4-Factor-criticality of vertex-transitive graphs

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Submitted: Nov 3, 2014; Accepted: Jun 18, 2016; Published: Jul 8, 2016
Mathematics Subject Classifications: 05C70

Abstract

A graph of order n is *p-factor-critical*, where p is an integer of the same parity as n, if the removal of any set of p vertices results in a graph with a perfect matching. 1-factor-critical graphs and 2-factor-critical graphs are well-known factor-critical graphs and bicritical graphs, respectively. It is known that if a connected vertex-transitive graph has odd order, then it is factor-critical, otherwise it is elementary bipartite or bicritical. In this paper, we show that a connected vertex-transitive non-bipartite graph of even order at least 6 is 4-factor-critical if and only if its degree is at least 5. This result implies that each connected non-bipartite Cayley graph of even order and degree at least 5 is 2-extendable.

Keywords: Vertex-transitive graph; 4-Factor-criticality; Matching; Connectivity

1 Introduction

Only finite and simple graphs are considered in this paper. A *matching* of a graph is a set of its mutually nonadjacent edges. A *perfect matching* of a graph is a matching covering all its vertices. A graph is called *factor-critical* if the removal of any one of its vertices results in a graph with a perfect matching. A graph is called *bicritical* if the removal of any pair of its distinct vertices results in a graph with a perfect matching. The concepts of factor-critical and bicritical graphs were introduced by Gallai [9] and by Lovász [12], respectively. In matching theory, factor-critical graphs and bicritical graphs are two basic bricks in matching structures of graphs [17]. Later on, the two concepts were generalized to the concept of p-factor-critical graphs by Favaron [7] and Yu [21], independently. A graph of order n is said to be *p-factor-critical*, where p is an integer of the same parity as n, if the removal of any p vertices results in a graph with a perfect matching.

*Supported by NSFC grant 11371180.

q-extendable graphs was proposed by Plummer [17] in 1980. A connected graph of even order n is *q-extendable*, where q is an integer with $0 \leqslant q < n/2$, if it has a perfect matching and every matching of size q is contained in one of its perfect matchings. Favaron [8] showed that for q even, every connected non-bipartite q-extendable graph is q-factor-critical. In 1993 Yu [21] introduced an analogous concept for graphs of odd order. A connected graph of odd order is *$q\frac{1}{2}$-extendable*, if the removal of any one of its vertices results in a q-extendable graph.

A graph G is said to be *vertex-transitive* if for any two vertices x and y in G there is an automorphism φ of G such that $y = \varphi(x)$. A graph with a perfect matching is *elementary* if the union of its all perfect matchings forms a connected subgraphs. In [13], there is a following classic result about the factor-criticality and bicriticality of vertex-transitive graphs.

Theorem 1 ([13]). *Let G be a connected vertex-transitive graph of order n. Then*
(a) G is factor-critical if n is odd;
(b) G is either elementary bipartite or bicritical if n is even.

A question arises naturally: Does a vertex-transitive non-bipartite graph has larger p-factor-criticality?

In fact, the q-extendability and $q\frac{1}{2}$-extendability of Cayley graphs, an important class of vertex-transitive graphs, have been investigated in literature. It was proved in papers [3, 4, 16] that a connected Cayley graph of even order on an abelian group, a dihedral group or a generalized dihedral group is 2-extendable except for several circulant graphs of degree at most 4. Miklavič and Šparl [16] also showed that a connected Cayley graph on an abelian group of odd order $n \geqslant 3$ either is a cycle or is $1\frac{1}{2}$-extendable. Chan et al. [3] raised the problem of characterizing 2-extendable Cayley graphs.

In [22], we showed that a connected vertex-transitive graph of odd order $n \geqslant 3$ is 3-factor-critical if and only if it is not a cycle. This general result is stronger than $1\frac{1}{2}$-extendability of Cayley graphs. In this paper, we obtain the following main result which gives a simple characterization of 4-factor-critical vertex-transitive non-bipartite graphs.

Theorem 2. *Let G be a connected vertex-transitive non-bipartite graph of degree k and of even order at least 6. Then G is 4-factor-critical if and only if $k \geqslant 5$.*

By Theorem 2, we know that all connected non-bipartite Cayley graphs of even order and of degree at least 5 is 2-extendable.

The necessity of Theorem 2 is clear. Our main task is to show the sufficiency of Theorem 2 by contradiction. Suppose that G is a connected non-bipartite vertex-transitive graph G of even order at least 6 and of degree at least 5 but G is not 4-factor-critical. By the s-restricted edge-connectivity of G, we find that in most cases a suitable integer s can be chosen such that every λ_s-atom of G is an imprimitive block. Then we can deduce contradictions. Some preliminary results are presented in Section 2 and some properties of λ_s-atoms of G which are used to show their imprimitivity are proved in Section 3. Finally, we complete the proof of Theorem 2 in Section 4.

2 Preliminaries

In this section, we introduce some notations and results needed in this paper.

Let $G = (V(G), E(G))$ be a graph. For $X \subseteq V(G)$, let $\overline{X} = V(G)\backslash X$. For $Y \subseteq \overline{X}$, denote by $[X, Y]$ the set of edges of G with one end in X and the other in Y. In particular, we denote $[X, \overline{X}]$ by $\nabla(X)$ and denote $|\nabla(X)|$ by $d_G(X)$. Denote by $N_G(X)$ the set of vertices in $\overline{X}$ which are ends of some edges in $\nabla(X)$. If $X = \{v\}$, then X is usually written to v. Vertices in $N_G(v)$ are called the neighbors of v. If no confusion exists, the subscript G are usually omitted. Denote by $G[X]$ the subgraph induced by X and denote by $G-X$ the subgraph induced by $\overline{X}$. The set of edges in $G[X]$ is denoted by $E(X)$. Denote by $c_0(G)$ the number of the components of G which have odd order. For a subgraph H of G, we denote $d_G(V(H_i))$ and $\nabla(V(H_i))$ by $d_G(H_i)$ and $\nabla(H_i)$, respectively.

For a connected graph G, a subset $F \subseteq E(G)$ is said to be an *edge-cut* of G if $G-F$ is disconnected, where $G-F$ is the graph with vertex-set $V(G)$ and edge-set $E(G)\backslash F$. The *edge-connectivity* of G is the minimum cardinality over all the edge-cuts of G, denoted by $\lambda(G)$. A subset $X \subseteq V(G)$ is called a *vertex-cut* of G if $G-X$ is disconnected. The *vertex-connectivity* of G of order n, denoted by $\kappa(G)$, is $n-1$ if G is the complete graph K_n and is the minimum cardinality over all the vertex-cuts of G otherwise. It is well known that $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$, where $\delta(G)$ is the minimum vertex-degree of G.

There are two properties of p-factor-critical graphs.

Theorem 3 ([7, 21]). *A graph G is p-factor-critical if and only if $c_0(G-X) \leqslant |X|-p$ for all $X \subseteq V(G)$ with $|X| \geqslant p$.*

Theorem 4 ([7]). *If a graph G is p-factor-critical with $1 \leqslant p < |V(G)|$, then $\kappa(G) \geqslant p$ and $\lambda(G) \geqslant p+1$.*

Let X be a subset of $V(G)$. Denoted by $\mathscr{C}_{G-X}$ the set of the components of $G-X$. X is called to be *matchable* to $\mathscr{C}_{G-X}$ if the bipartite graph G_X, which arises from G by contracting the components in $\mathscr{C}_{G-X}$ to single vertices and deleting all the edges in $E(X)$, contains a matching covering X. The following general result will be used.

Theorem 5 ([5]). *Every graph G contains a set X of vertices with the following properties:*
(a) X is matchable to $\mathscr{C}_{G-X}$;
(b) Every component of $G-X$ is factor-critical.
Given any such set X, the graph G contains a perfect matching if and only if $|X| = |\mathscr{C}_{G-X}|$.

The *girth* of a graph G with a cycle is the length of a shortest cycle in G and the *odd girth* of a non-bipartite graph G is the length of a shortest odd cycle in G. The girth and odd girth of G are denoted by $g(G)$ and $g_0(G)$, respectively. l-cycle means a cycle of length l. We present two useful lemmas as follows.

Lemma 6 ([15]). *Let G be a graph with $g_0(G) > 3$. Then $|E(G)| \leqslant \frac{1}{4}|V(G)|^2$.*

Lemma 7 ([1]). *Let G be a k-regular graph. If $g_0(G) > 3$, then $|V(G)| \geqslant kg_0(G)/2$.*

Now we list some useful properties of vertex-transitive graphs as follows.

Theorem 8 ([14]). *Let G be a connected vertex-transitive k-regular graph. Then $\lambda(G) = k$.*

Theorem 9 ([19]). *Let G be a connected vertex-transitive k-regular graph. Then $\kappa(G) > \frac{2}{3}k$.*

Lemma 10 ([19]). *Let G be a connected vertex-transitive k-regular graph. If $\kappa(G) < k$, then $\kappa(G) = m\tau(G)$ for some integer $m \geqslant 2$, where*

$$\tau(G) = min\{min\{|V(P)| : P \text{ is a component of } G - X\}: X \text{ is a minimum vertex-cut of } G\}.$$

Lemma 11 ([19]). *Let G be a connected vertex-transitive k-regular graph with $k = 4$ or 6. Then $\kappa(G) = k$.*

An *imprimitive block* of G is a proper non-empty subset X of $V(G)$ such that for any automorphism φ of G, either $\varphi(X) = X$ or $\varphi(X) \cap X = \emptyset$.

Lemma 12 ([18]). *Let G be a vertex-transitive graph and X be an imprimitive block of G. Then $G[X]$ is also vertex-transitive.*

Theorem 13 ([10]). *Let G be a connected vertex-transitive k-regular graph of order n. Let S be a subset of $V(G)$ chosen such that $\frac{1}{3}(k+1) \leqslant |S| \leqslant \frac{1}{2}n$, $d(S)$ is as small as possible, and, subject to these conditions, $|S|$ is as small as possible. If $d(S) < \frac{2}{9}(k+1)^2$, then S is an imprimitive block of G.*

Corollary 14 ([10]). *Let G be a connected vertex-transitive k-regular graph of order n. Let S be a subset of $V(G)$ chosen such that $1 < |S| \leqslant \frac{1}{2}n$, $d_G(S)$ is as small as possible, and, subject to these conditions, $|S|$ is as small as possible. If $d_G(S) < 2(k-1)$, then $d_G(S) = |S| \geqslant k$ and $d_{G[S]}(v) = k-1$ for all $v \in S$.*

Corollary 15. *Let G be a connected vertex-transitive k-regular graph. Suppose $g(G) > 3$ or $|V(G)| < 2k$. Then $d_G(X) \geqslant 2k-2$ for every $X \subseteq V(G)$ with $2 \leqslant |X| \leqslant |V(G)| - 2$.*

Proof. If $k = 2$, then it is trivial. Now suppose $k \geqslant 3$ and that there is a subset $X \subseteq V(G)$ with $2 \leqslant |X| \leqslant |V(G)| - 2$ such that $d_G(X) < 2k-2$. Let S be a subset of $V(G)$ chosen such that $1 < |S| \leqslant \frac{1}{2}|V(G)|$, $d_G(S)$ is as small as possible, and, subject to these conditions, $|S|$ is as small as possible. Then $d_G(S) \leqslant d_G(X) < 2k-2$. By Corollary 14, $d_G(S) = |S| \geqslant k$ and $d_{G[S]}(v) = k-1$ for all $v \in S$. As $2k-3 < \frac{2}{9}(k+1)^2$, S is an imprimitive block of G by Theorem 13. Then $|S|$ is a divisor of $|V(G)|$, which implies $|V(G)| \geqslant 2|S| \geqslant 2k$. Thus $g(G) > 3$. Noting that $|E(S)| = \frac{1}{2}(k-1)|S| \leqslant \frac{1}{4}|S|^2$ by Lemma 6, we have $d_G(S) = |S| \geqslant 2k-2$, a contradiction. $\square$

A subset X of $V(G)$ is called an independent set of G if any two vertices in X are not adjacent. The maximum cardinality of independent sets of G is the independent number of G, denoted by $\alpha(G)$.

Lemma 16. *Let G be a non-bipartite vertex-transitive k-regular graph. Then $\alpha(G) \leqslant \frac{1}{2}|V(G)| - \frac{k}{4}$ if $g_0(G) > 3$, and $\alpha(G) \leqslant \frac{1}{3}|V(G)|$ if $g_0(G) = 3$.*

Proof. Let X be a maximum independent set of G and set $g_0 := g_0(G)$. Noting that G is regular and non-bipartite, we have $|X| < |\overline{X}|$. Set $t = |\overline{X}| - |X|$. Since G is vertex-transitive, the number of g_0-cycles of G containing any given vertex in G is constant. Let q be this constant number and let m be the number of all the g_0-cycles of G. Note that each g_0-cycle of G contains at most $(g_0-1)/2$ vertices in X and at least $(g_0+1)/2$ vertices in $\overline{X}$. We have $q|X| \leqslant \frac{1}{2}m(g_0-1)$ and $q|\overline{X}| \geqslant \frac{1}{2}m(g_0+1)$, which implies $qt = q(|\overline{X}|-|X|) \geqslant m$.

