# A Linear Bound towards the Traceability Conjecture

Susan A. van Aardt*
Department of Mathematical Sciences
University of South Africa
Gauteng, South Africa
vaardsa@unisa.ac.za

Jean E. Dunbar
Department of Mathematics
Converse College
South Carolina, USA
jean.dunbar@converse.edu

Marietjie Frick †
Department of Mathematical Sciences
University of South Africa
Gauteng, South Africa
marietjie.frick@gmail.com

Nicolas Lichiardopol
Lycee A. de Capronne
Salon
France
nicolas.lichiardopol@neuf.fr

Submitted: Oct 4, 2014; Accepted: Nov 3, 2015; Published: Nov 13, 2015
Mathematics Subject Classifications: 05C20, 05C38

**Abstract**

A digraph is $k$-traceable if its order is at least $k$ and each of its subdigraphs of order $k$ is traceable. An oriented graph is a digraph without 2-cycles. The 2-traceable oriented graphs are exactly the nontrivial tournaments, so $k$-traceable oriented graphs may be regarded as generalized tournaments. It is well-known that all tournaments are traceable. We denote by $t(k)$ the smallest integer bigger than or equal to $k$ such that every $k$-traceable oriented graph of order at least $t(k)$ is traceable. The Traceability Conjecture states that $t(k) \leqslant 2k-1$ for every $k \geqslant 2$ [van Aardt, Dunbar, Frick, Nielsen and Oellermann, A traceability conjecture for oriented graphs, Electron. J. Combin., 15(1):#R150, 2008]. We show that for $k \geqslant 2$, every $k$-traceable oriented graph with independence number 2 and order at least $4k-12$ is traceable. This is the last open case in giving an upper bound for $t(k)$ that is linear in $k$.

**Keywords:** Oriented graph, Generalized tournament, $k$-traceable, Traceability Conjecture, Path Partition Conjecture

*Supported by the National Research Foundation of S.A, Grant 81075.
†Supported by the National Research Foundation of S.A, Grant 81004.

## 1 Introduction and Background

A digraph is *hamiltonian* if it contains a cycle that visits every vertex, *traceable* if it contains a path that visits every vertex.

A digraph is $k$-*traceable* if its order is at least $k$ and each of its subdigraphs of order $k$ is traceable. A digraph without 2-cycles is called an *oriented graph*. It is easily seen that an oriented graph is 2-traceable if and only if it is a nontrivial tournament. Thus $k$-traceable oriented graphs may be regarded as generalized tournaments. It is well-known that every nontrivial strong tournament is hamiltonian and every tournament is traceable. The following theorem, which follows from results in [3, 5, 12], shows that these properties are retained by $k$-traceable oriented graphs for small values of $k$.

**Theorem 1.1. [3, 5, 12]**

1. *For* $k = 2, 3, 4$, *every strong* $k$-*traceable oriented graph of order at least* $k + 1$ *is hamiltonian.*

2. *For* $k = 2, 3, 4, 5, 6$, *every* $k$-*traceable oriented graph is traceable.*

However, it is shown in [5] that for $k \geqslant 5$ there exists a nonhamiltonian strong $k$-traceable oriented graph of order $n$ for every $n \geqslant k$. Furthermore, it is shown in [7] that for $k = 7$ and for every $k \geqslant 9$ there exist $k$-traceable oriented graphs of order $k + 1$ that are nontraceable. (Such graphs are called *hypotraceable*). There also exist nontraceable $k$-traceable oriented graphs of order $k + 2$ for infinitely many $k$, as shown in [6]. These observations lead naturally to the following question, posed in [3].

**Question 1.** *For* $k \geqslant 2$, *what is the smallest integer* $t(k)$ *such that* $t(k) \geqslant k$ *and every* $k$-*traceable oriented graph of order at least* $t(k)$ *is traceable?*

The Traceability Conjecture (or TC for short), which is studied in [1, 3, 4, 5, 12] may be stated as follows.

**Conjecture 1. (TC)** $t(k) \leqslant 2k - 1$ *for every* $k \geqslant 2$.

As explained in [5], settling the TC could be an important step towards settling the Path Partition Conjecture for Digraphs. The latter conjecture was motivated by the paper [14] by Laborde, Payan and Xuong and is discussed in [2, 8, 9].

