

An improved bound on the sizes of matchings guaranteeing a rainbow matching

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Abstract

A conjecture by Aharoni and Berger states that every family of n matchings of size $n + 1$ in a bipartite multigraph contains a rainbow matching of size n . In this paper we prove that matching sizes of $(\frac{3}{2} + o(1))n$ suffice to guarantee such a rainbow matching, which is asymptotically the same bound as the best-known one in the case where we only aim to find a rainbow matching of size $n - 1$. This improves previous results by Aharoni, Charbit and Howard, and Kotlar and Ziv.

1 Introduction

In this paper we are concerned with the question which sizes of n matchings in a bipartite multigraph suffice in order to guarantee a rainbow matching of size n .

One motivation for considering these kinds of problems is due to some well-known conjectures on Latin squares. A *Latin square* of order n is an $n \times n$ matrix in which each symbol appears exactly once in every row and exactly once in every column. A *partial transversal* in a Latin square is a set of entries with distinct symbols such that from each row and each column at most one entry is contained in this set. We call a partial transversal of size n in a Latin square of order n simply a *transversal*. A famous conjecture of Ryser [11] states that for every odd integer n any Latin square of order n contains a transversal. The conjecture is known to be true for $n \leq 9$. Omitting the restriction to odd numbers yields a false statement. Brualdi [7, 8] and Stein [12] independently formulated the following conjecture for all orders n .

Conjecture 1. For every $n \geq 1$ any Latin square of order n has a partial transversal of size $n - 1$.

A natural way to transfer this problem to graphs is the following. Let $L = (\ell_{i,j})_{i,j \in [n]}$ be a Latin square of order n . We define $G_L := (A \cup B, E)$ as the complete bipartite edge-coloured graph with partite sets $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$, where $a_i b_j$ is coloured $\ell_{i,j}$. That is, A and B represent the columns and rows of L , respectively. Moreover, a transversal of L corresponds to a perfect matching in G_L that uses each edge colour exactly once, which we call a *rainbow matching* of size n . Using this notion, Conjecture 1 is equivalent to the following: For every $n \geq 1$ any complete bipartite edge-coloured graph, the colour classes of which are perfect matchings, contains a rainbow matching of size $n - 1$.

One may wonder whether this might even be true in the more general setting of bipartite edge-coloured multigraphs: Following Aharoni, Charbit and Howard [2], we define $f(n)$ to be the smallest integer m such that every bipartite edge-coloured multigraph with exactly n colour classes, each being a matching of size at least m , contains a rainbow matching of size n . Aharoni and Berger [1] conjectured the following generalization of Conjecture 1.

Conjecture 2. For every $n \geq 1$ we have $f(n) = n + 1$.

The first approaches towards this conjecture are given by the bounds $f(n) \leq \lfloor \frac{7}{4}n \rfloor$ due to Aharoni, Charbit and Howard [2] and $f(n) \leq \lfloor \frac{5}{3}n \rfloor$ due to Kotlar and Ziv [10]. Here, we give an improved bound, which is asymptotically the same as the best-known bound on the sizes of the colour classes in the case where we aim to find a rainbow matchings of size $n - 1$ [10]. In particular, we prove the following.

Theorem 3. For every $\varepsilon > 0$ there exists an integer $n_0 \geq 1$ such that for every $n \geq n_0$ we have $f(n) \leq (\frac{3}{2} + \varepsilon)n$.

Subsequently, we use the following notation. Let G be a bipartite multigraph with partite sets A and B . For sets $X \subseteq A$ and $Y \subseteq B$ we write $E_G[X, Y]$ to denote the set of edges in G between X and Y . Given some matching R in G , $N_G(X|R) := \{y \in B : \exists xy \in R \text{ with } x \in X\}$ denotes the neighbourhood of X with respect to R . For the sake of readability, we omit floor and ceiling signs and do not intend to optimize constants in the proofs.

