

0-Sum and 1-Sum Flows in Regular Graphs

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Abstract

Let G be a graph. Assume that l and k are two natural numbers. An l -sum flow on a graph G is an assignment of non-zero real numbers to the edges of G such that for every vertex v of G the sum of values of all edges incident with v equals l . An l -sum k -flow is an l -sum flow with values from the set $\{\pm 1, \dots, \pm(k-1)\}$. Recently, it was proved that for every $r, r \geq 3, r \neq 5$, every r -regular graph admits a 0-sum 5-flow. In this paper we settle a conjecture by showing that every 5-regular graph admits a 0-sum 5-flow. Moreover, we prove that every r -regular graph of even order admits a 1-sum 5-flow.

Keywords: 0-sum flow, regular graph, 1-sum flow, factor

1. Introduction

Throughout this paper a graph means a finite undirected graph without loop or multiple edges. Let G be a multigraph with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices and the number of edges of G are called the *order* and the *size* of G , respectively. A k -regular graph is a graph where each vertex is of degree k . The degree of vertex v in G is denoted by $d_G(v)$ and $N_G(v)$ denotes the set of all vertices adjacent to v . A graph G is called k -edge connected if the minimum number of edges whose removal would disconnect the graph is at least k . A *pendant edge* is an edge incident with a vertex of degree 1.

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For a set $\{a_1, \dots, a_r\}$ of non-negative integers an $\{a_1, \dots, a_r\}$ -graph is a graph each of whose vertices has degree from the set $\{a_1, \dots, a_r\}$. For integers a and b , $1 \leq a \leq b$, an $[a, b]$ -graph is defined to be a graph G such that for every $v \in V(G)$, $a \leq d_G(v) \leq b$. An $[a, b]$ -factor is a spanning subgraph of G in which the degree of each vertex is in the interval $[a, b]$. When $a = b$, we call it an a -factor.

Assume that l and k are two natural numbers. An l -sum flow on a graph G is an assignment of non-zero real numbers to each edge of G such that for every vertex v in $V(G)$ the sum of values of all edges incident with v equals l and call it l -sum rule. An l -sum k -flow is an l -sum flow with values from the set $\{\pm 1, \dots, \pm(k-1)\}$.

Let G be a graph. A k -flow of G is an assignment of integers with maximum value at most $k-1$ to each edge of G together with its orientation (or direction) such that for each vertex of G , the sum of the labels of incoming edges is equal to that of the labels of outgoing edges. A nowhere-zero k -flow is a k -flow with no zeros.

Tutte proposed the following interesting conjecture.

Conjecture A.[8](**Tutte's 5-flow Conjecture**) If G is 2-edge connected, then it has a nowhere-zero 5-flow.

In [2], it was proved that Tutte's 5-flow Conjecture is equivalent to show that every 2-edge connected bipartite graph admits a 0-sum 5-flow.

In 2009, an analagous version of Tutte's Conjecture proposed for undirected graphs.

Conjecture B.[2] (**0-Sum Conjecture (ZSC)**). If a graph G admits a 0-sum flow, then G admits a 0-sum 6-flow.

For r -regular graphs it was conjectured that 6 can be reduced to 5.

Conjecture C.[1] Every r -regular graphs ($r \geq 3$) admits a 0-sum 5-flow.

Conjecture C has been settled for cubic graphs in [2] and for every positive integer r , $r \neq 5$ in [3]. In [10], the authors proved that every r -regular graph ($r \geq 3$) admits a 0-sum 7-flow. Also in [9], for some r, k, l , the existence of l -sum k -flow for r -regular graphs has been studied.

In the present manuscript using strong tools in factorization of graphs, we show that Conjecture C holds in general. Also, we prove that every r -regular graph of even order admits a 1-sum 5-flow.

1 0-sum 5-flow for 5-regular graphs

The main goal of this section is showing that Conjecture C is true. We would like to prove the next result which settles Conjecture C.

Theorem 1. Every 5-regular graph admits a 0-sum 5-flow.

Proof. First let us state five lemmas.