Theorem 1.1 and results in [1, 3, 10] imply the following.

**Theorem 1.2. [1, 3, 10]**
$t(k) = k$ *for* $2 \leqslant k \leqslant 6$
$t(7) = 9$
$t(8) \leqslant 14$
$t(k) \leqslant 2k^2 - 20k + 59$ *for every* $k \geqslant 9$.

The TC motivated us to search for an upper bound for $t(k)$ that is linear in $k$. Van Aardt, Dunbar, Frick and Nielsen [5] proved the following result with respect to oriented graphs with independence number greater than 2.

**Theorem 1.3.** [5] *If $k \geqslant 4$ and $D$ is a $k$-traceable oriented graph with $\alpha(D) \geqslant 3$ and $n(D) \geqslant 6k-20$, then $D$ is traceable.*

In this paper we show that for $k \geqslant 4$, every $k$-traceable oriented graph with independence number 2 and order at least $4k-12$ is traceable. This then proves that $t(k) \leqslant 6k-20$ for every $k \geqslant 4$ and thus brings us significantly closer to settling the TC.

## 2 Notation and Auxilliary Results

For undefined concepts we refer the reader to [8].

The set of vertices and the set of arcs of a digraph $D$ are denoted by $V(D)$ and $A(D)$, respectively, and the order of $D$ is denoted by $n(D)$. If $X \subset V(D)$, then $\langle X \rangle$ denotes the subdigraph induced by $X$ in $D$. The *independence number* of $D$, denoted by $\alpha(D)$, is the cardinality of a largest set $X \subset V(D)$ such that $\langle X \rangle$ has no arcs.

If $v \in V(D)$, we denote the sets of *out-neighbours* and *in-neighbours* of $v$ in $D$ by $N^+(v)$ and $N^-(v)$, respectively. The set $N(v) = N^+(v) \cup N^-(v)$ is simply called the *neighbourhood* of $v$. If $S$ is a subset of $V(D)$ or a subdigraph of $D$, we denote the set of neighbours, in-neighbours and out-neighbours of $v$ in $S$ by $N_S(v)$, $N_S^-(v)$ and $N_S^+(v)$, respectively.

A digraph $D$ is *strong* (or *strongly connected*) if for every pair of distinct vertices $u, v$ in $D$ there is a path from $u$ to $v$. A maximal strong subdigraph of a digraph $D$ is called a *strong component* of $D$. The strong components of $D$ have an acyclic ordering $D_1, D_2, \ldots, D_h$ such that if there is an arc from $D_i$ to $D_j$, then $i \leqslant j$. If $D$ is $k$-traceable for some $k \geqslant 2$, this acyclic ordering is unique since there is at least one arc from $D_i$ to $D_{i+1}$ for $i = 1, 2, \ldots, h-1$. Throughout this paper we label the strong components of a $k$-traceable digraph in accordance with this acyclic ordering. We denote by $D_r^s$ the subdigraph of $D$ induced by the vertex set $\bigcup_{i=r}^{s} V(D_i)$.

Chen and Manalastas [11] proved that every strong digraph with independence number two is traceable. Havet [13] strengthened their result by proving that if $D$ is a strong digraph with $\alpha(D) = 2$, then $D$ has two nonadjacent vertices that are terminal vertices of Hamilton paths in $D$ and two nonadjacent vertices that are initial vertices of Hamilton paths in $D$. The following theorem, which follows from Havet's result, is proved in [3].

**Theorem 2.1.** [3] *If $D$ is a connected digraph with $\alpha(D) = 2$ and at most two strong components, then $D$ is traceable.*

We shall frequently use the following result.

**Lemma 2.2.** [1] *Let $G$ be a $k$-traceable oriented graph of order $n$. Then the following hold.*

1. *$|N(x)| \geqslant n-k+1$ for every $x \in V(G)$.*
2. *$|N^-(x) \cup N^-(y)| \geqslant n-k+1$ and $|N^+(x) \cup N^+(y)| \geqslant n-k+1$ for every pair of nonadjacent vertices $x$ and $y$ in $G$.*

The following theorem follows from [1], Lemma 10 and Corollary 12.