2 Proof of Theorem 3

In this section we give a proof of Theorem 3 the idea of which can be summarized as follows. We start with assuming for a contradiction that a maximum rainbow matching in the given graph $G = (A \cup B, E)$ is of size $n - 1$. A rainbow matching of this size is known to exist [10]. We fix such a matching R and find two sequences e_1, \dots, e_k and g_1, \dots, g_k of edges, the first consisting of edges from R and the second consisting of edges outside R . We then show that either we can switch some of the edges between the edge sequences to produce a rainbow matching of size n (see the proofs of the Claims 4, 5 and 7), or the matchings represented by the edges e_1, \dots, e_k need to touch at least n vertices in B

that are saturated by R , both leading to a contradiction. To make the second case more precise we additionally introduce certain sequences $X_1, \dots, X_k \subseteq A$ and $Y_1, \dots, Y_k \subseteq B$ in the proof.

Proof. Let $\varepsilon > 0$ be given and whenever necessary we may assume that n is large enough. Let $\mathcal{F} = \{F_0, F_1, \dots, F_{n-1}\}$ be a family of n matchings of size at least $(3/2 + \varepsilon)n$ in a bipartite multigraph $G = (A \cup B, E)$ with partite sets A and B . We aim to find a rainbow matching of size n .

For a contradiction, let us assume that there is no such matching. As shown in [10], there must exist a rainbow matching R of size $n - 1$. We may assume without loss of generality that none of the edges of F_0 appears in R . Let t be the smallest positive integer with $1/(2t - 1) \leq \varepsilon$. Moreover, let $X \subseteq A$ and $Y \subseteq B$ be the sets of vertices that are saturated by R , i.e. incident with some edge of R .

In the following we show that for every $k \in [t]$ we can construct sequences

(S1) e_1, \dots, e_k of k distinct edges $e_i = x_i y_i$ in R with $x_i \in X$ and $y_i \in Y$,

(S2) g_1, \dots, g_k of k distinct edges $g_i = z_i y_i$ with $z_i \in A \setminus X$,

(S3) X_1, \dots, X_k of subsets of X ,

(S4) $Y_1 = N_G(X_1|R), \dots, Y_k = N_G(X_k|R)$ of subsets of Y ,

and an injective function $\pi : \{0, 1, \dots, k\} \rightarrow \{0, 1, \dots, n - 1\}$ with $\pi(0) := 0$ such that the following properties hold:

(P1) for each $i \in [k]$ we have $e_i \in F_{\pi(i)}$,

(P2) for each $i \in [k]$ we have $g_i \in \bigcup_{j=0}^{i-1} F_{\pi(j)}$,

(P3) $(e_1 \cup \dots \cup e_k) \cap (X_k \cup Y_k) = \emptyset$,

(P4) $|X_k| = |Y_k| = s_k := 2k\varepsilon n + k(7 - 3k)/2$,

(P5) for each $i \in [k]$ and each $j \in \{0, \dots, n - 1\}$ it holds that if R contains an edge of the matching F_j between X_i and Y_i , then there is also an edge of F_j between x_i and $B \setminus Y$,

(P6) for each $i \in [k]$ and each $w \in Y_i \setminus Y_{i-1}$ there exists a vertex $v \in A \setminus (X \cup \{z_1, \dots, z_{i-1}\})$ such that $vw \in F_{\pi(i-1)}$ (where $Y_0 := \emptyset$), and

(P7) for each $i \in [k]$ and each $j \in [i - 1]$ it holds that if $g_i \in F_{\pi(j)}$, then $z_i \in A \setminus (X \cup \{z_1, \dots, z_j\})$.

Let us explain some of the ideas behind Properties (P1)-(P7). Let $i \in [k]$, then Properties (P1), (P2) and (P7) ensure subsequences of the sequences in (S1) and (S2) such that switching between them produces another rainbow matching of size $n - 1$ where the matching $F_{\pi(i)}$ is not present. Properties (P5) and (P6) guarantee a large number of

switchings that produce different rainbow matchings of size $n - 1$. This will help us to prove restrictions on where the edges of $F_{\pi(i)}$ can occur in the multigraph. In particular we show that $F_{\pi(i)}$ cannot have edges between some parts of the graph and the set Y_i from (S4), the size of which increases with i .

Before we start with the construction, let us first observe that by Property (P4) we have a set $Y_t \subseteq Y$ which satisfies $2t\epsilon n + t(7 - 3t)/2 = |Y_t| \leq |Y| < n$. However, for large enough n and by the choice of t we have that $2t\epsilon n + t(7 - 3t)/2 > n$, a contradiction.