We know $q|V(G)| = mg_0$ by the vertex-transitivity of G. Then $qt \geqslant m = \frac{q}{g_0}|V(G)|$, implying $t \geqslant \frac{|V(G)|}{g_0}$. If $g_0 = 3$, then $\alpha(G) = \frac{1}{2}(|V(G)| - t) \leqslant \frac{1}{3}|V(G)|$. If $g_0 > 3$, then $|V(G)| \geqslant kg_0/2$ by Lemma 7, which implies $\alpha(G) = \frac{1}{2}(|V(G)| - t) \leqslant \frac{1}{2}|V(G)| - \frac{k}{4}$. □

A graph G is called *trivial* if $|V(G)| = 1$.

Lemma 17. *Let G be a connected vertex-transitive non-bipartite graph. Let X be an independent set of G. Suppose that $G - X$ has $|X| - t$ trivial components, where t is a positive integer. Then $g_0(G) \geqslant \frac{2|X|}{t} + 1$.*

Proof. Let Y be the set of vertices in the trivial components of $G-X$ and set $g_0 := g_0(G)$. Let $n_{i,j}$ be the number of g_0-cycles of G which contain exactly i vertices in X and j vertices in Y. Set $s = \frac{1}{2}(g_0 - 1)$. Since X and Y are independent sets of G, each g_0-cycle of G contains at most s vertices in X and contains less vertices in Y than in X. Let q be the number of g_0-cycles of G containing any given vertex in G. We have $\sum_{0\leqslant j<i\leqslant s} in_{i,j} = q|X|$ and $\sum_{0\leqslant j<i\leqslant s} jn_{i,j} = q|Y| = q(|X| - t)$. Then $q|X| = \sum_{0\leqslant j<i\leqslant s} in_{i,j} \leqslant \sum_{0\leqslant j<i\leqslant s} s(i-j)n_{i,j} = s(\sum_{0\leqslant j<i\leqslant s} in_{i,j} - \sum_{0\leqslant j<i\leqslant s} jn_{i,j}) = sqt = \frac{1}{2}(g_0 - 1)qt$, which implies $g_0 \geqslant \frac{2|X|}{t} + 1$. □

Lemma 18. *Let G be a vertex-transitive graph with a triangle. Then the number of trivial components of $G - X$ is not larger than $|E(X)|$ for each subset $X \subseteq V(G)$.*

Proof. Let Y be the set of vertices in the trivial components of $G - X$. Suppose $|Y| > |E(X)|$. Let q be the number of triangles of G containing any given vertex in G. Note that there are $q|Y|$ triangles of G containing vertices in Y. As $|Y| > |E(X)|$, it implies that $G[X]$ has an edge e which is contained in more than q triangles. This means that more than q triangles containing both ends of e, a contradiction. □

Lemma 19. *Let G be a connected triangle-free vertex-transitive 6-regular graph of even order. Suppose that there are 3 distinct vertices with the same neighbors. Then G is bipartite.*

Proof. Suppose, to the contrary, that G is non-bipartite. Then $g_0 := g_0(G) \geqslant 5$. Let $C = u_0u_1 \dots u_{g_0-1}u_0$ be a g_0-cycle of G. For any pair of vertices u and v in $V(C)$, we know $N(u) \neq N(v)$. So for each $u_i \in V(C)$ there are two distinct vertices u_i' and u_i'' in $\overline{V(C)}$ such that $N(u_i) = N(u_i') = N(u_i'')$ by the vertex-transitivity of G. Set

$U_i = \{u_i, u_i', u_i''\}$. Then U_i is an independent set of G and $U_i \cap U_j = \emptyset$ for $j \neq i$. Noting that u_i and u_{i+1} are adjacent, we have that every vertex in U_i is adjacent to every vertex in U_{i+1}, where $i+1$ is an arithmetic on modular g_0. Since G is 6-regular and connected, $|V(G)| = |\bigcup_{i=0}^{g_0-1} U_i| = 3g_0$, which implies that $|V(G)|$ is odd, a contradiction. □

3 λ_s-atoms of vertex-transitive graphs

In this section, we will introduce some properties of the λ_s-atoms of vertex-transitive graphs. The concept of λ_s-atoms [11, 20] of graphs is used in investigating the s-restricted edge-connectivity of graphs. The s-restricted edge-connectivity of graphs was proposed by Fàbrega and Fiol [6].

For a connected graph G and some positive integer s, an edge-cut F of G is said to be an *s-restricted edge-cut* of G if every component of $G-F$ has at least s vertices. The minimum cardinality of s-restricted edge-cuts of G is the *s-restricted edge-connectivity* of G, denoted by $\lambda_s(G)$. By the definition of $\lambda_s(G)$, we can see that $\lambda(G) = \lambda_1(G) \leqslant \lambda_2(G) \leqslant \lambda_3(G) \cdots$ as long as these parameters exists.

A proper subset X of $V(G)$ is called a *λ_s-fragment* of G if $\nabla(X)$ is an s-restricted edge-cut of G with minimum cardinality. We can see that for every λ_s-fragment X of G, $G[X]$ and $G[\overline{X}]$ are connected graphs of order at least s. A λ_s-fragment of G with minimum cardinality is called a *λ_s-atom* of G.

Lemma 20. *Let G be a connected triangle-free vertex-transitive graph of degree $k \geqslant 5$. For an integer s with $4 \leqslant s \leqslant 8$, suppose $\lambda_s(G) \leqslant 3k$. Let S be a λ_s-atom of G.*
(a) For $X \subseteq V(G)$ with $|X| \geqslant s$ and $|\overline{X}| \geqslant s$, we have $d_G(X) \geqslant \lambda_s(G)$. Furthermore, $d_G(X) > \lambda_s(G)$ if $G[X]$ or $G[\overline{X}]$ is disconnected.
(b) For $A \subseteq S$ with $1 \leqslant |A| \leqslant |S| - s$, we have $d_{G[S]}(A) > \frac{1}{2} d_G(A)$.
(c) For each λ_s-atom T of G with $S \neq T$ and $S \cap T \neq \emptyset$, we have $d_G(S\cap T) + d_G(S \cup T) \leqslant 2\lambda_s(G)$, $d_G(S\backslash T) + d_G(T\backslash S) \leqslant 2\lambda_s(G)$, $|S \cap T| \leqslant s-1$ and $|S\backslash T| \leqslant s-1$.

Proof. (a) If $G[X]$ and $G[\overline{X}]$ are connected, then $\nabla(X)$ is an s-restricted edge-cut of G and hence $d_G(X) \geqslant \lambda_s(G)$. Thus it only needs to show $d_G(X) > \lambda_s(G)$ if $G[X]$ or $G[\overline{X}]$ is disconnected.

Suppose that $G[X]$ is disconnected. If each component of $G[X]$ has less than 4 vertices, then $d_G(X) = k|X| - 2|E(X)| \geqslant k|X| - 2(|X|-2) \geqslant (k-2)s + 4 > 3k \geqslant \lambda_s(G)$. Then we assume that $G[X]$ has a component H_1 with at least 4 vertices. If each component of $G[\overline{V(H_1)}]$ has less than 4 vertices, then $d_G(X) > d_G(H_1) = d_G(\overline{V(H_1)}) > \lambda_s(G)$. Then we assume further that $G[\overline{V(H_1)}]$ has a component H_2 with at least 4 vertices. Noting that both G and H_1 are connected, we have that $G[\overline{V(H_2)}]$ is connected, which implies that $\nabla(H_2)$ is a 4-restricted edge-cut of G. Noting that $\lambda(G) = k$ by Theorem 8, we have $d_G(X) \geqslant \lambda(G) + d_G(H_1) \geqslant k + d(V(H_2)) \geqslant k + \lambda_4(G)$.

So $d(X) > \lambda_4(G)$. Next we consider the case that $5 \leqslant s \leqslant 8$. Set $\tau_s(G) = \min\{d(A) : A \subseteq V(G), 4 \leqslant |A| \leqslant s-1\}$. Then $\lambda_4(G) \geqslant \min\{\lambda_s(G), \tau_s(G)\}$. For each subset $A \subseteq V(G)$ with $4 \leqslant |A| \leqslant 7$, noting that $|E(A)| \leqslant \frac{1}{4}|A|^2$ by Lemma 6, we have $d(A) = k|A| - 2|E(A)| \geqslant k|A| - \frac{1}{2}|A|^2 > 2k$. Hence $\tau_s(G) > 2k$. If $\lambda_s(G) > 2k$, then $d(X) \geqslant$

$k + \lambda_4(G) > k + 2k \geqslant \lambda_s(G)$. If $\lambda_s(G) \leqslant 2k$, then, noting $\min\{\lambda_s(G), \tau_s(G)\} \leqslant \lambda_4(G) \leqslant \lambda_s(G)$, we have $d(X) \geqslant k + \lambda_4(G) = k + \lambda_s(G) > \lambda_s(G)$.

(b) To the contrary, suppose $d_{G[S]}(A) \leqslant \frac{1}{2}d_G(A)$. Then $d_G(S\backslash A) = d_G(S) - (d_G(A) - 2d_{G[S]}(A)) \leqslant d_G(S) = \lambda_s(G)$. By (a), $G[S\backslash A]$ and $G[\overline{S} \cup A]$ are connected. Hence $\nabla(S\backslash A)$ is an s-restricted edge-cut of G. By the minimality of λ_s-atoms of G, we have $d_G(S\backslash A) > \lambda_s(G)$, a contradiction.

(c) By the well-known submodular inequality (see [2] for example), we have that $d_G(S \cap T) + d_G(S \cup T) \leqslant d_G(S) + d_G(T) = 2\lambda_s(G)$ and $d_G(S\backslash T) + d_G(T\backslash S) = d_G(S \cap \overline{T}) + d_G(S \cup \overline{T}) \leqslant d_G(S) + d_G(\overline{T}) = 2\lambda_s(G)$. Next we show $|S \cap T| \leqslant s - 1$ and $|S\backslash T| \leqslant s - 1$. Clearly, they hold if $|S| = s$. So we may assume $|S| > s$.

Suppose $|S \cap T| \geqslant s$. Then $d_G(S \cap T) = d_G(S) + 2d_{G[S]}(S\backslash T) - d_G(S\backslash T) > d_G(S) = \lambda_s(G)$ by (b). Noting $|\overline{S \cup T}| \geqslant |V(G)| - |S| - |T| + |S \cap T| \geqslant s$, we have $d_G(S \cup T) \geqslant \lambda_s(G)$ by (a). Hence $d_G(S \cap T) + d_G(S \cup T) > 2\lambda_s(G)$, a contradiction. Thus $|S \cap T| \leqslant s - 1$.

If $|S\backslash T| = |T\backslash S| \geqslant s$, then we can similarly obtain $d_G(S\backslash T) > \lambda_s(G)$ and $d_G(T\backslash S) > \lambda_s(G)$ by (b), which implies $d_G(S\backslash T) + d_G(T\backslash S) > 2\lambda_s(G)$, a contradiction. Thus $|S\backslash T| \leqslant s - 1$. □

Lemma 21. *Let G be a connected triangle-free vertex-transitive 5-regular graph of even order. For $s = 5$ or 6, suppose $\lambda_s(G) = s + 9$. Then $|S| \geqslant s + 5$ for a λ_s-atom S of G.*

Proof. Suppose, to the contrary, that $|S| < s + 5$. As $s + 9 = d_G(S) = 5|S| - 2|E(S)|$, $|S|$ and s have different parities. Hence $|S| \geqslant s + 1$. By Lemma 20(b), $\delta(G[S]) \geqslant 3$. If $|S| = s + 1$, then $2|E(S)| \geqslant \delta(G[S])|S| \geqslant 3|S|$, which implies $d_G(S) = 5|S| - 2|E(S)| \leqslant 2|S| = 2s + 2 < s + 9$, a contradiction. Thus $|S| = s + 3$. Let R be the set of vertices u in S with $d_{G[S]}(u) = 3$. By Lemma 20(b), $E(R) = \emptyset$. Noting $3s + 9 \leqslant \sum_{u \in S} d_{G[S]}(u) = 2|E(S)| = 5|S| - \lambda_s(G) = 4s + 6$, we have $|R| \geqslant |S| - (4s + 6 - 3s - 9) = 6$. Since $s = 5$ or 6, $d_{G[S]}(R) = 3|R| \geqslant 18 > 5(s - 3) \geqslant d_{G[S]}(S\backslash R)$, a contradiction. □

Lemma 22. *Let G be a bicritical graph. If G is not 4-factor-critical, then there is a subset $X \subseteq V(G)$ with $|X| \geqslant 4$ such that $c_0(G - X) = |X| - 2$ and every component of $G - X$ is factor-critical.*

Proof. Since G is not 4-factor-critical, there is a set X_1 of four vertices of G such that $G - X_1$ has no perfect matchings. By Theorem 5, $G - X_1$ has a vertex set X_2 such that X_2 is matchable to $\mathscr{C}_{G-X_1-X_2}$ and every component of $G - X_1 - X_2$ is factor-critical. Set $X = X_1 \cup X_2$. Then $c_0(G - X) = |\mathscr{C}_{G-X}| > |X_2| = |X| - 4$. Since G is bicritical, we have $c_0(G - X) \leqslant |X| - 2$ by Theorem 3. Hence $|X| - 4 < c_0(G - X) \leqslant |X| - 2$. Noting that $c_0(G - X)$ and $|X|$ have the same parity, we have $c_0(G - X) = |X| - 2$. □

In the rest of this section, we always suppose that G is a connected non-bipartite vertex-transitive graph of degree $k \geqslant 5$ and even order, but G is not 4-factor-critical. Also we always use the following notation. Let X be a subset of $V(G)$ with $|X| \geqslant 4$ such that $c_0(G - X) = |X| - 2$ and every component of $G - X$ is factor-critical. By Theorem 1 and Lemma 22, such subset X exists. Let $H = H_1, H_2, \ldots, H_p$ be the nontrivial components of $G - X$. For a positive integer m, let $[m]$ denote the set $\{1, 2, \ldots, m\}$.