**Theorem 2.3.** *Let $k \geqslant 7$ and suppose $D$ is a nontraceable $k$-traceable oriented graph of order $n \geqslant 2k-3$ with independence number 2. Let $D_1, \ldots, D_h$ be the strong components of $D$. Then $h \geqslant 3$ and there exists a $t \in \{2, \ldots, h-1\}$ such that $D_t$ is nonhamiltonian, while $D_1^{t-1}$ as well as $D_{t+1}^h$ are tournaments. Moreover, $n(D_t) \geqslant n-k+5$.*

Next we state a lemma for the particular case $h = 3$, which is used in our main theorem. It follows from results in [1, 3, 5], but for ease of reference we provide a proof.

**Lemma 2.4.** *Let $k \geqslant 7$ and suppose $D$ is a nontraceable $k$-traceable oriented graph of order $n \geqslant 2k-3$ with independence number 2 and exactly three strong components $D_1, D_2, D_3$. Let $n(D_i) = n_i$, $i = 1, 2, 3$. Then the following hold.*

1. *If $P$ is a Hamilton path in $D_2$ whose initial vertex has an in-neighbour in $D_1$, then the terminal vertex of $P$ does not have an out-neighbour in $D_3$.*
2. *$D_2$ is $(k - n_1 - n_3)$-traceable.*
3. *$|N_{D_2}(x)| \geqslant n-k+1$ for every $x \in V(D_2)$.*
4. *If $x$ and $y$ are two nonadjacent vertices in $D_2$, then*
   *(a) $|N^+_{D_2}(x) \cup N^+_{D_2}(y)| \geqslant n-k+1$,*
   *(b) $|N^-_{D_2}(x) \cup N^-_{D_2}(y)| \geqslant n-k+1$.*
5. *(a) $|N^+_{D_2}(D_1)| \geqslant n-k+1$,*
   *(b) $|N^-_{D_2}(D_3)| \geqslant n-k+1$.*
6. *(a) If $x \in V(D_2)$ and $x \notin N^+(D_1)$, then $|N^-_{D_2}(x)| \geqslant n-k+1$,*
   *(b) If $x \in V(D_2)$ and $x \notin N^-(D_3)$, then $|N^+_{D_2}(x)| \geqslant n-k+1$.*

*Proof.*

1. Suppose the initial vertex of $P$ has an in-neighbour $y$ in $D_1$ and the terminal vertex of $P$ has an out-neighbour $z$ in $D_3$. By Theorem 2.3, each of $D_1$ and $D_3$ is a strong tournament and hence is either hamiltonian or a single vertex. Thus $D_1$ has a path $Q$ with $y$ as terminal vertex, and $D_3$ has a path $R$ with $z$ as initial vertex. But then the path $QPR$ is a Hamilton path of $D$, contradicting our assumption that $D$ is nontraceable.

2. From Theorem 2.3 and our assumption that $n \geqslant 2k-3$ it follows that $0 < k - n_1 - n_3 < n_2$. Now consider any subdigraph $H$ of $D_2$ with $n(H) = k - n_1 - n_3$. Let $H^* = \langle V(H) \cup V(D_1) \cup V(D_3) \rangle$. Then $n(H^*) = k$, so $H^*$ is traceable since $D$ is $k$-traceable. Let $P = v_1 \ldots v_k$ be a Hamilton path of $H^*$. Then, due to the acyclic ordering of the strong components, the intersection of the path $P$ with the strong component $D_2$ is a Hamilton path of $H$. This proves that $D_2$ is $(k - n_1 - n_3)$-traceable.

3. It follows from (2) above and Lemma 2.2(1) that $|N_{D_2}(x)| \geqslant n_2 - (k - n_1 - n_3) + 1 = n - k + 1$.

4. This follows directly from (2) and Lemma 2.2(2).

5. If $|N^+_{D_2}(D_1)| \leqslant n - k$, then $|V(D_2) - N^+_{D_2}(D_1)| \geqslant n_2 - (n - k) = k - n_1 - n_3$, so we can choose a set $S \subseteq (V(D_2) - N^+_{D_2}(D_1))$ such that $|S| = k - n_1 - n_3$. Then the subdigraph $\langle V(D_1) \cup S \cup V(D_3) \rangle$ has order $k$ but is nontraceable, contradicting that $D$ is $k$-traceable. This proves 5(a). The proof of 5(b) is similar.