In order to find the sequences described above, we proceed by induction on k . For the base case, let us argue why we find edges e_1, g_1 , sets X_1, Y_1 , and an injective function π with Properties (P1)-(P7). First observe that F_0 does not have any edges between $A \setminus X$ and $B \setminus Y$, by assumption on R . As $|F_0| \geq (3/2 + \epsilon)n$, there are at least $(1/2 + \epsilon)n + 1$ edges of F_0 between $A \setminus X$ and Y . Let $N_0 \subseteq Y$ denote a set of size $(1/2 + \epsilon)n + 1$ such that for every vertex $w \in N_0$ there exists a vertex $v \in A \setminus X$ such that $vw \in F_0$. Furthermore, let $X'_1 := N_G(N_0|R)$ and let $\mathcal{R}_1 := \{F_j \in \mathcal{F} : F_j \cap R \cap E_G[X'_1, N_0] \neq \emptyset\}$, i.e. \mathcal{R}_1 consists of all matchings that have an edge in R between X'_1 and N_0 .

As a next step we show that there exists an edge $e_1 = x_1y_1 \in R \cap E_G[X'_1, N_0]$ such that there are $2\epsilon n + 2$ matchings in \mathcal{R}_1 that have an edge between x_1 and $B \setminus Y$. The sets $X_1 \subseteq X$ and $Y_1 = N_G(X_1|R)$ will then be defined as the vertices that are incident with edges in R that are contained in these $2\epsilon n + 2$ matchings.

Let F be any matching in \mathcal{R}_1 , let vw be the unique edge in $F \cap R \cap E_G[X'_1, N_0]$ and let $z \in A \setminus X$ be the unique vertex such that $zw \in F_0$. Notice that there cannot be any edge g of F between $A \setminus (X \cup \{z\})$ and $B \setminus Y$, since otherwise $(R \setminus \{vw\}) \cup \{zw, g\}$ would give a rainbow matching of size n , in contradiction with R being a maximum rainbow matching. Therefore, there are at least $(1/2 + \epsilon)n + 1$ edges of F between $B \setminus Y$ and $X \cup \{z\}$. Since $|X'_1| = (1/2 + \epsilon)n + 1$, there are at least $2\epsilon n + 2$ edges of F between $B \setminus Y$ and X'_1 . Since this is true for any $F \in \mathcal{R}_1$, we know by the pigeonhole principle that there is a vertex $x_1 \in X'_1$ and a subset $X_1 \subseteq X'_1$ of size $2\epsilon n + 2$ such that, for every $F_j \in \mathcal{F}$, if $F_j \cap R \cap E_G[X_1, B] \neq \emptyset$, then F_j has an edge between x_1 and $B \setminus Y$. Note that $x_1 \notin X_1$. Let $e_1 = x_1y_1$ be the unique edge in R incident with x_1 and let $g_1 = z_1y_1$ be the unique edge of F_0 incident with $y_1 \in N_0$. Set $\pi(1)$ to be the unique index $j \in [k]$ such that $e_1 \in F_j$. One can easily verify that $e_1 = x_1y_1, g_1 = z_1y_1, X_1, Y_1 := N_G(X_1|R)$, and π satisfy Properties (P1)-(P7).

For the inductive step let us assume that for some $k \in [t - 1]$ the above sequences are given with Properties (P1)-(P7). We now aim to extend these by edges e_{k+1}, g_{k+1} , sets X_{k+1}, Y_{k+1} , and a value $\pi(k + 1)$ while maintaining Properties (P1)-(P7). The approach that we use is similar to the base case. We start with some useful claims that will guarantee us in Corollary 6 that the matching $F_{\pi(k)}$ has $(1/2 + \epsilon)n + 1 - 2k$ edges between $A \setminus (X \cup \{z_1, \dots, z_k\})$ and $Y \setminus (Y_k \cup \{y_1, \dots, y_k\})$.

Claim 4. $F_{\pi(k)}$ has no edge between $A \setminus (X \cup \{z_1, \dots, z_k\})$ and $B \setminus Y$.