Lemma 23. *We have $p \geqslant 1$. Furthermore, if $g(G) > 3$, then*
(a) $p = 1$ if $\lambda_5(G) > 2k$,
(b) $|X| \geqslant 7$ and $|V(H)| \geqslant 9$ if $\lambda_5(G) > 4k-8$ and $5 \leqslant k \leqslant 6$, and
(c) $|X| \geqslant 10$ and $|V(H)| \geqslant 15$ if $\lambda_6(G) \geqslant 14$ and $k = 5$.

Proof. If $p = 0$, then $|V(G)| = 2|X| - 2 \geqslant 2k - 2 \geqslant 8$ and $\alpha(G) \geqslant |\overline{X}| = \frac{1}{2}|V(G)| - 1 > \max\{\frac{1}{3}|V(G)|, \frac{1}{2}|V(G)| - \frac{k}{4}\}$, which contradicts Lemma 16. Thus $p \geqslant 1$.

Next we suppose $g(G) > 3$. For each $i \in [p]$, we have $|V(H_i)| \geqslant 5$ as H_i is triangle-free and factor-critical.

Suppose $\lambda_5(G) > 2k$. By Lemma 20(a), $d(H_i) \geqslant \lambda_5(G)$ for each $i \in [p]$. We have $2pk < p\lambda_5(G) \leqslant \sum_{i=1}^{p} d(H_i) = d(X) - k(c_0(G-X) - p) \leqslant k(p+2)$, which implies $p < 2$. Thus $p = 1$. (a) is proved.

Suppose $\lambda_5(G) > 4k-8$ and $5 \leqslant k \leqslant 6$. We know $p = 1$ by (a). Assume $k = 6$. Notice that G is non-bipartite. It follows from Lemma 19 that $|X| \geqslant 7$. As $d(H) \leqslant 3k$ and H is triangle-free and factor-critical, we have $|V(H)| \geqslant 9$. Assume next $k = 5$. Notice that $|V(G)| = |V(H)| + 2|X| - 3 \geqslant 12$. By Lemma 20(a), $d(A) \geqslant \lambda_5(G) > 12$ for every subset $A \subseteq V(G)$ with $|A| = 6$, which implies that G has no subgraphs which are isomorphic to the complete bipartite graph $K_{3,3}$. By the vertex-transitivity of G, it follows that G has also no subgraphs which are isomorphic to $K_{2,5}$. So $|X| \geqslant 7$. If $E(X) = \emptyset$, then $g_0(G) \geqslant 7$ by Lemma 17, which implies $|V(H)| \geqslant 13$. If $E(X) \neq \emptyset$, then $d(H) = 13$, which implies $|V(H)| \geqslant 9$. Hence the statement (b) holds.

Now we suppose $\lambda_6(G) \geqslant 14$ and $k = 5$. Then $\lambda_5(G) \geqslant \min\{\lambda_6(G), 5k-12\} = 13$. We know $p = 1$ by (a). By the above argument, we know $|X| \geqslant 7$, $|V(H)| \geqslant 9$ and that G has no subgraphs which are isomorphic to $K_{2,5}$ or $K_{3,3}$. By Lemma 20(a), $d(V(H) \cup A) \geqslant \lambda_6(G)$ and $d(V(H) \backslash A) \geqslant \lambda_6(G)$ for every subset $A \subseteq V(G)$ with $|A| \leqslant 2$. It implies that $E(X) = \emptyset$, $|\nabla(u) \cap \nabla(H)| \leqslant 3$ for each $u \in V(G)$ and each of X and $V(H)$ has at most one vertex v with $|\nabla(v) \cap \nabla(H)| = 3$. Set $Y = \overline{V(H) \cup X}$.

Suppose $|X| = 7$. Then X has one vertex u_1 with 3 neighbors in $V(H)$ and other vertices in X has exactly two neighbors in $V(H)$. Choose a vertex $u_2 \in X \backslash \{u_1\}$ and a vertex $u_3 \in Y \backslash N(u_1)$. Since G is vertex-transitive, there is an automorphism φ_1 of G such that $\varphi_1(u_3) = u_2$. Noting that $|N(v) \cap N(u_3)| \geqslant 3$ for each $v \in Y$, we have $\varphi_1(Y) \subseteq X$, which implies $|\nabla(v) \cap \nabla(H)| = 3$ for each $v \in N(u_2) \cap V(H)$, a contradiction.

Suppose $8 \leqslant |X| \leqslant 9$. Then there are two vertices u_4 and u_5 in X with $|N(u_4) \cap V(H)| = 2$ and $|N(u_5) \cap V(H)| \leqslant 1$. Since G is vertex-transitive, there is an automorphism φ_2 of G such that $\varphi_2(u_5) = u_4$. Then $\varphi_2(Y) \cap V(H) \neq \emptyset$ and $|\varphi_2(Y) \cap Y| \geqslant 2$. As G has no subgraphs which are isomorphic to $K_{2,5}$ or $K_{3,3}$, it follows that $|\varphi_2(X) \cap X| \geqslant 6$. Hence $\varphi_2(Y) \subseteq V(H) \cup Y$ and $\varphi_2(X) \subseteq V(H) \cup X$. Noting that $|\nabla(u) \cap \nabla(H)| \leqslant 3$ and $N(u) \subseteq \varphi_2(X)$ for each $u \in \varphi_2(Y) \cap V(H)$, we have $|\varphi_2(X) \cap V(H)| \geqslant 2$. Notice that each of X and $V(H)$ has at most one vertex v with $|\nabla(v) \cap \nabla(H)| = 3$. We know $d_{G[\varphi_2(X \cup Y)]}(\varphi_2(X) \cap V(H)) \geqslant 3$, which implies $|\varphi_2(Y) \cap V(H)| \geqslant 3$. It follows that $N_H(\varphi_2(Y) \cap V(H)) \geqslant 3$. Now we have $|\varphi_2(X) \cap X| = 6$ and $|\varphi_2(X) \cap V(H)| = 3$ as $|\varphi_2(X)| = |X| \leqslant 9$. It follows that $G[\varphi_2(X \cup Y) \cap V(H)]$ contains a subgraph isomorphic to $K_{3,3}$ if $|\varphi_2(Y) \cap V(H)| \geqslant 4$ and $G[\varphi_2(X \cup Y) \backslash V(H)]$ contains a subgraph isomorphic to $K_{3,3}$ otherwise, a contradiction.

Thus $|X| \geqslant 10$. Then $g_0(G) \geqslant 9$ by Lemma 17. Let C be a $g_0(H)$-cycle of H. Then $g_0(H) \geqslant g_0(G) \geqslant 9$ and $|N_H(v) \cap V(C)| \leqslant 2$ for each $v \in V(H) \backslash V(C)$. Noting $15 = d(V(H)) = 5|V(H)| - 2|E(H)|$, we can easily verify $|V(H)| \geqslant 15$. (c) is proved. $\square$

Lemma 24. *Suppose $k = 5$, $\lambda_6(G) = \lambda_7(G) = 12$ and $g(G) > 3$. For a λ_7-atom S of G, we have that S is an imprimitive block of G.*

Proof. Suppose, to the contrary, that S is not an imprimitive block of G. Then there is an automorphism φ_1 of G such that $\varphi_1(S) \neq S$ and $\varphi_1(S) \cap S \neq \emptyset$. Set $T = \varphi_1(S)$. By Lemma 20(c), we have $|S \cap T| \leqslant 6$ and $|S \backslash T| \leqslant 6$, which implies $|S| \leqslant 12$. As $12 = d(S) = 5|S| - 2|E(S)|$, $|S|$ is an even integer. By Lemma 20(b), $\delta(G[S]) \geqslant 3$. For each $u \in \overline{S}$, we have $d_G(S \cup \{u\}) \geqslant \lambda_6(G)$ by Lemma 20(a), which implies $|N_G(u) \cap S| \leqslant 2$. Noting that $\lambda_6(G) \geqslant \lambda_5(G) \geqslant \lambda_4(G) \geqslant \min\{4k-8, 5k-12, \lambda_6(G)\} = 12$, we have $\lambda_5(G) = \lambda_4(G) = 12$. By Lemma 23, $p = 1$. By Lemma 20(a), we have $d_G(H) \geqslant \lambda_5(G) = 12$. Then either $d_G(H) = 13$ and $|E(X)| = 1$, or $d_G(H) = 15$ and $E(X) = \emptyset$.

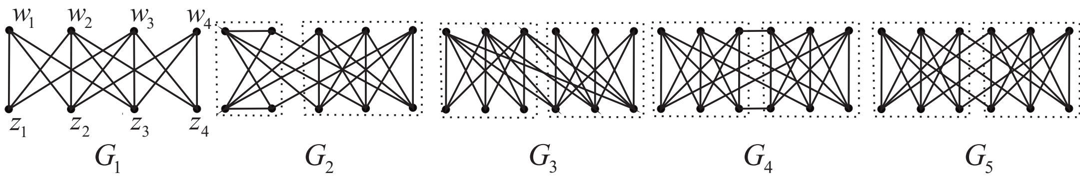

Figure 1. Some possible cases of $G[S]$. In each G_i, $2 \leqslant i \leqslant 5$, the two graphs in the virtual boxes correspond to $G[S \cap T]$ and $G[S \backslash T]$.

Case 1. $|S| = 8$.

We have $|E(S)| = \frac{1}{2}(5|S| - \lambda_6(G)) = 14$. It is easy to verify that $G[S]$ is isomorphic to G_1 in Figure 1. Label $G[S]$ as in G_1 and set $W = \{w_1, w_2, w_3, w_4\}$. As $|N_G(u) \cap S| \leqslant 2$ for each $u \in \overline{S}$, G has no vertex v different from w_1 such that $N_G(v) = N_G(w_1)$. Hence G has no subgraphs isomorphic to $K_{2,5}$ by the vertex-transitivity of G.

Claim 1. *Each edge in G is contained in a 4-cycle of G.*

Suppose that G has an edge contained in no 4-cycles of G. Since G is vertex-transitive, each vertex in G is incident with an edge contained in no 4-cycles of G and there is an automorphism φ_2 of G such that $\varphi_2(w_1) = w_2$. As each edge in $G[S]$ is contained in a 4-cycle, we have $\varphi_2(N_{G[S]}(w_1)) \subseteq N_{G[S]}(w_2)$ and $N_{G[S]}(\varphi_2(z_i)) \subseteq \varphi_2(S)$ for each $i \in \{2, 3\}$. It implies $|S \cap \varphi_2(S)| \geqslant 7$. On the other hand, noting $\varphi_2(S) \neq S$, we have $|S \cap \varphi_2(S)| \leqslant 6$ by Lemma 20(c), a contradiction. Thus Claim 1 holds.