6. If $|N^-_{D_2}(x)| \leqslant n - k$, then we choose a subset $S$ with $|S| = k - n_1 - n_3$ such that $x \in S \subseteq (V(D_2) - N^-_{D_2}(x))$. But then the subdigraph $\langle V(D_1) \cup S \cup V(D_3) \rangle$ has order $k$ but is nontraceable, since there are no arcs from $D_1$ to $S$. This proves 6(a). The proof of 6(b) is similar.

□

# 3 Main Result

**Theorem 3.1.** *Let $k \geqslant 2$ and suppose $D$ is a $k$-traceable oriented graph such that $\alpha(D) = 2$ and $n(D) \geqslant 4k - 12$. Then $D$ is traceable.*

*Proof.* The proof is by induction on $k$. By Theorem 1.2, the result holds for $k \leqslant 8$. Now let $k \geqslant 9$ and let $D$ be a $k$-traceable oriented graph with independence number 2 and order $n \geqslant 4k - 12$. Suppose $D$ is nontraceable and let $D_1, \dots, D_h$ be the strong components of $D$, with $n(D_i) = n_i$, $i = 1, \dots, h$. Then, by Theorem 2.3, $h \geqslant 3$ and $D$ has a nonhamiltonian strong component $D_t$ of order at least $n - k + 5$ such that $2 \leqslant t \leqslant h - 1$. In particular, $n_i < k - 5$ for $i \neq t$. Moreover, $D_1^{t-1}$ and $D_{t+1}^h$ are tournaments.

Now $D_2^h$ is a $(k - n_1)$-traceable oriented graph with independence number 2 and $n(D_2^h) \geqslant 4k - 12 - n_1 > 4(k - n_1) - 12$. Hence it follows from our induction hypothesis that $D_2^h$ is traceable and thus has a Hamilton path with initial vertex $x$ in $D_2$.

Now suppose $h \geqslant 4$. Then if $t \geqslant 3$, Theorem 2.3 implies that $\langle D_1^2 \rangle$ is a tournament. Since $D_1$ is hamiltonian or a single vertex and every vertex in $D_1$ is adjacent to $x$, it follows that $D$ is traceable. If $t < 3$, we consider $D_1^{h-1}$ instead of $D_2^h$ and deduce in a similar manner that $D$ is traceable. We may therefore assume that $h = 3$. Thus $D_1$ and $D_3$ are tournaments, while $D_2$ is nonhamiltonian and $n(D_2) \geqslant n - k + 5$.

By Theorem 2.1, $D_1^2$ is traceable, so $D_2$ has a Hamilton path $x_1 \dots x_{n_2}$ such that $x_1 \in N^+(D_1)$. By Lemma 2.4(1), $x_{n_2} \notin N^-(D_3)$, so it follows from Lemma 2.4(6b) that $d^+_{D_2}(x_{n_2}) \geqslant n - k + 1 \geqslant 3k - 11$, since $n \geqslant 4k - 12$. Let $x_j$ be the out-neighbour of $x_{n_2}$ such that $x_{n_2}$ has exactly $k - 3$ out-neighbours in $\{x_1, \dots, x_j\}$. Then $x_{n_2}$ has at least $n - 2k + 4$ out-neighbours in $\{x_{j+1}, \dots, x_{n_2}\}$. Hence $n_2 - 2 - j \geqslant n - 2k + 4$. Since $n_2 \leqslant n - 2$, it follows that $j \leqslant 2k - 8$.