Proof of Claim 4. Assume for a contradiction that there exists an edge $g \in F_{\pi(k)}$ between the sets $A \setminus (X \cup \{z_1, \dots, z_k\})$ and $B \setminus Y$. (See Figure 1 for an illustration.) By

Property (P2) we find a sequence $k > j_1 > j_2 > \cdots > j_s = 0$ with $1 \leq s \leq k$ such that

$$\begin{aligned} g_k &\in F_{\pi(j_1)}, \\ g_{j_i} &\in F_{\pi(j_{i+1})} \text{ for } i < s. \end{aligned}$$

Moreover, according to Property (P7) we know that $z_k, z_{j_1}, \dots, z_{j_{s-1}}$ are distinct, and thus, also using Property (P1), we conclude that

$$(R \setminus \{e_k, e_{j_1}, \dots, e_{j_{s-1}}\}) \cup \{g_k, g_{j_1}, \dots, g_{j_{s-1}}, g\}$$

forms a rainbow matching which is larger than R , a contradiction. \square

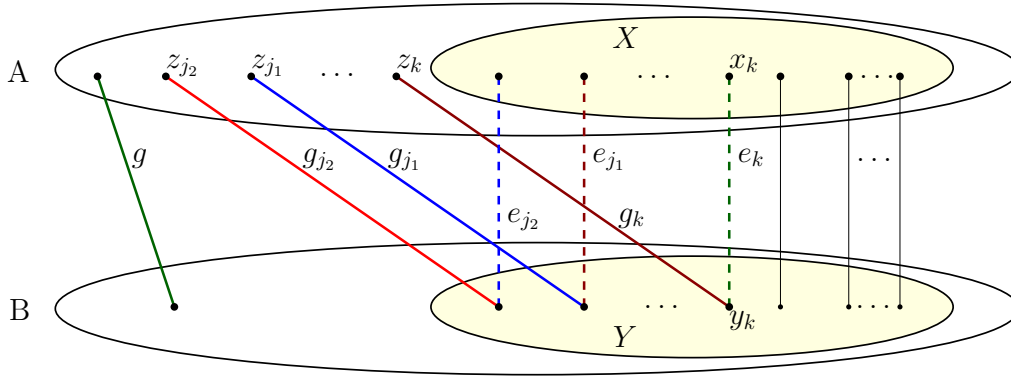


Figure 1: *Example with $g_{j_2} \in F_{\pi(0)}$ ($s = 3$). The dotted edges $\{e_k, e_{j_1}, e_{j_2}\}$ are replaced by the edges $\{g_k, g_{j_1}, g_{j_2}, g\}$ to obtain a larger rainbow matching.*

Claim 5. $F_{\pi(k)}$ has no edge between $A \setminus (X \cup \{z_1, \dots, z_k\})$ and Y_k .

Proof of Claim 5. Assume for a contradiction that there is an edge $g \in F_{\pi(k)}$ between the sets $A \setminus (X \cup \{z_1, \dots, z_k\})$ and Y_k . (See Figure 2 for an illustration.) Let e be the unique edge in R which is adjacent to g . Observe that e lies between X_k and Y_k by assumption. Let $j \in [n-1]$ be such that $e \in F_j$. By Property (P3) we have $e \notin \{e_1, \dots, e_k\}$. Thus, using Property (P1) and the fact that R is a rainbow matching, we can conclude that $j \notin \{\pi(i) : 1 \leq i \leq k\}$. Now, by Property (P5) it holds that there is an edge $\bar{e} \in F_j$ between x_k and $B \setminus Y$. Moreover, by Properties (P2) and (P7), we find a sequence $k > j_1 > j_2 > \cdots > j_s = 0$ with $1 \leq s \leq k$ such that

$$\begin{aligned} g_k &\in F_{\pi(j_1)}, \\ g_{j_i} &\in F_{\pi(j_{i+1})} \text{ for } i < s \end{aligned}$$

and all vertices $z_k, z_{j_1}, \dots, z_{j_{s-1}}$ are distinct. Therefore, using Property (P1), we conclude that

$$(R \setminus \{e_k, e_{j_1}, \dots, e_{j_{s-1}}, e\}) \cup \{g_k, g_{j_1}, \dots, g_{j_{s-1}}, \bar{e}, g\}$$