Claim 2. *For any vertex $x \in V(G)$ with $2 \leqslant |\nabla(x) \cap \nabla(H)| \leqslant 3$ such that $d_{G[X]}(u) = 0$ for each $u \in (\{x\} \cup N_G(x)) \cap X$, there is a subset $A \subseteq N_G(x)$ with $|A| \geqslant |\nabla(x) \cap \nabla(H)| - 1$ and a vertex $y \in V(G) \backslash \{x\}$ such that $\{xu, yu\} \subseteq \nabla(H)$ and $|\nabla(u) \cap \nabla(H)| \geqslant 3$ for each $u \in A$.*

Since G is vertex-transitive, there is an automorphism φ_3 of G such that $\varphi_3(w_2) = x$. Let T_1 be one of X and $V(H)$ such that $x \in T_1$, and let T_2 be the other of X and $V(H)$.

Then $\varphi_3(w_3) \in T_1$ and $|\varphi_3(N_{G[S]}(w_2)) \cap T_2| \geqslant |\nabla(x) \cap \nabla(H)| - 1$. If $|\varphi_3(N_{G[S]}(w_2)) \cap T_2| \leqslant 2$ or $\varphi_3(W) \subseteq T_1$, then we choose A to be $\varphi_3(N_{G[S]}(w_2)) \cap T_2$. If $|\varphi_3(N_{G[S]}(w_2)) \cap T_2| = 3$ and $\varphi_3(W) \backslash T_1 \neq \emptyset$, then $|\varphi_3(W) \cap T_1| = 3$ and $\{\varphi_3(z_2), \varphi_3(z_3)\} \subseteq T_2$. In the second case, we choose A to be $\{\varphi_3(z_2), \varphi_3(z_3)\}$. Then A and $\varphi_3(w_3)$ are a subset and a vertex which satisfy the condition. Thus Claim 2 holds.

Subcase 1.1. $d_G(H) = 13$.

Let x_1x_2 be the edge in $E(X)$. We know $|X| \geqslant 6$ and $|V(H)| \geqslant 7$. By Lemma 20(a), $d_G(V(H) \cup A) \geqslant \lambda_4(G)$ and $d_G(V(H) \backslash A) \geqslant \lambda_4(G)$ for each subset $A \subseteq V(G)$ with $|A| \leqslant 2$, which implies that $|\nabla(u) \cap \nabla(H)| \leqslant 3$ for each $u \in V(G)$ and each of X and $V(H)$ has at most one vertex v with $|\nabla(v) \cap \nabla(H)| = 3$. Hence it follows from Claim 2 that $|\nabla(u) \cap \nabla(H)| \leqslant 2$ for each $u \in X \backslash \{x_1, x_2\}$. This together with Claim 2 implies that $|\nabla(u) \cap \nabla(H)| \leqslant 1$ for each $u \in V(H) \backslash N_G(\{x_1, x_2\})$.

We claim $|\nabla(u) \cap \nabla(H)| \leqslant 2$ for each $u \in N_G(\{x_1, x_2\}) \cap V(H)$. Otherwise, suppose that there is a vertex $u_0 \in N_G(\{x_1, x_2\}) \cap V(H)$ with $|\nabla(u_0) \cap \nabla(H)| = 3$. Since G is vertex-transitive, there is an automorphism φ_4 of G such that $\varphi_4(w_2) = u_0$. It implies that there is a vertex $u_1 \in \varphi_4(N_{G[S]}(w_2) \cap (X \backslash \{x_1, x_2\})$ such that $|\nabla(u_1) \cap \nabla(H)| = 3$, a contradiction.

Thus it follows from Claim 2 that $|\nabla(u) \cap \nabla(H)| \leqslant 1$ for each $u \in X \backslash \{x_1, x_2\}$. Noting $|N_G(\{x_1, x_2\}) \cap V(H)| \leqslant 5$, we have $|\nabla(N_G(\{x_1, x_2\}) \cap V(H)) \cap \nabla(H)| \leqslant 10$ by the claim in the previous paragraph. Hence there is an edge $x_3x_4 \in \nabla(H)$ such that $x_3 \in X \backslash \{x_1, x_2\}$ and $|\nabla(x_3) \cap \nabla(H)| = |\nabla(x_4) \cap \nabla(H)| = 1$. Then x_3x_4 is contained in no 4-cycles of G, contradicting Claim 1. Hence Subcase 1.1 cannot occur.

Subcase 1.2. $d_G(H) = 15$.

Notice that G has no subgraphs which are isomorphic to $K_{2,5}$. We know $|X| \geqslant 6$. Next we show $|V(H)| \geqslant 9$. Let O_i be the set of vertices u in G with $|\nabla(u) \cap \nabla(H)| = i$ for $1 \leqslant i \leqslant 5$. If $|X| \geqslant 7$, then $g_0(G) \geqslant 7$ by Lemma 17, which implies $|V(H)| \geqslant 13$. Then we assume $|X| = 6$. As G has no subgraphs which are isomorphic to $K_{2,5}$, we have $|O_3 \cap X| = 3$ and $|O_2 \cap X| = 3$. Noting $g(G) > 3$, we can obtain $|V(H)| \neq 5$. By Claim 2, $|O_3 \cap V(H)| \geqslant 2$, which implies $|V(H)| \neq 7$. Hence $|V(H)| \geqslant 9$.

By Lemma 20(a), $d_G(V(H) \cup A) \geqslant \lambda_4(G)$ and $d_G(V(H) \backslash A) \geqslant \lambda_4(G)$ for each subset $A \subseteq V(G)$ with $|A| \leqslant 4$. It implies $O_5 = \emptyset$, $|O_4 \cap X| \leqslant 1$, $|O_3 \cap X| \leqslant 3$, $|O_3 \cap V(H)| \leqslant 3$ and $|O_4 \cap X| \cdot |O_3 \cap X| = 0$.

We claim $O_4 = \emptyset$. Otherwise, suppose $O_4 \neq \emptyset$. Noting that $\delta(H) \geqslant 2$ as H is factor-critical, we have $O_4 \subseteq X$. Now we know $|O_4| = 1$ and $O_3 \cap X = \emptyset$. It follows from Claim 2 that $O_3 \cap V(H) = \emptyset$ and $O_2 \subseteq N_G(O_4)$. As $|\nabla(N_G(O_4) \cap V(H))| \leqslant 8$, there is an edge $x_5x_6 \in \nabla(H)$ with $\{x_5, x_6\} \subseteq O_1$. Then x_5x_6 is contained in no 4-cycles of G, contradicting Claim 1.

Let F_1 be the subgraph of G with vertex set $\bigcup_{i=1}^{3} O_i$ and edge set $\nabla(H)$ and let F_2 be the subgraph of F_1 which is induced by O_3. By Claim 2, $\delta(F_2) \geqslant 2$. Hence F_2 is connected. Then F_1 is connected by Claims 1 and 2. Let t be the number of vertices u in F_2 with $d_{F_2}(u) = 2$. We have $15 = |E(F_1)| \leqslant |E(F_2)| + 2t = 6|O_3| - 3|E(F_2)|$ by Claim 2. It follows that $|O_3| = 6$ and $6 \leqslant |E(F_2)| \leqslant 7$.

Assume $|E(F_2)| = 6$. Then F_2 is a 6-cycle. For each $u \in O_3 \cap X$, there is a vertex

$y_u \in X\backslash O_3$ such that $N_{F_2}(u) \subseteq N_G(y_u)$ by Claim 2. It implies that there is a vertex $y \in X\backslash O_3$ such that $O_3 \cap V(H) \subseteq N_G(y)$, which contradicts $|O_3 \cap X| \leqslant 3$.

Assume $|E(F_2)| = 7$. As $|E(F_1)\backslash E(F_2)| = 8$, it follows from Claim 2 that there is a vertex $u_2 \in V(F_1)\backslash O_3$ with $d_{F_1}(u_2) = 2$ and we know $|N_{F_1}(u_2) \cap O_3| = 1$ and $d_{F_1}(N_{F_1}(u_2)\backslash O_3) = 1$. It is easy to see that there is no vertex u' in G such that $|N_G(u') \cap N_G(u_2)| = 4$. Noting $|N_G(w_2) \cap N_G(w_3)| = 4$, we have that there is no automorphism φ of G such that $\varphi(w_2) = u_2$, which contradicts the vertex-transitivity of G.

Case 2. $|S| = 10$ *or* 12.

Claim 3. *For any given two distinct λ_7-atoms S_1 and S_2 of G with $S_1 \cap S_2 \neq \emptyset$, $G[S_1 \cap S_2]$ and $G[S_1\backslash S_2]$ are isomorphic to $K_{3,3}$ or $K_{2,2}$.*

By Lemma 20(c), we have $d_G(S_1 \cap S_2) + d_G(S_1 \cup S_2) \leqslant 2\lambda_7(G)$, $d_G(S_1\backslash S_2) + d_G(S_2\backslash S_1) \leqslant 2\lambda_7(G)$, $|S_1 \cap S_2| \leqslant 6$ and $|S_1\backslash S_2| \leqslant 6$. Then $|S_1 \cap S_2| \geqslant 4$ and $|S_1\backslash S_2| \geqslant 4$. By Lemma 20(a), each of $d_G(S_1 \cap S_2)$, $d_G(S_1 \cup S_2)$, $d_G(S_1\backslash S_2)$ and $d_G(S_2\backslash S_1)$ is not less than $\lambda_4(G)$. Noting $\lambda_4(G) = \lambda_7(G) = 12$, we have $d_G(S_1 \cap S_2) = d_G(S_1\backslash S_2) = 12$. Then $G[S_1 \cap S_2]$ and $G[S_1\backslash S_2]$ are isomorphic to $K_{3,3}$ or $K_{2,2}$. So Claim 3 holds.

Let R_i be the set of vertices u in S with $d_{G[S]}(u) = i$ for $3 \leqslant i \leqslant 5$. By Lemma 20(b), $E(G[R_3]) = \emptyset$.

Claim 4. *$R_5 = \emptyset$, or $G[R_5]$ is a 6-cycle and $|S| = 12$.*

Suppose $R_5 \neq \emptyset$. It only needs to show that $|S| = 12$ and $G[R_5]$ is a 6-cycle. Assume $R_4 \neq \emptyset$. Choose a vertex $u \in R_4$ and a vertex $v \in R_5$. Let φ_5 be an automorphism of G such that $\varphi_5(u) = v$. Then $\varphi_5(N_{G[S]}(u)) \subseteq N_{G[S]}(v)$, which contradicts that $G[\varphi_5(S) \cap S]$ is isomorphic to $K_{3,3}$ or $K_{2,2}$ by Claim 3. Thus $R_4 = \emptyset$. Noting $|R_3| + |R_5| = |S|$ and $3|R_3| + 5|R_5| = 2|E(S)| = 5|S| - 12$, we have $|R_3| = 6$. For any two vertices $u', u'' \in R_5$, it follows from Claim 3 that $\varphi(S) = S$ for every automorphism φ of G with $\varphi(u') = u''$. Hence $G[R_5]$ is r-regular, for some integer r. Then $18 = 3|R_3| = d_{G[S]}(R_3) = d_{G[S]}(R_5) = (5 - r)(|S| - 6)$, which implies $|S| = 12$ and $r = 2$. Hence $G[R_5]$ is a 6-cycle and Claim 4 is proved.

By Claim 3, $G[S \cap T]$ and $G[S\backslash T]$ are isomorphic to $K_{3,3}$ or $K_{2,2}$. Noting $E(G[R_3]) = \emptyset$, we have by Claim 4 that $G[S]$ is isomorphic to G_2, G_3, G_4 or G_5 in Figure 1.

Claim 5. *Each vertex in G is contained in exactly two distinct λ_7-atoms of G.*

By the vertex-transitivity of G, it only needs to show that $S' = S$ or $S' = T$ for a λ_7-atom S' of G with $S' \cap S \cap T \neq \emptyset$. Suppose $S' \neq S$ and $S' \neq T$. From Figure 1, we can see that S has no subset A different from $S \cap T$ and $S\backslash T$ such that $G[A]$ is isomorphic to $K_{3,3}$. Hence it follows from Claim 3 that $S' \cap S = S \cap T = S' \cap T$. Then $12 = d_G(S \cap T) \geqslant d_{G[S]}(S \cap T) + d_{G[T]}(S \cap T) + d_{G[S']}(S \cap T) = 18$, a contradiction. So Claim 5 holds.

Suppose $|S| = 10$. Then $G[S]$ is isomorphic to G_2. By Claims 3 and 5, there is a λ_7-atom S'' of G such that $S'' \cap S = S\backslash T$. Choose a vertex $u_3 \in S\backslash T$ and a vertex $u_4 \in S \cap T$. Noting that $G[S\backslash T]$ is not isomorphic to $G[S \cap T]$, we know by Claim 5 that there is no automorphism φ of G such that $\varphi(u_3) = u_4$, a contradiction.