Lemma 2. [5 and 6, p.91 and p.203] *Let G be an n -edge connected multigraph ($n \geq 1$), θ be a real number such that $0 < \theta < 1$ and $f : V(G) \rightarrow \{0, 1, 2, \dots\}$. If (i), (ii) and one of (iiia), (iiib) hold, then G has an f -factor.*

(i) $\sum_{x \in V(G)} f(x)$ is even.

(ii) $\sum_{x \in V(G)} |f(x) - \theta d_G(x)| < 2$.

(iiia) $n\theta \geq 1$ and $n(1 - \theta) \geq 1$.

(iiib) The set $\{f(x)\}$ consists of even numbers and $m(1 - \theta) \geq 1$, where $m \in \{n, n + 1\}$ and $m \equiv 1 \pmod{2}$.

Now, we prove the following lemma.

Lemma 3. *Let G be a 2-edge connected $[2, 5]$ -multigraph. If*

$$3|\{x \in V(G) : d_G(x) = 2\}| + 2|\{x \in V(G) : d_G(x) = 3\}| + |\{x \in V(G) : d_G(x) = 4\}| \leq 4,$$

then G has a 2-factor.

Proof. Define a function f on $V(G)$ as $f(x) = 2$, for all $x \in V(G)$, and let $\theta = \frac{2}{5}$. Then

$$\begin{aligned} & \sum_{x \in V(G)} |f(x) - \theta d_G(x)| \\ &= \frac{6}{5}|\{x : d_G(x) = 2\}| + \frac{4}{5}|\{x : d_G(x) = 3\}| + \frac{2}{5}|\{x : d_G(x) = 4\}| \\ &\leq \frac{8}{5} < 2. \end{aligned}$$

Hence Parts (i), (ii) and (iiib) of Lemma 2 are satisfied with $m = 3$, and thus G has a 2-factor. \square

In [2] the following result was proved.

Lemma 4. *If G is a connected $\{1, 3\}$ -graph and the subgraph of G induced by vertices of degree 3 is 2-edge connected, then there is a function f on $E(G)$ with $f(e) \in \{-2, 1, 4\}$ so that the 0-sum rule holds for each vertex of degree 3, and each pendant edge e has $f(e) \in \{-2, 4\}$. Moreover, one pendant edge e may have its value pre-assigned.*

The next lemma shows that Lemma 4 can be generalized to every $\{1, 3\}$ -graph.

Lemma 5. *Let G be a connected $\{1, 3\}$ -graph and let h be a pendant edge of G . For any arbitrary $\alpha \in \{-2, 4\}$, there exists a function $f : E(G) \rightarrow \{-2, 1, 4\}$ such that $f(h) = \alpha$ and 0-sum rule holds in each vertex of degree 3 and the value of any pendant edge is in the set $\{-2, 4\}$.*

Proof. Consider a rooted tree T obtained from G such that every maximal 2-edge connected subgraph of G is considered as a vertex of T and $E(T)$ consists of all cut edges of G , where the root is the maximal 2-edge connected subgraph one of whose vertices incident with the given pendant edge h , and a subgraph with one vertex is considered as a 2-edge connected subgraph. Now, we start by a root of T . If the root consists of one vertex, then we can easily assign the desired values to the three edges. So, we may assume that the maximal 2-edge connected subgraph, say H , of G corresponding to the root of T has order at least 2. Thus the subgraph of G obtained from H by adding all cut edges of G incident with H is a graph that satisfies the conditions of Lemma 4. Then apply Lemma 4 to obtain an edge assignment f for the root with values form $\{-2, 1, 4\}$ in which the pendant edges have even value and $f(h) = \alpha$. Consider a maximal 2-edge connected subgraph K of G corresponding to a child of the root of T and apply again Lemma 4, where the edge joining K to the root corresponds the given pendant edge in Lemma 4. By continuing this procedure we can find the desired function on the edge set of G . \square

Lemma 6. *If G is a connected $\{1, 5\}$ -graph, $\{e_1, \dots, e_s\} \subseteq E(G)$ is the set of all pendant edges of G and $G - \{e_1, \dots, e_s\}$ is 2-edge connected, then there is a function f on $E(G)$ with $f(e) \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ so that 0-sum rule holds for each vertex of degree 5 and for $i = 1, \dots, s$, $f(e_i) \in \{-2, 2, 4\}$. Moreover, one pendant edge e_i may have value pre-assigned.*