forms a rainbow matching which is larger than R , a contradiction. \square

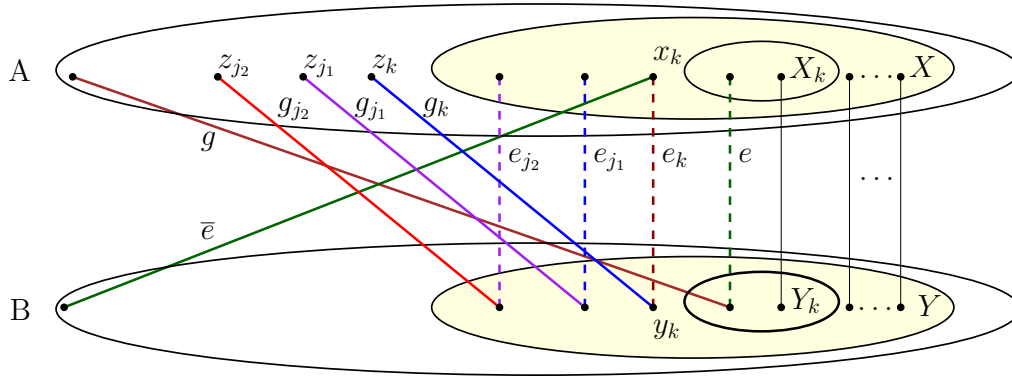


Figure 2: Example with $g_{j_2} \in F_{\pi(0)}$ ($s = 3$). The dotted edges $\{e_k, e_{j_1}, e_{j_2}, e\}$ are replaced by the edges $\{g_k, g_{j_1}, g_{j_2}, \bar{e}, g\}$ to obtain a larger rainbow matching.

Corollary 6. The matching $F_{\pi(k)}$ has at least $(\frac{1}{2} + \varepsilon)n + 1 - 2k$ edges between $A \setminus (X \cup \{z_1, \dots, z_k\})$ and $Y \setminus (Y_k \cup \{y_1, \dots, y_k\})$.

Proof of Corollary 6. As $|F_{\pi(k)}| \geq (3/2 + \varepsilon)n$ and $|X \cup \{z_1, \dots, z_k\}| \leq n - 1 + k$, we conclude that at least $(1/2 + \varepsilon)n + 1 - k$ edges of $F_{\pi(k)}$ are incident with vertices in $A \setminus (X \cup \{z_1, \dots, z_k\})$. Each of these edges intersects $Y \setminus Y_k$ by the previous claims and thus the statement follows. \square

In the following, let $N_k \subseteq Y \setminus (Y_k \cup \{y_1, \dots, y_k\})$ be a set of size $(1/2 + \varepsilon)n + 1 - 2k$ such that for each vertex $w \in N_k$ there is a vertex $v \in A \setminus (X \cup \{z_1, \dots, z_k\})$ with $vw \in F_{\pi(k)}$. Such a set exists by the previous corollary. Moreover, let

$$Y'_{k+1} := Y_k \cup N_k$$

and let $X'_{k+1} := N_G(Y'_{k+1} | R)$ be the neighbourhood of Y'_{k+1} with respect to R . By Property (P4), and as $N_k \cap Y_k = \emptyset$, we obtain

$$\begin{aligned} |X'_{k+1}| &= |Y'_{k+1}| = 2k\varepsilon n + \frac{k(7-3k)}{2} + \left(\frac{1}{2} + \varepsilon\right)n + 1 - 2k \\ &= \frac{1}{2}n + (2k+1)\varepsilon n + \frac{-3k^2 + 3k + 2}{2}. \end{aligned} \quad (*)$$

We now look at all matchings that have an edge in R between X'_{k+1} and Y'_{k+1} . Formally, we consider

$$\mathcal{R}_{k+1} := \{F_j \in \mathcal{F} : F_j \cap R \cap E_G[X'_{k+1}, Y'_{k+1}] \neq \emptyset\}.$$

Similarly to the base case we now aim to show that there exists an edge $e_{k+1} = x_{k+1}y_{k+1} \in R \cap E_G[X'_{k+1}, Y'_{k+1}]$ such that there are s_{k+1} matchings in \mathcal{R}_{k+1} that have an edge between x_{k+1} and $B \setminus Y$. This will follow immediately from the following claim and is stated in Corollary 8.

Claim 7. Every $F_j \in \mathcal{R}_{k+1}$ has at least s_{k+1} edges between X'_{k+1} and $B \setminus Y$.