**Claim 1.** $x_{j-1} \in N^-(D_3)$.

*Proof.* If $x_s \in N^+(x_{j-1})$ for some $s \geqslant j+1$, then $x_1 \dots x_{j-1} x_s \dots x_{n_2} x_j \dots x_{s-1}$ is a Hamilton path of $D_2$ and hence, by Lemma 2.4(1), $x_{s-1}$ has no out-neighbour in $D_3$. But, by Lemma 2.4(5b), the number of vertices in $D_2$ that have no out-neighbours in $D_3$ is at most $n_2-(n-k+1) \leqslant k-3$, since $n_2 \leqslant n-2$. Hence $|N^+(x_{j-1}) \cap \{x_{j+1}, \dots, x_{n_2}\}| \leqslant k-3$ and hence $N^+_{D_2}(x_{j-1}) \leqslant j-1+k-3 \leqslant 3k-12 \leqslant n-k$, since $j \leqslant 2k-8$ and $n \geqslant 4k-12$. Thus $x_{j-1} \in N^-(D_3)$ by Lemma 2.4(6b).

**Claim 2.** *If $x_i \in N^-(x_1)$, then $i < j$.*

*Proof.* Suppose, to the contrary that $i \geqslant j$. Since $D_2$ is nonhamiltonian, $i \neq n_2$. If $x_s \in N^+(x_{n_2})$, with $s \leqslant j$, then $x_{s-1} \notin N^-(x_{i+1})$, since otherwise $x_{i+1} \dots x_{n_2} x_s \dots x_i x_1 \dots x_{s-1} x_{i+1}$ is a Hamilton cycle of $D_2$. But $x_{n_2}$ has $k-3$ out-neighbours in $\{x_2, \dots, x_j\}$ (by our choice of $j$), so at least $k-3$ vertices in $\{x_1, \dots, x_{j-1}\}$ are not in $N^-(x_{i+1})$. Hence $|N^-_{D_2}(x_{i+1})| \leqslant n_2-1-(k-3) \leqslant n-k$. Hence, by Lemma 2.4(6a), $x_{i+1} \in N^+(D_1)$. But $x_{i+1} \dots x_{n_2} x_j \dots x_i x_1 \dots x_{j-1}$ is a Hamilton path of $D_2$ and, by Claim 1, $x_{j-1} \in N^-(D_3)$. This contradicts Lemma 2.4(1) and thus proves the claim.

**Claim 3.** $|N^+(x_1) \cap \{x_{j+1}, \dots, x_{n_2}\}| \geqslant n-3k+10$.

*Proof.* By Lemma 2.4(3), $x_1$ has at least $n-k+1$ neighbours in $D_2$. But $x_1$ has at most $j-1$ neighbours in $\{x_2, \dots, x_j\}$ and, by Claim 2, $x_1$ has no in-neighbours in $\{x_{j+1}, \dots, x_{n_2}\}$. Hence, $|N^+(x_1) \cap \{x_{j+1}, \dots, x_{n_2}\}| \geqslant n-k+1-(j-1) \geqslant n-3k+10$, since $j \leqslant 2k-8$.

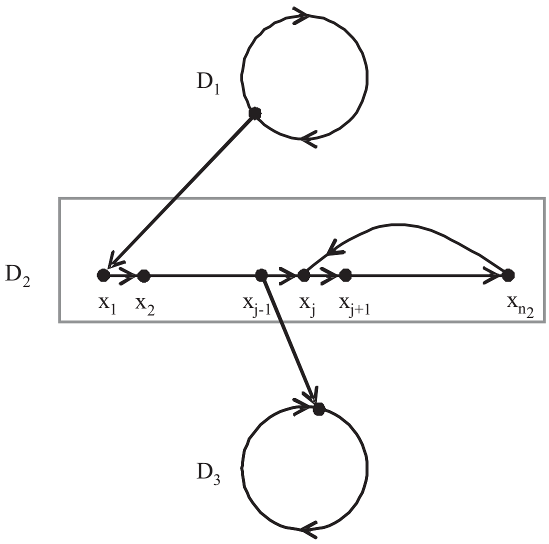


Figure 1: Structure of $D$

**Claim 4.** $x_2 \in N^+(D_1)$.

*Proof.* If $x_i \in N^+(x_1)$ with $i \geqslant j+1$, then $x_{i-1} \notin N^-(x_2)$, since otherwise $x_1 x_i \dots x_{n_2} x_j \dots x_{i-1} x_2 \dots x_{j-1}$ is a Hamilton path of $D_2$ with initial vertex in $N^+(D_1)$ and terminal vertex in $N^-(D_3)$ (by Claim 1), contradicting Lemma 2.4(1). Thus it follows from

Claim 3 that $|N^-_{D_2}(x_2)| \leqslant n_2 - 1 - (n-3k+10) \leqslant 3k-13 < n-k$ since $n \geqslant 4k-12$. Hence, by Lemma 2.4(6a), $x_2 \in N^+(D_1)$.