Suppose next $|S| = 12$. Then $G[S]$ is isomorphic to G_3, G_4 or G_5. Let $V_1, V_2, \ldots, V_m$ be all subsets of $V(G)$ which induce subgraphs of G isomorphic to $K_{3,3}$. Noting that $G[S \cap T]$ and $G[S \backslash T]$ are isomorphic to $K_{3,3}$, we can obtain by Claims 3 and 5 that $V_1, V_2, \ldots, V_m$ form a partition of $V(G)$ and for each V_i there are exactly two elements $j_1, j_2 \in \{1, 2, \ldots, m\} \backslash \{i\}$ such that $G[V_i \cup V_{j_1}]$ and $G[V_i \cup V_{j_2}]$ are isomorphic to $G[S]$. We denote $V_i \sim V_j$ if $G[V_i \cup V_j]$ is isomorphic to $G[S]$, and assume $V_1 \sim V_2 \sim \cdots \sim V_m \sim V_1$. If $G[S]$ is isomorphic to G_3, then it is easy to verify that G is bipartite, a contradiction. Thus $G[S]$ is isomorphic to G_4 or G_5.

Assume that there is some $V_q \subseteq \overline{V(H)}$. If $G[S]$ is isomorphic to G_4, then $|V_q \cap X| = 3$ and $N_G(V_q \backslash X) \cap V_{q-1} \subseteq X$, which implies $|E(X)| \geqslant |E(V_{q-1}) \cap E(X)| \geqslant 2$, a contradiction. Thus $G[S]$ is isomorphic to G_5. Let V_j be chosen such that $V_j \cap V(H) \neq \emptyset$ and $|j - q|$ is as small as possible. Then $|V_j \cap X| = 3$ and $|N(u) \cap X| \geqslant 4$ for each $u \in V_j \cap V(H)$, which contradicts that $\delta(H) \geqslant 2$.

We now assume that $V_i \cap V(H) \neq \emptyset$ for $1 \leqslant i \leqslant m$. Then $|V_i \cap X| > |V_i \backslash (V(H) \cup X)|$ if $V_i \cap X \neq \emptyset$. Choose some $V_{q'}$ which contains vertices in $V(G) \backslash (V(H) \cup X)$. Then $V_{q'-1} \cap X \neq \emptyset$ and $V_{q'+1} \cap X \neq \emptyset$. Noting $c_0(G - X) = |X| - 2$, we can obtain that for each $i \in [m]$, $|V_i \cap X| = |V_i \backslash (V(H) \cup X)| + 1$ if $i \in \{q' - 1, q', q' + 1\}$ and $|V_i \cap X| = \emptyset$ otherwise. Then $|V_{q'} \backslash (V(H) \cup X)| = 2$. Hence $|V_{q'-1} \cap X)| = |V_{q'+1} \cap X)| = 3$. Now we have $V_{q'-1} \sim V_{q'} \sim V_{q'+1} \sim V_{q'-1}$, which implies $V(G) = V_{q'-1} \cup V_{q'} \cup V_{q'+1}$ and $|V(H)| = 3$. It follows that $g(G) = 3$, a contradiction. □

Lemma 25. *Suppose $k = 5$, $\lambda_5(G) = \lambda_6(G) = 13$ and $g(G) > 3$. For a λ_6-atom S of G, we have $|S| \geqslant 11$.*

Proof. To the contrary, suppose $|S| < 11$. As $13 = d(S) = 5|S| - 2|E(S)|$, $|S|$ is odd. Then $|S| \geqslant 7$. By Lemma 20(b), $\delta(G[S]) \geqslant 3$. By Lemma 23, we have $p = 1$, $|X| \geqslant 7$ and $|V(H)| \geqslant 9$. Hence $|V(G)| \geqslant 20$.

Assume $|S| = 7$. Then $|E(S)| = \frac{1}{2}(5|S| - 13) = 11$. If $G[S]$ is bipartite, then $|E(S)| \geqslant \frac{1}{2}(|S| + 1)\delta(G[S]) \geqslant 12$, a contradiction. Thus $G[S]$ is non-bipartite. Let C be a shortest cycle of odd length in $G[S]$. Then $5 \leqslant |V(C)| \leqslant 7$. Noting that $|N_{G[S]}(u) \cap V(C)| \leqslant 2$ for each $u \in S \backslash V(C)$, we have $|E(S)| \leqslant 10$, a contradiction.

So $|S| = 9$. Let R_i be the set of vertices u in S with $d_{G[S]}(u) = i$ for $3 \leqslant i \leqslant 5$.

Claim 1. *For any automorphism φ of G with $\varphi(R_4 \cup R_5) \cap (R_4 \cup R_5) \neq \emptyset$, either $\varphi(S) = S$ or $G[S \cap \varphi(S)]$ is isomorphic to $K_{2,3}$.*

Suppose $\varphi(S) \neq S$. By Lemma 20(c), $|S \cap \varphi(S)| \leqslant 5$, $|S \backslash \varphi(S)| \leqslant 5$ and $d(S \cap \varphi(S)) + d(S \cup \varphi(S)) \leqslant 2\lambda_6(G)$. Then $4 \leqslant |S \cap \varphi(S)| \leqslant 5$ and $|S \cup \varphi(S)| = |S| + |\varphi(S)| - |S \cap \varphi(S)| \leqslant 14$. As $|V(G)| \geqslant 20$, we have $d(S \cup \varphi(S)) \geqslant \lambda_6(G)$ by Lemma 20(a). Hence $d(S \cap \varphi(S)) \leqslant \lambda_6(G) = 13$. This, together with the fact that $|N_{G[\varphi(S)]}(u) \cap N_{G[S]}(u)| \geqslant 3$ for each $u \in \varphi(R_4 \cup R_5) \cap (R_4 \cup R_5)$, implies that $G[S \cap \varphi(S)]$ is isomorphic to $K_{2,3}$. So Claim 1 holds.

By Claim 1, it follows that G has no automorphism φ such that $\varphi(R_4) \cap R_5 \neq \emptyset$. It implies $R_4 = \emptyset$ or $R_5 = \emptyset$. Noting $\sum_{i=3}^{5} i|R_i| = 2|E(S)| = 32$ and $\sum_{i=3}^{5} |R_i| = |S| = 9$, we have $|R_3| = 4$, $|R_4| = 5$ and $R_5 = \emptyset$. By Lemma 20(b), $E(R_3) = \emptyset$. Hence $|E(R_4)| = 4$.

As $g(G[S]) \geqslant g(G) > 3$, it is easy to verify that either $G[R_4]$ has a 4-cycle or $G[R_4]$ is isomorphic to $K_{1,4}$. Let u_1 and u_2 be two vertices in R_4 with $d_{G[R_4]}(u_1) < d_{G[R_4]}(u_2)$. Since G is vertex-transitive, there is an automorphism ψ of G such that $\psi(u_2) = u_1$. By Claim 1, $G[\psi(S) \cap S]$ is isomorphic to $K_{2,3}$. As $u_1, u_2 \in R_4$, we know $d_{G[\psi(S)\cap S]}(u_1) = 3$. Notice that $|N_{G[S]}(u) \cap N_{G[S]}(u_1)| \leqslant 2$ for each $u \in S\backslash\{u_1\}$ if $G[R_4]$ has a 4-cycle. It follows that $G[R_4]$ is isomorphic to $K_{1,4}$. Since $d_{G[\psi(S)\cap S]}(v) = 2$ for each $v \in N_{G[\psi(S)\cap S]}(u_1)$, it follows that $N_{G[\psi(S)\cap S]}(u_1) \subseteq R_3$. It implies that the vertex in $R_3\backslash N_{G[S]}(u_1)$ has only two neighbors in S, which contradicts $\delta(G[S]) \geqslant 3$. □

Lemma 26. *Suppose $k = 5$, $\lambda_6(G) = \lambda_7(G) = 14$ and $g(G) > 3$. For a λ_7-atom S of G, we have $|S| \geqslant 14$.*

Proof. By Lemma 23, we have $p = 1$, $|X| \geqslant 10$ and $|V(H)| \geqslant 15$. Hence $|V(G)| \geqslant 32$. For $1 \leqslant i \leqslant 5$, let O_i be the set of vertices u in G with $|\nabla(u) \cap (V(H))| = i$, and set $m_i = |O_i \cap X|$ and $n_i = |O_i \cap V(H)|$. By Lemma 20(a), $d_G(V(H) \cup A) \geqslant \lambda_6(G)$ and $d_G(V(H)\backslash A) \geqslant \lambda_6(G)$ for each subset A of $V(G)$ with $|A| \leqslant 2$. This, together with the fact that $d_G(H)$ is odd, implies that $d_G(H) = 15$, $O_4 \cup O_5 = \emptyset$, $m_3 \leqslant 1$ and $n_3 \leqslant 1$. Hence $E(X) = \emptyset$. Then $g_0(G) \geqslant 9$ by Lemma 17.

Suppose $|S| < 14$. As $5|S| - 2|E(G[S])| = 14$, $|S|$ is an even integer with $8 \leqslant |S| \leqslant 12$. By Lemma 20(b), $\delta(G[S]) \geqslant 3$. As $g_0(G) \geqslant 9$, it follows that $G[S]$ is bipartite. By Lemma 20(a), $d_G(S \cup \{u\}) \geqslant \lambda_6(G)$ for each $u \in \overline{S}$ and $d_G(A) \geqslant \lambda_6(G)$ for each subset $A \subseteq V(G)$ with $|A| = 6$. Hence $|N_G(u) \cap S| \leqslant 2$ for each $u \in \overline{S}$ and G has no subgraphs which are isomorphic to $K_{3,3}$.

Claim 1. *For any two distinct λ_7-atoms S_1 and S_2 of G with $S_1 \cap S_2 \neq \emptyset$, we have $d_G(S_1 \cap S_2) \leqslant 14$ and furthermore, $G[S_1 \cap S_2]$ and $G[S_1\backslash S_2]$ are isomorphic to $K_{2,4}$ or $K_{3,3} - e$ if $|S| = 12$, where $K_{3,3} - e$ is a subgraph of $K_{3,3}$ obtained by deleting an edge e from $K_{3,3}$.*

By Lemma 20(c), we have $|S_1 \cap S_2| \leqslant 6$, $|S_1\backslash S_2| \leqslant 6$, $d_G(S_1 \cap S_2) + d_G(S_1 \cup S_2) \leqslant 2\lambda_7(G)$ and $d_G(S_1\backslash S_2) + d_G(S_2\backslash S_1) \leqslant 2\lambda_7(G)$. Noting $|V(G)| \geqslant 32$, we have $d_G(S_1 \cup S_2) \geqslant \lambda_7(G)$ by Lemma 20(a). Hence $d_G(S_1 \cap S_2) \leqslant \lambda_7(G) = 14$. Next assume $|S| = 12$. Then $|S_1 \cap S_2| = |S_1\backslash S_2| = 6$. By Lemma 20(a), each of $d_G(S_1 \cap S_2)$, $d_G(S_1\backslash S_2)$ and $d_G(S_2\backslash S_1)$ is not less than $\lambda_6(G)$. Hence $d_G(S_1 \cap S_2) = d_G(S_1\backslash S_2) = 14$. It implies that $G[S_1 \cap S_2]$ and $G[S_1\backslash S_2]$ are isomorphic to $K_{2,4}$ or $K_{3,3} - e$. So Claim 1 holds.

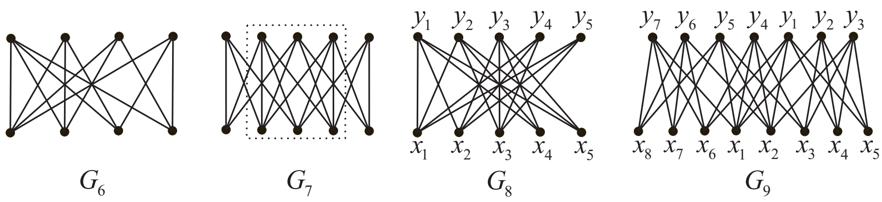

Figure 2. The illustration in the proof of Lemma 26.

Case 1. $|S| = 8$.

As $G[S]$ is a bipartite graph with $|E(S)| = 13$ and $\delta(G[S]) \geqslant 3$, $G[S]$ is isomorphic to G_6 in Figure 2. Let v_1, v_2 be the two vertices in S with $d_{G[S]}(v_1) = d_{G[S]}(v_2) = 4$ and choose a vertex $v_3 \in N_{G[S]}(v_1)\backslash\{v_2\}$.