Proof. Let $s = 5p + q$, where $p \geq 0$ and $0 \leq q \leq 4$ are integers. We divide $5p$ pendant edges of G into p groups each of which contains 5 edges, and identify the end points of every group to obtain the new p vertices of degree 5, called v_1, \dots, v_p . Remove q remaining pendant edges from G and call the resultant multigraph by H . Since H is obtained from G by removing q pendant edges, H has the following property:

$$3|\{x : d_H(x) = 2\}| + 2|\{x : d_H(x) = 3\}| + |\{x : d_H(x) = 4\}| \leq 4.$$

Thus by Lemma 3, H has a 2-factor F . Now, we define a function $f : E(G) \rightarrow \{-2, 2, 3, 4\}$ for G so that the 0-sum rule holds for each vertex of degree 5. Assign value 3 to all edges of F , and assign value -2 to all remaining edges of H . Also assign value -2 to q removed pendant edges of G . Now, if a cycle C of F contains at least one vertex in $\{v_1, \dots, v_p\}$, choose one vertex, say v_t , and change the values of edges of this cycle alternatively by 2 and 4 starting at an edge incident with v_t and ending at the other edge incident with v_t . Note that if a cycle C contains no vertex in $\{v_1, \dots, v_p\}$, do not change the values of edges of C . Then we split $5p$ edges incident with $\{v_1, \dots, v_p\}$ of H into $5p$ pendant edges of G . Thus we obtain a function f with the desired property.

For the last part of lemma we consider 3 cases:

(i) $f(e_\ell) = -2$. Consider the graph H as before. We know that H contains a 2-factor F . If e_ℓ is not contained in F , then the previous assignment works. If e_ℓ is contained in F and e_ℓ is incident with v_r , assign the value -3 to each edge of F , and assign 2 to all other edges. Then change the values of edges of the cycle C alternatively by -2 and

-4 , starting at e_ℓ . Note that we do the same procedure for every cycle of F containing a vertex in $\{v_1, \dots, v_p\}$.

(ii) $f(e_\ell) = 2$. If e_ℓ is contained in a 2-factor F of H , then the previous assignment works. If e_ℓ is not contained in F , then assign -3 to all edges of F and assign 2 to all remaining edges, and do the same procedure for every cycle F containing at least one vertex in $\{v_1, \dots, v_p\}$.

(iii) $f(e_\ell) = 4$. Consider the first assignment of edges of H . If e_ℓ is contained in F , then we are done. If e_ℓ is not contained in 2-factor F of H , then by removing all edges of F from H , we obtain a $[0, 3]$ -graph which is not necessarily connected. We have two possibilities: e_ℓ is an edge of $H - F$ or e_ℓ is not in H , i.e., e_ℓ is a removed pendant edge when H is obtained. In the first case suppose that v_t is a vertex of degree 3 in $H - F$ incident with e_ℓ . Now, for every $v_i \in \{v_1, \dots, v_p\}$, we split 3 edges of $H - F$ incident with v_i to make 3 pendant edges. Add q removed pendant edges of G to $H - F$. Then the resultant graph is a $\{1, 3\}$ -graph, say K , in which e_ℓ is a pendant edge.

By Lemma 5, we have a function $g : E(K) \rightarrow \{-2, 1, 4\}$ such that $g(e_\ell) = -2$ and the values of every pendant edge is in the set $\{-2, 4\}$ and moreover the 0-sum rule holds in each vertex of degree 3. Now, subtract 2 from all values of $E(K)$ and then multiply -1 to the values of all edges of K . Then assign -3 to all edges of F , and change the values of all edges of every cycle of F containing a vertex in $\{v_1, \dots, v_p\}$ alternatively by -2 and -4 . Clearly, the value of e_ℓ is 4 and 0-sum rule holds for each vertex of degree 5, as desired.

If e_ℓ is not in H , we add q removed pendant edges of G including e_ℓ to $H - F$ to obtain a $\{1, 3\}$ -graph, say K . Now, a similar method given above completes the proof. \square

Now, we are in a position to prove Theorem 1.