**Claim 5.** *$x_{j-1} \notin N^-(x_1)$.*

*Proof.* Since $n \geqslant 4k-12$, Claim 3 implies that $x_1$ has at least $k-2$ out-neighbours in $\{x_{j+1}, \dots, x_{n_2}\}$. But Lemma 2.4(5b) implies that the number of vertices in $D_2$ that are not in $N^-(D_3)$ is at most $n_2 - (n-k+1) \leqslant k-3$. Hence there is an out-neighbour $x_s$ of $x_1$, with $x_s \in \{x_{j+1}, \dots, x_{n_2}\}$, such that $x_{s-1} \in N^-(D_3)$. Now suppose $x_{j-1} \in N^-(x_1)$. Then $x_2 \dots x_{j-1}x_1x_s \dots x_{n_2}x_j \dots x_{s-1}$ is a Hamilton path of $D_2$. But $x_2 \in N^+(D_1)$ by Claim 4, so this contradicts Lemma 2.4(1).

**Claim 6.** *Let $r$ be the largest integer such that $x_r \in N^-(x_1)$. Then $x_{r+1} \in N^+(x_1)$.*

*Proof.* By Claims 2 and 5, $r \leqslant j-2$. If $x_s \in N^+_{D_2}(x_1) \cap \{x_{j+1}, \dots, x_{n_2}\}$, then $x_{s-1} \notin N^-(x_{r+1})$, since otherwise $x_2 \dots x_r x_1 x_s \dots x_{n_2} x_j \dots x_{s-1} x_{r+1} \dots x_{j-1}$ is a Hamilton path of $D_2$ with initial vertex in $N^+(D_1)$ and terminal vertex in $N^-(D_3)$. Hence, by Claim 3, at least $n-3k+10$ vertices in $\{x_j, \dots, x_{n_2-1}\}$ are not in $N^-(x_{r+1})$. By Claim 2, those vertices are also not in $N^-(x_1)$. Hence $|N^-_{D_2}(x_1) \cup N^-_{D_2}(x_{r+1})| \leqslant n_2 - (n-3k+10) \leqslant 3k-12 \leqslant n-k$. Hence, by Lemma 2.4(4b), $x_1$ and $x_{r+1}$ are neighbours. But $x_{r+1} \notin N^-(x_1)$ by our assumption on $r$, so Claim 6 is proved.

Now, let $\mathcal{P}$ consist of all Hamilton paths in $D_2$ whose initial vertices are in $N^+(D_1)$. Among all paths in $\mathcal{P}$, choose one that has the largest possible number of vertices between the initial vertex and its last in-neighbour. Denote this path by $Q_1 = x_1 \dots x_{n_2}$ and let $x_r$ be the last in-neighbour of $x_1$ on $Q_1$. As $D_2$ is nonhamiltonian we have $r < n_2$. Let $C$ be the cycle $x_1 \dots x_r x_1$. Then $x_1$ has no in-neighbour in $D_2 - V(C)$. By Claim 6, $x_1 x_{r+1} \in A(D_2)$ and by Claim 4, $x_2 \in N^+(D_1)$. Hence $Q_2 = x_2 \dots x_r x_1 x_{r+1} \dots x_{n_2}$ is also a path in $\mathcal{P}$. Note that $x_1$ is the last in-neighbour of $x_2$ on $Q_2$, by the maximality of $r$. Thus $x_2$ has no in-neighbour in $D_2 - V(C)$. Repeated applications of this procedure show that no vertex on $C$ has an in-neighbour in $D_2 - V(C)$. This contradicts the fact that $D_2$ is strong and thus proves the theorem. □

By combining Theorems 1.3 and 3.1, we conclude the following.

**Corollary 3.2.** *$t(k) \leqslant 6k-20$ for every $k \geqslant 4$.*

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