Proof of Claim 7. The main argument is similar to that of Claim 4 - Corollary 6. For $F_j \in \mathcal{R}_{k+1}$ let $f = vw$, with $v \in X'_{k+1}$, $w \in Y'_{k+1}$, denote the unique edge in $F_j \cap R \cap E_G[X'_{k+1}, Y'_{k+1}]$. Since $Y'_{k+1} := Y_k \cup N_k$, we either have $w \in Y_k$ or $w \in N_k$. In particular, by Property (P3) from the hypothesis and by the definition of N_k , we know that $w \notin \{y_1, \dots, y_k\}$, and therefore $j \notin \{\pi(i) : 0 \leq i \leq k\}$.

If $w \in Y_k$, then we find an integer $j_1 \in [k]$ such that $w \in Y_{j_1} \setminus Y_{j_1-1}$ (where $Y_0 := \emptyset$), and by Property (P6) there is a vertex $z \in A \setminus (X \cup \{z_1, \dots, z_{j_1-1}\})$ such that $zw \in F_{\pi(j_1-1)}$. If otherwise $w \in N_k$, then we find a vertex $z \in A \setminus (X \cup \{z_1, \dots, z_k\})$ such that $zw \in F_{\pi(k)}$, by construction of N_k . In either case, let us fix this particular vertex z . We now prove the claim by showing first that (i) F_j has no edge between $A \setminus (X \cup \{z_1, \dots, z_k, z\})$ and $B \setminus Y$, and then we conclude that (ii) the statement holds for F_j .

We start with the discussion of (i). So, assume for a contradiction that F_j has an edge \bar{f} between $A \setminus (X \cup \{z_1, \dots, z_k, z\})$ and $B \setminus Y$.

If $w \in Y_k$, then by the definition of z we have $zw \in F_{\pi(j_1-1)}$, with j_1 being defined above. We can assume that $j_1 > 1$, as otherwise $zw \in F_0$ and thus $(R \setminus \{f\}) \cup \{\bar{f}, zw\}$ forms a full rainbow matching, in contradiction to our main assumption. Now, using Property (P2), we find a sequence $j_1 - 1 > j_2 > \dots > j_s = 0$ with $2 \leq s < k$ such that

$$\begin{aligned} g_{j_1-1} &\in F_{\pi(j_2)}, \\ g_{j_i} &\in F_{\pi(j_{i+1})} \quad \text{for } 2 \leq i \leq s-1 \end{aligned}$$

By Property (P7) and since $z \in A \setminus (X \cup \{z_1, \dots, z_{j_1-1}\})$, the vertices $z, z_{j_1-1}, z_{j_2}, \dots, z_{j_{s-1}}$ are distinct. We thus find the rainbow matching

$$(R \setminus \{e_{j_1-1}, e_{j_2}, \dots, e_{j_{s-1}}, f\}) \cup \{g_{j_1-1}, g_{j_2}, \dots, g_{j_{s-1}}, \bar{f}, zw\}$$

which is larger than R , a contradiction.

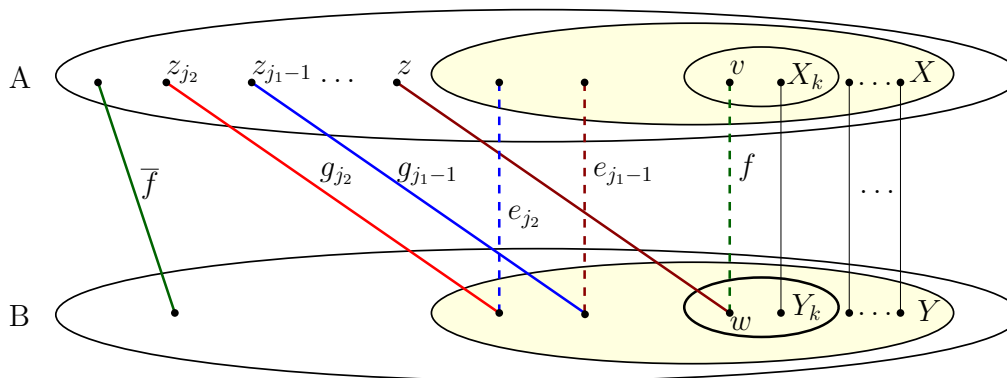


Figure 3: Example with $g_{j_2} \in F_{\pi(0)}$, in case $w \in Y_k$. The dotted edges $\{e_{j_1-1}, e_{j_2}, f\}$ are replaced by the edges $\{g_{j_1-1}, g_{j_2}, \bar{f}, zw\}$ to obtain a larger rainbow matching.