If G is 2-edge connected, then by Lemma 3, G has a 2-factor F . Then assign value 3 to all the edges of F , and assign value -2 to all remaining edges of G , which is the desired 0-sum 5-flow. Hence we may assume that G is not 2-edge connected. Consider a rooted tree T obtained from G such that every maximal 2-edge connected subgraph of G is considered as a vertex of T and $E(T)$ consists of all cut edges of G , where a subgraph consisting of one vertex is considered as a 2-edge connected subgraph. Now, we start by a root of T whose induced subgraph on the vertices of degree 5 is 2-edge connected. Let H be the maximal 2-edge connected subgraph of G corresponding to the root of T . Apply Lemma 6 to the subgraph of G obtained from H by adding all the cut edges of G incident with H to obtain an edge assignment for the root with values from the set $\{\pm 1, \pm 2, \pm 3, \pm 4\}$ in which every pendant edge has a value from the set $\{-2, 2, 4\}$. Consider a maximal 2-edge connected subgraph K of G corresponding to a child of the root of T and apply again Lemma 6 to the subgraph of G obtained from K by adding all cut edges of G incident with K to obtain an assignment on the root and K . By continuing this procedure we find a 0-sum 5-flow for G and the proof is complete.

2 1-sum flows in regular graphs

As we mentioned that before every r -regular graph $r \geq 3$, admits a 0-sum 5-flow. In this section we prove that every r -regular graph of even order $r \geq 3$, admits a 1-sum 5-flow. Before establishing our results we need some theorems.

Remark 1. We note that if a graph G admits a 1-sum k -flow, then G has even order. To see this assume that f is a 1-sum k -flow for G . We have

$$|V(G)| = \sum_{v \in V(G)} \sum_{u \in N_G(v)} f(uv) = 2 \sum_{e \in E(G)} f(e).$$

Thus $|V(G)|$ should be even.

In the sequel we need the following result.

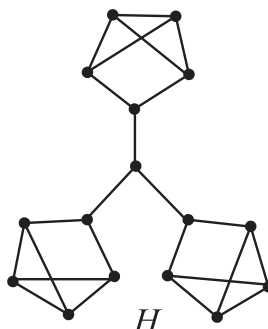
Theorem 7.[6 and 7, p. 184-190] *Let $r \geq 3$ be an odd integer and let k be an integer such that $1 \leq k \leq \frac{2r}{3}$. Then every r -regular graph has a $[k-1, k]$ -factor each component of which is regular.*

Also, we need the following theorem due to Petersen.

Theorem 8.[7] *Every $2k$ -regular multigraph admits a 2-factorization.*

The following remark shows that there are some regular graphs with no 1-sum 3-flow.

Remark 2. It is not hard to see that following 3-regular graph does not admit a 1-sum 3-flow.



Now, we are ready to show that every r -regular graph of even order admits a 1-sum 5-flow.

Theorem 9. *Let G be an r -regular connected graph of even order. Then the following hold:*

- (i) *If r is an odd integer or $r = 4k + 2$, for some integer $k \geq 0$, then G admits a 1-sum 4-flow.*
- (ii) *If $r = 4k$, for some integer $k \geq 1$, then G admits a 1-sum 5-flow.*

Proof. Assume that $V(G) = \{1, \dots, n\}$. First suppose that r is an odd integer. We define a bipartite graph from G , called B , with two parts $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ and $x_i y_j \in E(B)$ if and only if $ij \in E(G)$ for every i and j , $1 \leq i, j \leq n$. So, B is an r -regular graph and by Theorem [4, p.79], B has a 1-factorization F_1, \dots, F_r . Now, for every $e \in E(F_i)$, $1 \leq i \leq r$, define a function $g : E(B) \rightarrow \{\pm \frac{1}{2}, \pm \frac{3}{2}\}$ as follows.

For $r = 4k + 1$ define:

$$g(e) = \begin{cases} \frac{-3}{2}, & 1 \leq i \leq k; \\ \frac{1}{2}, & k < i \leq r. \end{cases}$$

Also, for $r = 4k + 3$ define:

$$g(e) = \begin{cases} \frac{3}{2}, & 1 \leq i \leq k + 1; \\ \frac{-1}{2}, & k + 1 < i \leq r. \end{cases}$$

Clearly, for each $u \in V(B)$, $\sum_{v \in N_B(u)} g(uv) = \frac{1}{2}$. Now, define a function $f : E(G) \rightarrow \{\pm 1, \pm 3\}$ such that for every $ij \in E(G)$, $f(ij) = g(x_i y_j) + g(x_j y_i)$. Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Now, suppose that r is an even integer. If G is a 2-regular graph, then by assigning the integers $-1, 2$ to the edges of G alternatively, we are done.