We claim that each edge in G is contained in a 4-cycle of G. Otherwise, suppose that G has an edge contained in no 4-cycles of G. Since G is vertex-transitive, each vertex in G is incident with an edge contained in no 4-cycles of G and there is an automorphism φ_1 of G such that $\varphi_1(v_3) = v_2$. Clearly, $\varphi_1(S) \neq S$. Noting that each edge in $G[S]$ is contained in a 4-cycle of $G[S]$, we have $d_{G[\varphi_1(S)\cup S]}(u) \leqslant 4$ for each $u \in \varphi_1(S) \cup S$. Then $\varphi_1(N_{G[S]}(v_3) \subseteq N_{G[S]}(v_2)$ and $N_{G[S]}(\varphi_1(v_1)) \subseteq \varphi_1(N_{G[S]}(v_1))$. Noting that $|\varphi_1(S) \cap S| \leqslant 6$ by Lemma 20(c) and $d_G(\varphi_1(S)\cap S) \leqslant 14$ by Claim 1, $G[S\cap\varphi_1(S)]$ is isomorphic to $K_{3,3}-e$. As $d_G(S\cup\varphi_1(S)) \geqslant \lambda_6(G) = 14$ by Lemma 20(a), it follows that $G[S\cup\varphi_1(S)]$ is isomorphic to G_7 in Figure 2, where the graph in the virtual box corresponds to $G[S\cap\varphi_1(S)]$. Choose a vertex $v_4 \in S\cap\varphi_1(S)$ with $d_{G[S\cap\varphi_1(S)]}(v_4) = 2$. Let φ_2 be an automorphism of G such that $\varphi_2(v_1) = v_4$. Then $\varphi_2(N_{G[S]}(v_1)) = N_{G[S\cup\varphi_1(S)]}(v_4)$ and $\varphi_2(N_{G[S]}(v_2))\backslash(S \cup \varphi_1(S)) \neq \emptyset$. Then $d_G(S \cup \varphi_2(S) \cup \varphi_2(N_{G[S]}(v_2))) < 14 = \lambda_6(G)$, contradicting Lemma 20(a). So this claim holds.

For each $uv \in \nabla(H)$, noting that uv is contained in a 4-cycle of G by the previous claim, we have $|\nabla(u) \cap \nabla(H)| + |\nabla(v) \cap \nabla(H)| \geqslant 3$. For each $u \in O_2 \cup O_3$, there is an automorphism φ_3 of G such that $\varphi_3(v_1) = u$, which implies that there is a vertex $v \in \varphi_3(N_{G[S]}(v_1))$ such that $uv \in \nabla(H)$ and $|\nabla(v) \cap \nabla(H)| = 3$. Hence $m_1 \leqslant n_2 + 2n_3$, $m_2 \leqslant 3n_3$ and $n_2 \leqslant 3m_3$. Noting $m_3 \leqslant 1$ and $n_3 \leqslant 1$, we have $15 = \sum_{i=1}^{3} im_i \leqslant n_2 + 2n_3 + 6n_3 + 3m_3 \leqslant 6m_3 + 8n_3 \leqslant 14$, a contradiction.

Case 2. $|S| = 10$ *or* 12.

Let R_i be the set of vertices u in S with $d_{G[S]}(u) = i$ for $3 \leqslant i \leqslant 5$. Then $E(R_3) = \emptyset$ by Lemma 20(b). Let Z and W be the bipartition of $G[S]$ such that $|Z| \leqslant |W|$. Noting $\frac{1}{2}(5|S| - 14) = |E(S)| \geqslant \delta(G[S])|W| \geqslant 3|W|$, we have $|W| < \frac{1}{2}|S| + 2$.

Claim 2. *If $R_5 \neq \emptyset$, then, for each $v \in R_4$, there is exactly one vertex w in $S\backslash\{v\}$ with $N_{G[S]}(v) \subseteq N_{G[S]}(w)$.*

Suppose $R_5 \neq \emptyset$. Choose a vertex $u \in R_5$ and a vertex $v \in R_4$. Let φ_4 be an automorphism of G such that $\varphi_4(u) = v$. Then $N_{G[S]}(v) \subseteq \varphi_4(N_{G[S]}(u))$. Noting that $|S \cap \varphi_4(S)| \leqslant 6$ by Lemma 20(c) and $d_G(S \cap \varphi_4(S)) \leqslant 14$ by Claim 1, we have that $G[S \cap \varphi_4(S)]$ is isomorphic to $K_{2,4}$. It implies that S has a vertex w different from v with $N_{G[S]}(v) \subseteq N_{G[S]}(w)$. As G has no subgraphs isomorphic to $K_{3,3}$, such vertex w is unique. So Claim 2 holds.

Claim 3. *$|W| = |Z|$ and $R_5 = \emptyset$.*

Suppose, to the contrary, that $|W| > |Z|$, or $|W| = |Z|$ and $R_5 \neq \emptyset$. As $E(R_3) = \emptyset$, it follows that $|W| = 6$ if $|S| = 10$.

Assume $|W| = |Z| + 2 = 7$. Noting $|E(S)| = 23$, there is a vertex $v_5 \in (R_4 \cup R_5) \cap W$ and a vertex $v_6 \in R_5 \cap Z$. Let φ_5 be an automorphism of G such that $\varphi_5(v_5) = v_6$. Then $\varphi_5(S) \neq S$ and $\varphi_5(N_{G[S]}(v_5)) \subseteq N_{G[S]}(v_6)$. Hence $G[S \cap \varphi_5(S)]$ is isomorphic to $K_{2,4}$ by Claim 1. It implies $|\varphi_5(W)\backslash S| = 5$, contradicting that $G[\varphi_5(S)\backslash S]$ is isomorphic to $K_{2,4}$

or $K_{3,3} - e$ by Claim 1.

Assume $|W| = 6$. If $|S| = 10$, we know $|R_4 \cap Z| = |R_5 \cap Z| = 2$ as $E(R_3) = \emptyset$ and $|E(S)| = 18$. If $|S| = 12$, we know either $|R_5 \cap Z| = 2 = |R_4 \cap Z| + 1$ or $|R_5 \cap Z| = 1 = |R_4 \cap Z| - 2$ as $|E(S)| = 23$. It follows from Claim 2 that there is a vertex $v_7 \in R_4 \cap Z$ and a vertex $v_8 \in (R_4 \cup R_5)\backslash\{v_7\}$ such that $N_{G[S]}(v_7) \subseteq N_{G[S]}(v_8)$ and $(R_5 \cap Z)\backslash\{v_8\} \neq \emptyset$. It implies that $G[S]$ has a subgraph which is isomorphic to $K_{3,3}$, a contradiction. So Claim 3 holds.

Subcase 2.1. $|S| = 10$.

Noting $E(R_3) = \emptyset$, we have by Claim 3 that $G[S]$ is isomorphic to G_8 in Figure 2. We label $G[S]$ as in G_8 and assume $x_1 \in Z$ and $y_1 \in W$.

Claim 4. $|N_G(u) \cap N_G(v)| \leqslant 3$ *for any two distinct vertices u and v in G.*

Suppose that there are two distinct vertices u and v in G with $|N_G(u) \cap N_G(v)| \geqslant 4$. Notice that $|N_G(u) \cap S| \leqslant 2$ for each $u \in \overline{S}$. By the vertex-transitivity of G, for each $y_i \in \{y_1, y_2, y_3\}$ there is a vertex $y_j \in \{y_1, y_2, y_3\}\backslash\{y_i\}$ such that $|N_G(y_i) \cap N_G(y_j)| \geqslant 4$. It follows that there is a vertex $w \in \overline{S}$ such that $\{y_1, y_2, y_3\} \subseteq N_G(w)$, a contradiction. So Claim 4 holds.

Let φ_6 be an automorphism of G such that $\varphi_6(y_5) = y_1$. Then $\varphi_6(S) \neq S$ and $|\varphi_6(N_{G[S]}(y_5)) \cap N_{G[S]}(y_1)| \geqslant 2$. Then $|\varphi_6(S) \cap S| \leqslant 6$ by Lemma 20(c) and $d_G(\varphi_6(S) \cap S) \leqslant 14$ by Claim 1. By Claim 4, $G[S]$ has no subgraphs isomorphic to $K_{2,4}$. It follows that $|\varphi_6(S) \cap W| \leqslant 3$ and $|\varphi_6(S) \cap Z| \leqslant 3$.

Assume that $\varphi_6(N_{G[S]}(y_5)) \cap \{x_1, x_2\} \neq \emptyset$ and $\varphi_6(N_{G[S]}(y_5)) \cap \{x_4, x_5\} \neq \emptyset$. Then $|N_{G[\varphi_6(S)]}(u) \cap N_{G[S]}(u)| = 3$ for each $u \in \varphi_6(N_{G[S]}(y_5)) \cap \{x_1, x_2\}$ and $|N_{G[\varphi_6(S)]}(v) \cap N_{G[S]}(v)| \geqslant 2$ for each $v \in \varphi_6(N_{G[S]}(y_5)) \cap \{x_4, x_5\}$. It follows that $|\varphi_6(S) \cap W| = 3$ and $|\varphi_6(S) \cap \{y_4, y_5\}| = 1$. Noting $2 \leqslant |\varphi_6(S) \cap Z| \leqslant 3$, we can see $d_G(\varphi_6(S) \cap S) > 14$, a contradiction.

Assume $\varphi_6(N_{G[S]}(y_5)) \cap N_{G[S]}(y_1) = \{x_4, x_5\}$. Then $\varphi_6(y_4) \in \{y_2, y_3\} \cup \overline{S}$, which implies that $|N_G(y_1) \cap N_G(\varphi_6(y_4))| \geqslant 4$ or $|N_G(x_4) \cap N_G(x_5)| \geqslant 4$. It contradicts Claim 4.

Thus $\varphi_6(N_{G[S]}(y_5)) \cap N_{G[S]}(y_1) = \{x_1, x_2\}$. By Claim 4, we have $\varphi_6(y_4) \in \{y_4, y_5\}$ and $\varphi_6(\{y_1, y_2, y_3\}) \cap W = \{y_4, y_5\}\backslash\varphi_6(y_4)$. Then $\{\varphi_6(x_4), \varphi_6(x_5)\} \subseteq \overline{S}$. Assume $\varphi_6(y_4) = y_4$. Set $\{y_6, y_7\} = \varphi_6(\{y_1, y_2, y_3\})\backslash W$, $\{x_6\} = \varphi_6(N_{G[S]}(y_5))\backslash N_{G[S]}(y_1)$ and $\{x_7, x_8\} = \{\varphi_6(x_4), \varphi_6(x_5)\}$. Then the graph G_9 showed in Figure 2 is a subgraph of G.

We can see that each edge incident with x_1 is contained in a 4-cycle of G. Then, by the vertex-transitivity of G, each edge $uv \in \nabla(H)$ is contained in a 4-cycle of G, which implies $|\nabla(u) \cap \nabla(H)| \geqslant 2$ or $|\nabla(v) \cap \nabla(H)| \geqslant 2$. Hence there is a vertex $u' \in G$ with $2 \leqslant |\nabla(u') \cap \nabla(H)| \leqslant 3$. Let φ_7 be an automorphism of G such that $\varphi_7(y_4) = u'$. It is easy to verify that either $\varphi_7(N_{G[\varphi_6(S) \cup S]}(y_4))$ has a vertex u with $|\nabla(u) \cap \nabla(H)| \geqslant 4$ or it has two vertices v' and v'' with $\{u'v', u'v''\} \subseteq \nabla(H)$ and $|\nabla(v') \cap \nabla(H)| = |\nabla(v'') \cap \nabla(H)| = 3$, contradicting the fact that $O_4 \cup O_5 = \emptyset$, $m_3 \leqslant 1$ and $n_3 \leqslant 1$.

Subcase 2.2. $|S| = 12$.

As $|E(G[S])| = 23$, $G[S]$ is not regular. Let φ_8 be an automorphism of G such that $\varphi_8(S) \neq S$ and $\varphi_8(S) \cap S \neq \emptyset$. Set $T = \varphi_8(S)$. It follows from Claims 1 and 3 that

$d_{G[S\cup T]}(u) = 5$ for each $u \in S\cap T$, each of $G[S\backslash T]$, $G[S\cap T]$ and $G[T\backslash S]$ is isomorphic to $K_{3,3}-e$ and $d_{G[S]}(v) = d_{G[T]}(v) = 4$ for each $v \in S\cap T$ with $d_{G[S\cap T]}(v) = 3$.

Let v_9 and v_{10} be two vertices in $W\cap T$ with $d_{G[S\cap T]}(v_9) = 3 = d_{G[S\cap T]}(v_{10}) + 1$. We know either $d_{G[S]}(v_{10}) = 4$ or $d_{G[T]}(v_{10}) = 4$ and assume, without loss of generality, that $d_{G[S]}(v_{10}) = 4$. Let φ_9 be an automorphism of G such that $\varphi_9(v_9) = v_{10}$. Let Q be one of $\varphi_9(S)$ and $\varphi_9(T)$ such that $Q \neq S$. Since $d_{G[Q]}(v_{10}) = 4$, we know $Q \neq T$. By Claims 1 and 3, each of $G[Q\cap S]$, $G[Q\backslash S]$, $G[Q\cap T]$ and $G[Q\backslash T]$ is isomorphic to $K_{3,3}-e$. Noting $d_{G[S]}(v_{10}) = d_{G[Q]}(v_{10}) = 4$, we have $|N_{G[Q]}(v_{10})\cap S| = 3$, which implies $2 \leqslant |Q\cap S\cap T| \leqslant 5$.