If otherwise $w \in N_k$, then $zw \in F_{\pi(k)}$. Analogously we find a sequence $k > j_1 > j_2 > \dots > j_s = 0$ with $1 \leq s \leq k$ such that $g_k \in F_{\pi(j_1)}$ and $g_{j_i} \in F_{\pi(j_{i+1})}$ for $i < s$, and we

obtain a contradiction as

$$(R \setminus \{e_k, e_{j_1}, \dots, e_{j_s}, f\}) \cup \{g_k, g_{j_1}, \dots, g_{j_s}, \bar{f}, zw\}$$

forms a rainbow matching which is larger than R . Thus, we are done with part (i).

Let us proceed with (ii): F_j needs to saturate at least $(1/2 + \varepsilon)n + 1$ vertices of $B \setminus Y$, as $|F_j| \geq (3/2 + \varepsilon)n$ and $|Y| \leq n - 1$. Thus, by part (i), we have at least $(1/2 + \varepsilon)n + 1$ edges of F_j between $X \cup \{z_1, \dots, z_k, z\}$ and $B \setminus Y$. Using (*), we further calculate that

$$\begin{aligned} |X \cup \{z_1, \dots, z_k, z\}| - |X'_{k+1}| &\leq (n + k) - \left(\frac{1}{2}n + (2k + 1)\varepsilon n + \frac{-3k^2 + 3k + 2}{2} \right) \\ &= \frac{1}{2}n - (2k + 1)\varepsilon n + \frac{3k^2 - k - 2}{2}. \end{aligned}$$

Thus, the number of edges in F_j between X'_{k+1} and $B \setminus Y$ needs to be at least

$$\left(\frac{1}{2} + \varepsilon \right) n + 1 - \left(\frac{1}{2}n - (2k + 1)\varepsilon n + \frac{3k^2 - k - 2}{2} \right) = s_{k+1},$$

as claimed. \square

We now proceed with the construction of the edges e_{k+1}, g_{k+1} and the sets X_{k+1}, Y_{k+1} , and afterwards we show that all required properties are maintained. The next corollary is by the pigeonhole principle an immediate consequence of Claim 7.

Corollary 8. *There exists a vertex $x_{k+1} \in X'_{k+1}$, a set $X_{k+1} \subseteq X'_{k+1}$ of size s_{k+1} and its neighborhood $Y_{k+1} \subseteq Y'_{k+1}$ with respect to R such that the following holds for every $j \in [n - 1]$: If $F_j \cap R \cap E_G[X_{k+1}, Y_{k+1}] \neq \emptyset$, then F_j has an edge between x_{k+1} and $B \setminus Y$.* \square

To extend the sequences, choose X_{k+1} and Y_{k+1} according to Corollary 8, and let $e_{k+1} = x_{k+1}y_{k+1}$ be the unique edge in R that is incident with x_{k+1} . Note that $x_{k+1} \notin X_{k+1}$, as otherwise x_{k+1} would need to be incident to two edges of the same matching F_j .

Observe that $y_{k+1} \notin \{y_1, \dots, y_k\}$. Indeed, $y_{k+1} \in Y'_{k+1} = Y_k \cup N_k$, and by construction we have $N_k \cap \{y_1, \dots, y_k\} = \emptyset$, while $Y_k \cap \{y_1, \dots, y_k\} = \emptyset$ holds by Property (P3).

Now, let $e_{k+1} \in F_j$. As $e_{k+1} \in R \setminus \{e_1, \dots, e_k\}$, we have $j \notin \{\pi(i) : 0 \leq i \leq k\}$. We extend the injective function π with $\pi(k + 1) = j$.

Finally, we choose g_{k+1} as follows: If $y_{k+1} \in N_k$, then by construction of N_k there is a vertex $z_{k+1} \in A \setminus (X \cup \{z_1, \dots, z_k\})$ with $z_{k+1}y_{k+1} \in F_{\pi(k)}$. Otherwise, if $y_{k+1} \in Y_k$, then there is an $i \in [k]$ with $y_{k+1} \in Y_i \setminus Y_{i-1}$, and by Property (P6) there is a vertex $z_{k+1} \in A \setminus (X \cup \{z_1, \dots, z_{i-1}\})$ such that $z_{k+1}y_{k+1} \in F_{\pi(i-1)}$. In any case, we set $g_{k+1} := z_{k+1}y_{k+1}$.