Let $r = 4k$. Double all edges of G to obtain an $8k$ -regular multigraph G' . Since G' contains two edge disjoint spanning subgraphs H_1 and H_2 isomorphic to G and H_1 is decomposed into 2-factors F_1, \dots, F_{2k} , we can obtain a $(4k + 2)$ -regular multigraph

$G'' = G' \setminus E(F_1) \cup \dots \cup E(F_{2k-1})$, which contains a $4k$ -regular graph H_2 . Since G'' is 2-edge connected, then by Parts (i), (ii) and (iiia) of Lemma 2, if we define $f(i) = 2k + 1$, for all $i \in V(G'')$ and $\theta = \frac{1}{2}$, G'' is decomposed into two $(2k + 1)$ -factors G''_1 and G''_2 . Now, for every $e \in E(G')$, we define a function $g : E(G') \rightarrow \{-2, 1, 3\}$ as follows:

$$g(e) = \begin{cases} -2, & e \in E(F_1) \cup \dots \cup E(F_{k-2}) \cup E(G''_1); \\ 1, & e \in E(F_{k-1}) \cup E(F_k) \cup E(F_{k+1}) \cup E(G''_2); \\ 3, & e \in E(F_{k+2}) \cup \dots \cup E(F_{2k-1}). \end{cases}$$

Clearly, for each $i \in V(G')$, $\sum_{j \in N_{G'}(i)} g(ij) = 1$. Now, define a function $f : E(G) \rightarrow \{-4, -1, 1, 2, 4\}$ such that for every $e \in E(G)$, $f(e) = g(e) + g(e')$, where e' is the copy of e in duplicating of this edge in G' . Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Now, assume that $r = 4k + 2$ and $r \neq 6, 10, 14, 22$. First note that every integer of the form $4k + 2$ can be written as $12k + 2$, $12k + 6$ or $12k + 10$, for some integer $k \geq 0$.

Let $r = 12k + 2$. Since G is 2-edge connected, then by Parts (i), (ii) and (iiia) of Lemma 2, if we define $f(i) = 6k + 1$, for all $i \in V(G)$ and $\theta = \frac{1}{2}$, then G has two $(6k + 1)$ -factors H_1 and H_2 . On the other hand, by Theorem 7, H_2 has a $[4k - 1, 4k]$ -factor, say T whose components are regular. Let T_1 be the union of the $(4k - 1)$ -regular components of T and let T_2 be the union of $4k$ -regular components of T . Note that by Theorem 8, T_2 has a 2-factorization with 2-factors F_1, \dots, F_{2k} . Now, we define a function $g : E(G) \setminus E(T_1) \rightarrow \{-3, -2, -1, 2\}$ as follows:

$$g(e) = \begin{cases} -3, & e \in E(H_2) \setminus E(T); \\ -2, & e \in E(F_i), 1 \leq i \leq k - 1; \\ -1, & e \in E(F_i), k \leq i \leq 2k; \\ 2, & e \in E(H_1). \end{cases}$$

Now, we want to assign some labels to the edges of T_1 . With no loss of generality one can assume that $V(T_1) = \{1, \dots, q\}$. We define a bipartite graph, call L , with two parts $X = \{x_1, \dots, x_q\}$ and $Y = \{y_1, \dots, y_q\}$ and $x_i y_j \in E(L)$ if and only if $ij \in E(T_1)$ for every i and j , $1 \leq i, j \leq q$. So, L is a $(4k - 1)$ -regular graph and by Theorem [4, p.79], L has a 1-factorization F'_1, \dots, F'_{4k-1} . Now, for every $e \in E(F'_i)$, $1 \leq i \leq 4k - 1$, define a function $g' : E(L) \rightarrow \{-\frac{1}{2}, -\frac{3}{2}\}$ as follows:

$$g'(e) = \begin{cases} -\frac{3}{2}, & 1 \leq i \leq k - 2; \\ -\frac{1}{2}, & k - 1 \leq i \leq 4k - 1. \end{cases}$$