Assume $2 \leqslant |Q\cap S\cap T| \leqslant 3$. Noting that $G[Q\cap T]$ is isomorphic to $K_{3,3}-e$, we have $d_{G[Q\cap T]}(Q\cap S\cap T) > |Q\cap S\cap T| \geqslant d_{G[T]}(Q\cap S\cap T)$, a contradiction.

Assume $4 \leqslant |Q\cap S\cap T| \leqslant 5$. Then $|N_{G[Q]}(v_{10})\cap S\cap T| = 2$. If $E(Q\cap S\cap \overline{T}) = \emptyset$, then $d_{G[Q\cap S]}(Q\cap S\cap\overline{T}) + d_{G[Q\backslash T]}(Q\cap S\cap\overline{T}) \geqslant 4|Q\cap S\cap\overline{T}| > |[Q\cap S\cap\overline{T}, \overline{S}\cup T]|$, a contradiction. Thus $|Q\cap S\cap\overline{T}| = 2$ and $|E(Q\cap S\cap\overline{T})| = 1$. Then $d_{G[Q\cap S]}(Q\cap S\cap\overline{T}) + d_{G[Q\backslash T]}(Q\cap S\cap\overline{T}) \geqslant 3+3 > 5 \geqslant |[Q\cap S\cap\overline{T}, \overline{S}\cup T]|$, a contradiction. □

Lemma 27. *Suppose $k = 5$, $\lambda_6(G) \geqslant 14$, $\lambda_8(G) = 15$ and $g(G) > 3$. For a λ_8-atom S of G, we have $|S| \geqslant 15$.*

Proof. By Lemma 23, we have $p = 1$, $|X| \geqslant 10$ and $|V(H)| \geqslant 15$. By Lemma 20(a), $d_G(A) \geqslant \lambda_6(G) \geqslant 14$, $d_G(V(H)\cup B) \geqslant \lambda_8(G)$ and $d_G(V(H)\backslash B) \geqslant \lambda_8(G)$ for any two subsets A and B of $V(G)$ with $|A| = 6$ and $|B| \leqslant 1$. It implies that G has no subgraphs isomorphic to $K_{3,3}$, $d_G(H) = 15$ and $|\nabla(u)\cap\nabla(H)| \leqslant 2$ for each $u \in V(G)$. Hence $E(X) = \emptyset$ and there is an edge $u_1u_2 \in \nabla(H)$ such that $N_G(u_1)\cap X = \{u_2\}$. By Lemma 17, $g_0(G) \geqslant 9$.

Suppose $|S| < 15$. As $g_0(G) \geqslant 9$ and $15 = \lambda_8(G) = d_G(S) = 5|S| - 2|E(G[S])|$, it follows that $|S|$ is odd and $G[S]$ is bipartite. By Lemma 20(b), $\delta(G[S]) \geqslant 3$. Let W and Z be the bipartition of $G[S]$ such that $|W| > |Z|$. We have $|W| = \frac{1}{2}(|S|+1)$ if $|S| \leqslant 11$, and $7 \leqslant |W| \leqslant 8$ if $|S| = 13$.

Case 1. *There is a vertex v_1 in S with $d_{G[S]}(v_1) = 5$.*

Let R be one of W and Z such that $v_1 \in R$. As $\delta(G[S]) \geqslant 3$ and $|E(S)| = \frac{1}{2}(5|S| - 2|E(G[S])|)$, it follows that $N_{G[S]}(N_{G[S]}(v_1)) = R$. Since G is vertex-transitive, there is an automorphism φ_1 of G such that $\varphi_1(v_1) = u_2$. Then $\varphi_1(R) \subseteq X\cup V(H)$. Noting that $|\nabla(u)\cap\nabla(H)| \leqslant 2$ for each $u \in V(G)$, we have $\varphi_1(S\backslash R)\cap X = \emptyset$. Notice that G has no subgraphs isomorphic to $K_{3,3}$. We have $|\varphi_1(R)\cap X| \geqslant 4$ as $|N_G(u_2)\backslash V(H)| \geqslant 3$ and $\delta(G[S]) \geqslant 3$. Then $|\varphi_1(S)\cap V(H)| \leqslant 6$ as $|S| \leqslant 13$. It follows that $d_{G[\varphi_1(S)]}(u_1) = 3$. Then $d_{G[\varphi_1(S)]}(v) \geqslant 4$ for each $v \in N_{G[\varphi_1(S)]}(u_1)$ by Lemma 20(b). Now we know $|S| = 13$, $|\varphi_1(R)\cap X| = 4 = |\varphi_1(R)\cap V(H)| + 2$ and $|\varphi_1(S\backslash R)\cap V(H)| = 4 = |\varphi_1(S\backslash R)\backslash V(H)| + 1$. Then $R = Z$ and $|N_G(u_2)\cap V(H)| = 2$.

Noting that $|\nabla(u)\cap\nabla(H)| \leqslant 2$ for each $u \in V(G)$, we have $d_{G[\varphi_1(S)]}(u) \leqslant 4$ for each $u \in \varphi_1(W)$. Since $\delta(G[S]) \geqslant 3$ and G has no subgraphs isomorphic to $K_{3,3}$, two vertices in $\varphi_1(W)\backslash V(H)$ has exactly 3 neighbors in $\varphi_1(Z)\cap X$. So $d_{G[\varphi_1(S)]}(u) = 4$ for each $u \in \varphi_1(W)\backslash N_G(u_2)$ as $|E(S)| = 25$. Then there is a vertex $u_3 \in \varphi_1(Z)\cap X$ such that $\varphi_1(W)\backslash N_G(u_2) \subseteq N_G(u_3)$.

Assume $\varphi_1(Z)\cap V(H) = \{u_4, u_5\}$. Let φ_2 be an automorphism of G such that $\varphi_2(u_4) = u_2$. Then $u_1 \notin \varphi_2(N_{G[\varphi_1(S)]}(u_4))$ and $\varphi_2(\{u_2, u_3, u_5\}) \subseteq X$, which implies $|\nabla(u)\cap\nabla(H)| \geqslant 3$ for the vertex $u \in (N_G(u_2) \cap V(H))\backslash\{u_1\}$, a contradiction.

Case 2. $d_{G[S]}(u) \leqslant 4$ *for each* $u \in S$.

If $|S| = 13$, then, noting $|E(G[S])| = 25$ and $5 \leqslant |Z| \leqslant 6$, we have that there is a vertex $u \in Z$ with $d_{G[S]}(u) = 5$, a contradiction. Thus $|S| \leqslant 11$. There is a vertex $w \in W$ with $d_{G[S]}(w) = |W| - 2$ such that $d_{G[S]}(u) = 4$ for each $u \in N_{G[S]}(w)$. Choose a vertex $z \in N_{G[S]}(w)$.

We claim that the edge u_1u_2 is contained in a 4-cycle of G. Suppose not. Since G is vertex-transitive, each vertex in G is incident with an edge contained in no 4-cycles of G and there is an automorphism φ_3 of G such that $\varphi_3(w) = z$. We know $\varphi_3(S) \neq S$. Noting that $|N_{G[S]}(u) \cap N_{G[S]}(v)| \geqslant 2$ for every subset $\{u, v\} \subseteq Z$, each edge in $G[S]$ is contained in a 4-cycle of $G[S]$. Hence $\varphi_3(N_{G[S]}(w)) \subseteq N_{G[S]}(z)$ and $N_{G[S]}(u) \subseteq \varphi_3(S)$ for each $u \in \varphi_3(N_{G[S]}(w))$. By Lemma 20(c), $|S\cap\varphi_3(S)| \leqslant 7$ and $d_G(S\cap\varphi_3(S))+d_G(S\cup\varphi_3(S)) \leqslant 2\lambda_8(G)$. If $|S| = 11$, then $|S \cap \varphi_3(S)| \geqslant |\varphi_3(N_{G[S]}(w)) \cup N_{G[S]}(w)| = 8$, a contradiction. Thus $|S| = 9$. As $\delta(G[S]) \geqslant 3$, we have $Z = \bigcup_{u\in\varphi_3(N_{G[S]}(w))} N_{G[S]}(u) \subseteq \varphi_3(S)$. Hence $|S \cap \varphi_3(S)| = 7$ and $d_G(S \cap \varphi_3(S)) = 17$. Noting that $d_G(S \cup \varphi_3(S)) \geqslant \lambda_8(G)$ by Lemma 20(a), we have $d_G(S \cap \varphi_3(S)) + d_G(S \cup \varphi_3(S)) > 2\lambda_8(G)$, a contradiction.

Thus $|N_G(u_2) \cap V(H)| = 2$. Let φ_4 be an automorphism of G such that $\varphi_4(z) = u_2$ if $|S| = 9$, and $\varphi_4(w) = u_2$ if $|S| = 11$. If $u_1 \in \varphi_4(S)$, then $|Z| \geqslant d_{G[\varphi_4(S)]}(u_1) - 1 + |N_{G[\varphi_4(S)]}(N_{G[\varphi_4(S)]}(u_2)\backslash V(H))| \geqslant 2 + 3 = 5$ if $|S| = 9$, and $|W| \geqslant 7$ if $|S| = 11$, a contradiction. Thus $u_1 \notin \varphi_4(S)$. Then $\varphi_5(Z) \subseteq X$ if $|S| = 9$ and $\varphi_5(W) \subseteq X$ if $|S| = 11$, which implies $|\nabla(u) \cap \nabla(H)| \geqslant 3$ for the vertex $u \in (N_G(u_2) \cap V(H))\backslash\{u_1\}$, a contradiction. $\square$

Lemma 28. *Suppose* $k = 6$, $\lambda_5(G) = 16$ *and* $g(G) > 3$. *For a* λ_5*-atom* S *of* G, *we have* $|S| \geqslant 9$.

Proof. To the contrary, suppose $|S| \leqslant 8$. As $\frac{1}{2}(6|S| - \lambda_5(G)) = |E(S)| \leqslant \frac{1}{4}|S|^2$ by Lemma 6, we have $|S| \geqslant 8$. Hence $|S| = 8$ and $G[S]$ is isomorphic to $K_{4,4}$.

By Lemma 23, $p = 1$. Then $|X| \geqslant 7$ by Lemma 19. Noting that $d(H) \leqslant 18$ and H is triangle-free and factor-critical, we have $|V(H)| \geqslant 11$. Let O_i be the set of vertices u in G with $|\nabla(u)\cap\nabla(H)| = i$ for $4 \leqslant i \leqslant 6$. By Lemma 20(a), we have $d(V(H)\cup A) \geqslant \lambda_5(G)$ and $d(V(H)\backslash A) \geqslant \lambda_5(G)$ for each subset $A \subseteq V(G)$ with $|A| \leqslant 3$, which implies $d(H) \geqslant 16$, $O_5 \cup O_6 = \emptyset$, $|O_4 \cap X| \leqslant 1$ and $|O_4 \cap V(H)| \leqslant 1$.

Suppose that S is an imprimitive block of G. Then the orbits $S = S_1, S_2, \ldots, S_m$ of S under the automorphism group of G form a partition of $V(G)$. If $E(S_i) \cap E(X) \neq \emptyset$ for some S_i, then $d(H) = 16$ and $|S_i \cap V(H)| = 6$, which implies $d(V(H) \cup S_i) \leqslant 14 < \lambda_5(G)$, a contradiction. Thus $E(S_j) \cap E(X) = \emptyset$ for each S_j. As $c_0(G - X) = |X| - 2$, it follows that $|O_4| \geqslant 3$, which contradicts the fact that $|O_4| = |O_4 \cap X| + |O_4 \cap V(H)| \leqslant 2$.

Suppose next that S is not an imprimitive block of G. Then there is an automorphism φ_1 of G such that $\varphi_1(S) \neq S$ and $\varphi_1(S) \cap S \neq \emptyset$. Set $T = \varphi_1(S)$. As G is 6-regular, we have $\delta(G[S \cap T]) \geqslant 2$. By Lemma 20(c), $|S \cap T| \leqslant 4$. Hence $G[S \cap T]$ is a 4-cycle of G. Assume $S \cap T = \{v_1, v_2, v_3, v_4\}$, where $N(v_1) = N(v_2)$ and $N(v_3) = N(v_4)$.

