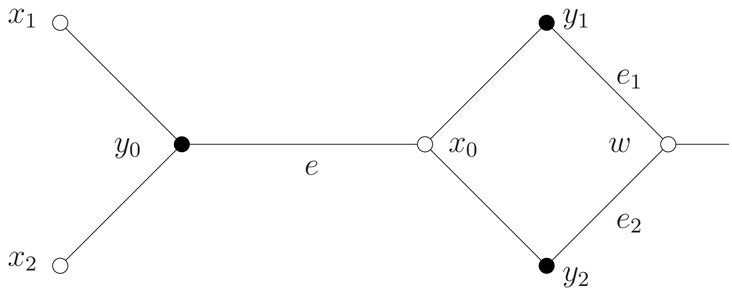


Figure 22: Brace G is the cube

The case in which x_1 is adjacent to both y_1 and y_2 has already been considered in Case 2.1, therefore x_1 has degree three in G. Likewise, x_2 has degree three in G. Thus, the four vertices of the part of G that contains vertex x_0 have degree three in G. Thus, G is cubic. The only cubic brace on eight vertices is the cube. This is a contradiction, as the cube is a prism. ☐

Consider first the case in which both e_1 and e_2 are in $\mathcal{T}$. Vertex w has degree four or more. Thus, e_1 has index zero or one. If e_1 is not strictly thin then it has index one, a case already considered (Case 1). Thus, e_1 is strictly thin. Likewise e_2 is strictly thin. The assertion holds.

We may thus assume that one of e_1 and e_2 does not belong to $\mathcal{T}$. Adjust notation so that e_1 is not in $\mathcal{T}$. By Lemma 40, edge e_2 lies in $\mathcal{T}$ and has index zero or one. Edges in $\mathcal{T} \setminus \mathcal{T}^*$ of index one have been considered in Case 1. We may thus assume that e_2 is strictly thin.

To complete the proof, we must now prove that G has an edge distinct from e_2 that also lies in $\mathcal{T}$ and is strictly thin. Edge e_1 does not lie in $\mathcal{T}$. By Lemma 40, G has an edge f in $\mathcal{T}$ distinct from e_2 and such that e_2 is not a multiple edge in the retract of $G - f$.

If f is strictly thin then the assertion holds, because f and e_2 are distinct. We may thus assume that f is not strictly thin. The case in which f has index one has already been considered (Case 1). We may thus assume that f has index two. Let f_1 and f_2 denote two parallel edges of the retract of $G - f$. The case in which f_1 and f_2 share an end adjacent with an end of f has already been considered (Case 2.1). The case in which f_1 and f_2 are not adjacent has already been considered (Case 2.2). We may thus assume that f_1 and f_2 share a common end not adjacent to an end of f. We have seen that in this case at least one of f_1 and f_2 lies in $\mathcal{T}$ and is strictly thin. Adjust notation so that f_2 lies in $\mathcal{T}$ and is strictly thin. Edge e_2 is not a parallel edge in the retract of $G - f$. Thus e_2 and f_2 are distinct strictly thin edges in $\mathcal{T}$. The assertion holds. $\square$

With the aid of the above lemma and Theorems 31 and 32, it is now straightforward to deduce the validity of Theorem 34.

Proof of the Main Theorem. Let us first prove the validity of item (i) of the Main Theorem. By Theorem 31, G has a thin edge e such that J is a matching minor of $G - e$. If e is strictly thin, then item (i) holds. We may thus assume that e is not strictly thin. By Lemma 41, G has two strictly thin edges f and g such that J is a matching minor of both $G - f$ and $G - g$. In both alternatives, item (i) holds.

Let us now prove that G has at least two strictly thin edges. By Theorem 32, G has two thin edges, e and f. If both e and f are strictly thin then the Theorem is proved. Adjust notation so that e is not strictly thin. Choose any simple brace J that is a matching minor of $G-e$. For instance, let $J := C_4$. By Lemma 41, G has two strictly thin edges e_1 and e_2 such that C_4 is a matching minor of both $G - e_1$ and $G - e_2$. In both alternatives, G has two strictly thin edges. In sum, the Main Theorem is reduced to the Key Lemma. $\square$

5 Braces with just Two Strictly Thin Edges

In this section we give examples of simple braces which have just two strictly thin edges thereby showing that our Main Theorem 34 provides the best possible lower bound on the number of strictly thin edges in a brace. Our constructions are based on the operation of 4-sum which appears in the works of Robertson, Seymour, and Thomas [14] and of McCuaig [12].

Let $G_1, G_2, \ldots, G_r$ be r distinct graphs, and let Q be a cycle of length four such that $G_i \cap G_j = Q$, for $1 \leqslant i < j \leqslant r$. Then, for any fixed subset R (possibly empty) of the

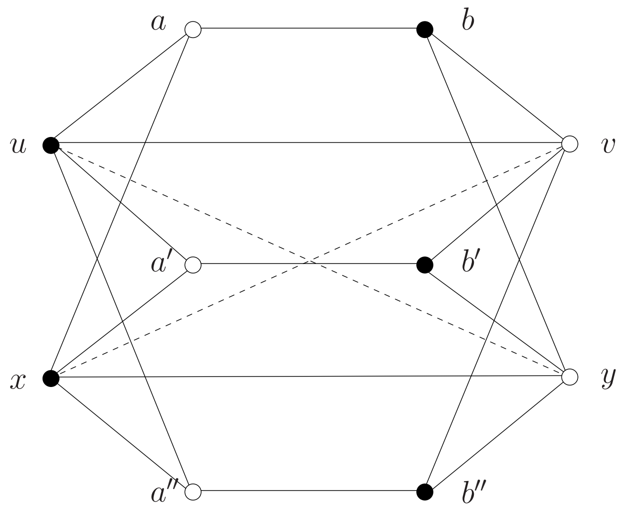


Figure 23: A 4-sum of three copies of $K_{3,3}$

edge set of Q, the graph $\cup(G_i - R)$ is called a *4-sum* of $G_1, G_2, \ldots, G_r$. McCuaig (see [12], Lemma 19) showed that if $r \geqslant 3$ and each G_i is a brace of order six or more then their 4-sum $\cup(G_i - R)$ is also a brace, with only one exception: $r = 3$, each G_i is $K_{3,3}$ and $R = E(Q)$. The graph G depicted in Figure 23 is the 4-sum of three copies of $K_{3,3}$, where R consists of two nonadjacent edges of their shared 4-cycle $Q = (u, v, x, y, u)$. (The edges in R are indicated by dotted lines).