Clearly, for each $i \in V(L)$, $\sum_{j \in N_L(i)} g'(ij) = \frac{-6k+5}{2}$. Now, define a function $f : E(G) \rightarrow \{-3, -2, -1, 2\}$ such that for every $e \in E(G) \setminus E(T_1)$, $f(e) = g(e)$ and for every $e = ij \in E(T_1)$, $f(e) = g'(x_i y_j) + g'(x_j y_i)$. Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Now, suppose that $r = 12k + 6$. Since G is 2-edge connected, then by Parts (i), (ii) and (iiia) of Lemma 2, if we define $f(i) = 6k + 3$, for all $i \in V(G)$ and $\theta = \frac{1}{2}$, then G has two $(6k + 3)$ -factors H_1 and H_2 . On the other hand, by Theorem 7, H_2 has a

$[4k+1, 4k+2]$ -factor, say T whose components are regular. Let T_1 be the union of the $(4k+1)$ -regular components of T and let T_2 be the union of $(4k+2)$ -regular components of T . Note that by Theorem 8, T_2 has a 2-factorization with 2-factors F_1, \dots, F_{2k+1} . Now, we define a function $g : E(G) \setminus E(T_1) \rightarrow \{-3, -2, -1, 2\}$ as follows:

$$g(e) = \begin{cases} -3, & e \in E(H_2) \setminus E(T); \\ -2, & e \in E(F_i), 1 \leq i \leq k; \\ -1, & e \in E(F_i), k+1 \leq i \leq 2k+1; \\ 2, & e \in E(H_1). \end{cases}$$

Now, we want to assign some labels to the edges of T_1 . With no loss of generality one can assume that $V(T_1) = \{1, \dots, q\}$. We define a bipartite graph, call L , with two parts $X = \{x_1, \dots, x_q\}$ and $Y = \{y_1, \dots, y_q\}$ and $x_i y_j \in E(L)$ if and only if $ij \in E(T_1)$ for every i and j , $1 \leq i, j \leq q$. So, L is a $(4k+1)$ -regular graph and by Theorem [4, p.79], L has a 1-factorization F'_1, \dots, F'_{4k+1} . Now, for every $e \in E(F'_i)$, $1 \leq i \leq 4k+1$, define a function $g' : E(L) \rightarrow \{-\frac{1}{2}, -\frac{3}{2}\}$ as follows:

$$g'(e) = \begin{cases} -\frac{3}{2}, & 1 \leq i \leq k-1; \\ -\frac{1}{2}, & k \leq i \leq 4k+1. \end{cases}$$

Clearly, for each $v \in V(L)$, $\sum_{u \in N_L(u)} g'(uv) = \frac{-6k+1}{2}$. Now, define a function $f : E(G) \rightarrow \{-3, -2, -1, 2\}$ such that for every $e \in E(G) \setminus E(T_1)$, $f(e) = g(e)$ and for every $e = ij \in E(T_1)$, $f(e) = g'(x_i y_j) + g'(x_j y_i)$. Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Next, assume that $r = 12k + 10$. Since G is 2-edge connected, then by Parts (i), (ii) and (iia) of Lemma 2, if we define $f(i) = 6k + 5$, for all $i \in V(G)$ and $\theta = \frac{1}{2}$, then G has two $(6k+5)$ -factors H_1 and H_2 . On the other hand, by Theorem 7, H_2 has a $[4k+1, 4k+2]$ -factor, say T whose components are regular. Let T_1 be the union of the $(4k+1)$ -regular components of T and let T_2 be the union of $(4k+2)$ -regular components of T . Note that by Theorem 8, T_2 has a 2-factorization with 2-factors F_1, \dots, F_{2k+1} . Now, we define a function $g : E(G) \setminus E(T_1) \rightarrow \{-3, -2, -1, 2\}$ as follows:

$$g(e) = \begin{cases} -3, & e \in E(H_2) \setminus E(T); \\ -2, & e \in E(F_i), 1 \leq i \leq k-1; \\ -1, & e \in E(F_i), k \leq i \leq 2k+1; \\ 2, & e \in E(H_1). \end{cases}$$