Claim 9. *The extended sequences satisfy Properties (P1)-(P7).*

Proof of Claim 9. Properties (P1) and (P2) follow immediately from the induction hypothesis and from the definition of $\pi(k + 1)$ and g_{k+1} . By construction, we have $Y_{k+1} \subseteq Y'_{k+1} = Y_k \cup N_k$. By Property (P3) of the induction hypothesis and by the definition of

N_k , we have $\{y_1, \dots, y_k\} \cap Y_{k+1} = \emptyset$. It follows from the construction of X_{k+1} (Corollary 8) that $y_{k+1} \notin Y_{k+1}$. By symmetry, we have $\{e_1, \dots, e_{k+1}\} \cap (X_{k+1} \cup Y_{k+1}) = \emptyset$, which shows Property (P3). Properties (P4) and (P5) hold by Corollary 8 and by Property (P5) of the induction hypothesis. Recall that $Y_{k+1} \setminus Y_k \subseteq N_k$. This means that for every $w \in Y_{k+1} \setminus Y_k$ there exists a vertex $v \in A \setminus (X \cup \{z_1, \dots, z_k\})$ such that $vw \in F_{\pi(k)}$, proving Property (P6). Finally, Property (P7) holds by the induction hypothesis and since we chose z_{k+1} from a set $A \setminus (X \cup \{z_1, \dots, z_{i-1}\})$ such that $z_{k+1}y_{k+1} \in F_{\pi(i-1)}$ for the appropriate $i \in [k+1]$. Consequently, all Properties (P1)-(P7) are fulfilled by the extended sequences. \square

Claim 9 concludes the induction and thus the proof of Theorem 3. \square

3 Open problems and concluding remarks

In this paper we proved that a collection of n matchings of size $(3/2 + o(1))n$ in a bipartite multigraph guarantees a rainbow matching of size n . For smaller matching sizes it is even unknown whether a rainbow matching of size $n-1$ exists. More generally, as suggested by Tibor Szabó (private communication), it would be interesting to determine upper bounds on the smallest integer $\mu(n, \ell)$ such that every family of n matchings of size $\mu(n, \ell)$ in a bipartite multigraph guarantees a rainbow matching of size $n - \ell$. One can verify that $\mu(n, \ell) \leq \frac{\ell+2}{\ell+1}n$. Moreover, it holds that $\mu(n, \sqrt{n}) \leq n$, which is a generalization (see e.g. [3]) of a result proved in the context of Latin squares by Woolbright [13], and independently by Brouwer, de Vries and Wieringa [6].

In order to approach Conjecture 2, one can also increase the number of matchings and fix their sizes to be equal to n instead of considering families of n matchings of sizes greater than n . Drisko [9] proved that a collection of $2n-1$ matchings of size n in a bipartite multigraph with partite sets of size n guarantees a rainbow matching of size n . He also showed that this result is sharp. This problem was investigated in the following two directions. Does the statement also hold if we omit the restriction on the sizes of the vertex classes? And how many matchings do we need to find a rainbow matching of size $n - \ell$ for every $\ell \geq 1$? Aharoni and Berger [1] affirmed the first question by showing that for any integers $s \leq t$, the maximal number of matchings of size t in a bipartite multigraph that do not contain a rainbow matching of size s is equal to $2(s-1)$. The second question was studied recently by Barát, Gyárfás and Sárközy in [4]. They proved that for every $\ell \geq 1$ any bipartite multigraph with $\lfloor \frac{\ell+2}{\ell+1}n \rfloor - (\ell+1)$ matchings of size n has a rainbow matching of size $n - \ell$. This result is sharp for $\ell = 0$ and $\lfloor n/2 \rfloor \leq \ell < n$. For $\ell = \lfloor \sqrt{n} \rfloor$ it extends the result $\mu(n, \sqrt{n}) \leq n$ to the setting of multigraphs.

Finally, in case Conjecture 2 turns out to be true, it is of interest to see how sharp it is. As shown by Barát and Wanless [5], one can find constructions of n matchings with $\lfloor \frac{n}{2} \rfloor - 1$ matchings of size $n+1$ and the remaining ones being of size n such that there is no rainbow matching of size n . We wonder whether the expression $\lfloor \frac{n}{2} \rfloor - 1$ above could also be replaced by $(1 - o(1))n$.

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