The graphs $G - uv$ and $G - xy$ are simple braces which are also 4-sums of three copies of $K_{3,3}$ (with different proper subsets of $E(Q)$ designated as the set R). Therefore both uv and xy are strictly thin edges (of index zero) of the brace G. However, no edge in $E(G) - \{uv, xy\}$ is strictly thin. To see this, using the symmetries of G, it suffices to check that the two edges au and ab are not strictly thin. The edge au is not strictly thin because, in the retract of $G - au$, the edges xy and by are parallel. And, the edge ab is not strictly thin because, in the retract of $G - ab$, the edges uv and xy are parallel.

In exactly the same manner as above, it can be shown that the 4-sum of any r copies of $K_{3,3}$, with $r \geqslant 3$, where R consists of two nonadjacent edges of their shared 4-cycle, is a brace with exactly two strictly thin edges.

6 Thin Edges in Bricks

The notions of thin edges and strictly thin edges extend in an obvious manner to bricks. A removable edge e of a brick G is *thin* if the retract $\widehat{G - e}$ of $G - e$ is a brick; and if $\widehat{G - e}$ is a simple brick, then e is a *strictly thin edge of* G. In this concluding section, we briefly review some results related to thin and strictly thin edges in bricks which are analogous to those concerning braces discussed in this article.

A removable edge e of a brick G is *b-invariant* if $G-e$ has at most one brick. Confirming a conjecture made by Lovász in 1987, we showed in [3] that every brick different from K_4, $\overline{C_6}$, R_8 (which is obtained by splicing K_4 and $\overline{C_6}$), and the Petersen graph has at least two

b-invariant removable edges. From this result, we deduced in [5] that every brick distinct from K_4, $\overline{C_6}$ and the Petersen graph has a thin edge, and used this conclusion to describe a generation procedure for bricks.

Theorem 43. *Given any brick G, there exists a sequence $(G_1, G_2, \ldots, G_r)$ of bricks such that:*

(i) $G_1 \in \{K_4,\ \overline{C_6},\ \textit{Petersen graph}\}$,

(ii) $G_r \cong G$, *and*

(iii) for $2 \leqslant i \leqslant r$, the brick G_i has a thin edge e_i such that $G_{i-1} \cong \widehat{G_i - e_i}$ implying that G_i can be obtained from G_{i-1} by one of four types of elementary expansion operations. □

Just as there are families of braces (prisms, Möbius ladders and biwheels) which do not have strictly thin edges, there are families of bricks which do not have strictly thin edges. Norine and Thomas [13] discovered that, apart from the Petersen graph, there are five infinite families of such bricks. They include prisms and Möbius ladders whose orders are divisible by four, odd wheels, and two other families which Norine and Thomas refer to as prismoids and staircases. For convenience, let us denote by $\mathcal{NT}$ the class of bricks consisting of the Petersen graph and members of these five infinite families. In the same paper cited above, Norine Thomas proved that any simple brick which does not belong to $\mathcal{NT}$ has a strictly thin edge. This significant work was quite independent of our work, and used methods entirely different from ours. They state their result as a generation procedure for simple brick which is analogous to the procedure for generating simple braces due to McCuaig [11]. (The interpretation in terms of strictly thin edges is ours.)

Subsequently, we were able to show that their result can be deduced from our theorem on thin edges [5] and described this in [6]. We submitted this paper to a leading journal; it was rejected on the grounds that it, in their opinion, merely presents a new proof of a known result. We urge the interested reader to take a look at this unpublished article for an alternative perspective on the important works of McCuaig [11] on braces and Norine and Thomas [13] on bricks.

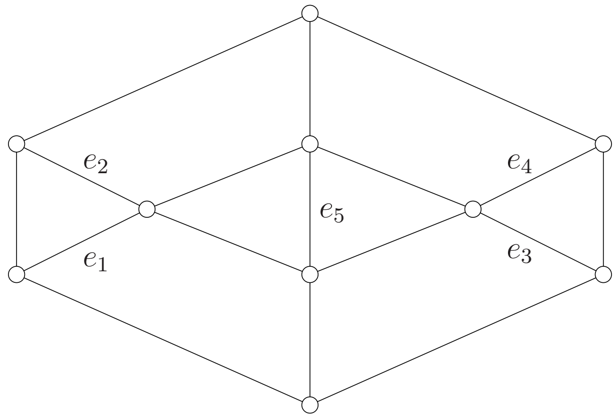


Figure 24: A brick with a unique strictly thin edge (e_5)

We do not know whether or not every brick (other than a few exceptions) has at least two thin edges. However, Nishad Kothari (a graduate student at the University of Waterloo) has found, by means of extensive computations, a number of bricks with just one strictly thin edge. One of the bricks he discovered is shown in Figure 24. This brick has five thin edges, e_i, $1 \leqslant i \leqslant 5$, of which only e_5 is strictly thin. Thus the natural analogue of Theorem 34 to bricks does not hold.

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