Now, we want to assign some labels to the edges of T_1 . With no loss of generality one can assume that $V(T_1) = \{1, \dots, q\}$. We define a bipartite graph, call L , with two parts $X = \{x_1, \dots, x_q\}$ and $Y = \{y_1, \dots, y_q\}$ and $x_i y_j \in E(L)$ if and only if $ij \in E(T_1)$ for every i and j , $1 \leq i, j \leq q$. So, L is a $(4k+1)$ -regular graph and by Theorem [4, p.79], L has a 1-factorization F'_1, \dots, F'_{4k+1} . Now, for every $e \in E(F'_i)$, $1 \leq i \leq 4k+1$, define a function $g' : E(L) \rightarrow \{-\frac{1}{2}, -\frac{3}{2}\}$ as follows:

$$g'(e) = \begin{cases} -\frac{3}{2}, & 1 \leq i \leq k-2; \\ -\frac{1}{2}, & k-1 \leq i \leq 4k+1. \end{cases}$$

Clearly, for each $v \in V(L)$, $\sum_{u \in N_L(v)} g'(uv) = \frac{-6k+3}{2}$. Now, define a function $f : E(G) \rightarrow \{-3, -2, -1, 2\}$ such that for every $e \in E(G) \setminus E(T_1)$, $f(e) = g(e)$ and for every $e = ij \in E(T_1)$, $f(e) = g'(x_i y_j) + g'(x_j y_i)$. Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Now, suppose that G is an r -regular graph such that $r \in \{6, 10, 14, 22\}$ and $r = 4k + 2$. Since G is 2-edge connected then by Parts (i), (ii) and (iiia) of Lemma 2, if we define $f(i) = 2k + 1$, for all $i \in V(G)$ and $\theta = \frac{1}{2}$, then G has two $(2k + 1)$ -factors G_1 and G_2 . Then by Theorem 7, G_2 has a $[t - 1, t]$ -factor T , for every t , $1 \leq t \leq \frac{2r}{3}$, whose components are regular. Let T_1 be the union of the $(t - 1)$ -regular components of T and let T_2 be the union of t -regular components of T .

If $r = 6$, then G_2 has a $[1, 2]$ -factor. Define a function $f : E(G) \rightarrow \{-2, 1, 2, 3\}$, where $f(e) = -2$ for $e \in E(G_1)$, $f(e) = 3$ for $e \in E(G_2) \setminus E(T)$, $f(e) = 1$ for $e \in E(T_1)$ and $f(e) = 2$ for $e \in E(T_2)$.

If $r = 10$, then G_2 has a $[1, 2]$ -factor. Define a function $f : E(G) \rightarrow \{-2, -1, 1, 3\}$, where $f(e) = -2$ for $e \in E(G_1)$, $f(e) = 3$ for $e \in E(G_2) \setminus E(T)$, $f(e) = -1$ for $e \in E(T_1)$ and $f(e) = 1$ for $e \in E(T_2)$.

If $r = 14$, then G_2 has a $[3, 4]$ -factor. Note that by Theorem 8, T_2 has two 2-factors, say T'_1 and T'_2 . Now, define a function $f : E(G) \rightarrow \{-3, -1, -2, 2\}$, where $f(e) = 2$ for $e \in E(G_1)$, $f(e) = -1$ for $e \in E(G_2) \setminus E(T)$, $f(e) = -3$ for $e \in E(T_1)$, $f(e) = -2$ for $e \in E(T'_1)$ and $f(e) = -3$ for $e \in E(T'_2)$.

If $r = 22$, then G_2 has a $[2, 3]$ -factor. Define a function $f : E(G) \rightarrow \{-3, 1, 2, 3\}$, where $f(e) = 2$ for $e \in E(G_1)$, $f(e) = -3$ for $e \in E(G_2) \setminus E(T)$, $f(e) = 3$ for $e \in E(T_1)$ and $f(e) = 1$ for $e \in E(T_2)$.

Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired. \square

We close the paper with the following conjecture.

Conjecture 10. *Every connected $4k$ -regular graph of even order admits a 1-sum 4-flow.*

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