By the vertex-transitivity of G, for each $u \in V(G)$ there is a vertex u' different from u such that $N(u') = N(u)$. Assume $E(X) \neq \emptyset$. Then $|E(X)| = 1$ and let u_1u_2 be the edge in $E(X)$. We know that there is a vertex u_1' in $V(H)$ with $N(u_1') = N(u_1)$, which implies $|N(u_1) \cap V(H)| = 5$. Then $O_5 \neq \emptyset$, a contradiction. Thus $E(X) = \emptyset$. As for each $u \in V(G)$ there is a vertex u' different from u such that $N(u') = N(u)$, it follows that there is a vertex $u_3 \in X$ with $2 \leqslant |N(u_3) \cap V(H)| \leqslant 4$. Let φ_2 be an automorphism of G such that $\varphi_2(v_1) = u_3$. If $\varphi_2(\{v_3, v_4\}) \backslash V(H) \neq \emptyset$, then $N(u_3) \cap V(H) = \varphi_2(N(v_1)) \cap V(H) \subseteq \bigcup_{i=4}^{6} O_i$. If $\varphi_2(\{v_3, v_4\}) \subseteq V(H)$, then $\varphi_2(\{v_3, v_4\}) \subseteq \bigcup_{i=4}^{6} O_i$. So $|(\bigcup_{i=4}^{6} O_i) \cap V(H)| \geqslant 2$, a contradiction. $\square$

Lemma 29. *Suppose $k = 6$, $\lambda_5(G) = \lambda_8(G) = 18$ and $g(G) > 3$. For a λ_8-atom S of G, we have $|S| \geqslant 15$.*

Proof. To the contrary, suppose $8 \leqslant |S| \leqslant 14$. By Lemma 23, we have $p = 1$, $|X| \geqslant 7$ and $|V(H)| \geqslant 9$. By Lemma 20(a), we have $d_G(V(H) \cup A) \geqslant \lambda_5(G)$ and $d_G(V(H) \backslash A) \geqslant \lambda_5(G)$ for each subset $A \subseteq V(G)$ with $|A| \leqslant 1$, which implies $d_G(H) = 18$ and $|\nabla(u) \cap \nabla(H)| \leqslant 3$ for each $u \in V(G)$. Then $g_0(G) \geqslant 7$ by Lemma 17. It follows that $G[A]$ is bipartite for each subset $A \subseteq V(G)$ with $|A| \leqslant 13$ and $d_G(A) = 18$. Hence $|V(H)| \geqslant 15$, and $G[S]$ is bipartite if $|S| \leqslant 13$. Then $|V(G)| \geqslant 26$.

Case 1. $|S| = 8$.

By Lemma 20(a), $d_G(A) \geqslant \lambda_5(G)$ for every subset $A \subseteq V(G)$ with $7 \leqslant |A| \leqslant 8$, which implies $\delta(G[S]) \geqslant 3$ and G has no subgraphs isomorphic to $K_{4,4}$. As $|E(G[S])| = \frac{1}{2}(6|S| - 18) = 15$ and $G[S]$ is bipartite, there is a vertex $u_0 \in S$ with $d_{G[S]}(u_0) = 3$ and $G[S \backslash \{u_0\}]$ is isomorphic to $K_{3,4}$.

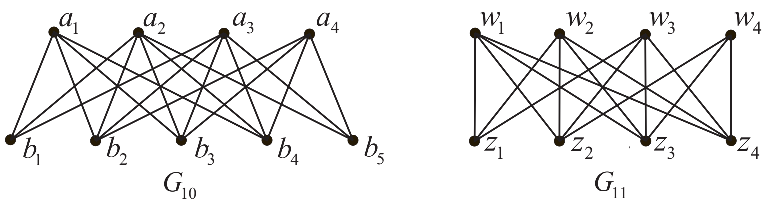

Figure 3. The illustration in the proof of Lemma 29.

Claim 1. *There are no two distinct vertices u and v in G with $N_G(u) = N_G(v)$.*

Suppose that u_1 and u_2 are two distinct vertices in G with $N_G(u_1) = N_G(u_2)$. Let x, y and z be the three vertices in S which have 4 neighbors in $S \backslash \{u_0\}$. Noting that G has no subgraphs isomorphic to $K_{4,4}$, we have, by the definition of the vertex-transitivity of G, that for each vertex $u \in \{x, y, z\}$ there is a vertex $u' \in \{x, y, z\} \backslash \{u\}$ such that $N_G(u) = N_G(u')$. It follows that $N_G(x) = N_G(y) = N_G(z)$. Then G is bipartite by Lemma 19, a contradiction. So Claim 1 holds.

Claim 2. *G has no subgraphs isomorphic to $K_{3,5}$.*

Suppose that u_3, u_4 and u_5 are three distinct vertices in G with $|N_G(u_3) \cap N_G(u_4) \cap N_G(u_5)| = 5$. By Claim 1 and the vertex-transitivity of G, it follows that for each $u \in$

$N_G(u_3) \cap N_G(u_4)$ there are two distinct vertices $u', u'' \in (N_G(u_3) \cap N_G(u_4)) \backslash \{u\}$ such that $|N_G(u) \cap N_G(u') \cap N_G(u'')| = 5$. It implies that there is a vertex $v \in V(G) \backslash (\{u_3, u_4, u_5\})$ such that $|N_G(v) \cap N_G(u_3) \cap N_G(u_4)| \geqslant 4$. So G has a subgraph isomorphic to $K_{4,4}$, a contradiction. Claim 2 is proved.

Claim 3. *G has no subgraphs isomorphic to G_{10} in Figure* 3.

Suppose that G_{10} is a subgraph of G. Let φ_1 be an automorphism of G such that $\varphi_1(a_2) = a_1$. Noting that $d_G(V(G_{10}) \cup A) \geqslant \lambda_5(G)$ for each subset $A \subseteq V(G)$ with $|A| \leqslant 1$ by Lemma 20(a), we have $G_{10} = G[V(G_{10})]$ and $|N_G(u) \cap V(G_{10})| \leqslant 3$ for each $u \in \overline{V(G_{10})}$. We know $\varphi_1(a_3) \in \{a_2, a_3\}$ if $|\varphi_1(N_{G_{10}}(a_2)) \cap N_{G_{10}}(a_1)| = 4$. Hence either each edge in $\nabla(a_1)$ or each edge in $\nabla(\varphi_1(a_3))$ is contained in a 4-cycle of G. By the vertex-transitivity of G, each edge in G is contained in a 4-cycle of G. It follows that $|\nabla(u) \cap \nabla(H)| + |\nabla(v) \cap \nabla(H)| \geqslant 3$ for each edge $uv \in \nabla(H)$.

We claim that $|\nabla(u) \cap \nabla(H)| \leqslant 2$ for each $u \in V(G)$. Otherwise, noting that $|\nabla(u) \cap \nabla(V(H))| \leqslant 3$ for each $u \in V(G)$, we suppose that there is a vertex u_6 in G with $|\nabla(u_6) \cap \nabla(H)| = 3$. Let φ_2 be an automorphism of G such that $\varphi_2(b_2) = u_6$. By considering the definition of $\varphi_2(V(G_{10}))$, we can obtain that there is a vertex $u \in \varphi_2(N_{G_{10}}(b_2))$ with $|\nabla(u) \cap \nabla(H)| \geqslant 4$, a contradiction.

Thus there a vertex $u_7 \in V(G)$ with $|\nabla(u_7) \cap \nabla(H)| = 2$. Let φ_3 be an automorphism of G such that $\varphi_3(a_2) = u_7$. Then there is a vertex $u \in \varphi_3(N_{G_{10}}(a_2))$ with $|\nabla(u) \cap \nabla(H)| \geqslant 3$, a contradiction. So Claim 3 holds.

By Claim 2, it follows that $G[S]$ is isomorphic to G_{11} in Figure 3 and we label $G[S]$ as in G_{11}. Then $|N_G(u) \cap S| \leqslant 2$ for each $u \in \overline{S}$ by Claims 2 and 3. Let φ_4 be an automorphism of G such that $\varphi_4(z_1) = z_4$. If $\varphi_4(N_{G[S]}(z_1)) \subseteq N_{G[S]}(z_4)$, then there is a vertex $u \in \varphi_4(S) \backslash S$ with $|N_G(u) \cap S| \geqslant 3$, a contradiction. Thus $\varphi_4(N_{G[S]}(z_1)) \backslash S \neq \emptyset$.

Assume $\varphi_4(N_{G[S]}(z_1)) \cap N_{G[S]}(z_1) = \{w_i, w_j\}$. As $|N_G(u) \cap S| \leqslant 2$ for each $u \in \varphi_4(N_{G[S]}(z_1)) \backslash S$, it follows that $|\varphi_4(\{z_2, z_3, z_4\}) \backslash S| = 2$. Then $N_G(w_i) = N_G(w_j)$, contradicting Claim 1.

Assume $\varphi_4(N_{G[S]}(z_1)) \cap N_{G[S]}(z_1) = \{w_{i'}\}$. Then $|\varphi_4(\{z_2, z_3, z_4\}) \backslash S| = 2$, which implies that each edge in $\nabla(w_{i'})$ is contained in a 4-cycle of G. Then each edge in G is contained in a 4-cycle of G by the vertex-transitivity of G. Thus there is a vertex $u_8 \in V(G)$ with $2 \leqslant |\nabla(u_8) \cap \nabla(H)| \leqslant 3$. Let φ_5 be an automorphism of G such that $\varphi_5(z_4) = u_8$. As $|N_G(w_1) \cap N_G(w_2) \cap N_G(w_3)| = 4$ and $|N_G(\varphi_4(w_1)) \cap N_G(\varphi_4(w_2)) \cap N_G(\varphi_4(w_3))| = 4$, it follows that there is a vertex $u \in \varphi_5(N_{G[S \cup \varphi_4(S)]}(z_4))$ with $|\nabla(u) \cap \nabla(H)| \geqslant 4$, a contradiction.

Thus $\varphi_4(N_{G[S]}(z_1)) \cap N_{G[S]}(z_1) = \emptyset$. By Claim 1, it follows that $\varphi_4(\{z_2, z_3, z_4\}) = N_G(w_4) \backslash S$. Let φ_6 be an automorphism of G such that $\varphi_6(z_1) = z_3$. Similarly, we have $\varphi_6(N_{G[S]}(z_1)) \cap N_{G[S]}(z_1) = \emptyset$ and $\varphi_6(\{z_2, z_3, z_4\}) = N_G(w_4) \backslash S$. It implies that $G[N_G(w_4) \cup \varphi_4(N_{G[S]}(z_1)) \cup \varphi_6(N_{G[S]}(z_1))$ has a subgraph isomorphic to $K_{3,5}$ or G_{10}, contradicting Claim 2 or Claim 3.

Case 2. $9 \leqslant |S| \leqslant 14$.

By Lemma 20(b), $\delta(G[S]) \geqslant 4$. If $|S| = 9$, then $18 = \frac{1}{2}(6|S| - \lambda_8(G)) = |E(S)| \geqslant \frac{1}{2}(|S| + 1)\delta(G[S]) \geqslant 20$, a contradiction. Thus $|S| \geqslant 10$. If $|S| \leqslant 13$, then let W and Z

be the bipartition of $G[S]$ with $|Z| \leqslant |W|$ and we have $|W| = |Z| + \frac{1}{2}(1 - (-1)^{|S|})$.

Subcase 2.1. $10 \leqslant |S| \leqslant 12$.

We claim that $d_{G[S]}(u) \leqslant 5$ for each $u \in S$. Otherwise, suppose that there is a vertex $v_1 \in S$ with $d_{G[S]}(v_1) = 6$. Choose a vertex $u_9 \in X$ with $\nabla(u_9) \cap \nabla(H) \neq \emptyset$. Let φ_7 be an automorphism of G such that $\varphi_7(v_1) = u_9$. As $\delta(G[S]) \geqslant 4$, it follows that $\varphi_7(S \backslash N_G(v_1)) \subseteq X$, which implies that $|\nabla(u) \cap \nabla(V(H))| \geqslant 4$ for each $u \in \varphi_7(N_G(v_1)) \cap V(H)$, a contradiction.

Noting that $4 \leqslant d_{G[S]}(u) \leqslant 5$ for each $u \in S$, and recalling $|E(S)| = 3|S| - 9$ and $|W| = |Z| + \frac{1}{2}(1-(-1)^{|S|})$, we know that there is a vertex $v_2 \in Z$ and $v_3 \in N_{G[S]}(v_2)$ such that $d_{G[S]}(v_2) = d_{G[S]}(v_3) + 1 = 5$.

Now we claim that each edge in G is contained in a 4-cycle of G. Otherwise, suppose that G has an edge contained in no 4-cycles. By the definition of the vertex-transitivity of G, each vertex in G is incident with an edge contained in no 4-cycles of G. Let φ_8 be an automorphism of G such that $\varphi_8(v_3) = v_2$. Then $\varphi_8(S) \neq S$. Notice that each edge in $G[S]$ is contained in a 4-cycle of $G[S]$. We have $\varphi_8(N_{G[S]}(v_3)) \subseteq N_{G[S]}(v_2)$ and $N_{G[S]}(\varphi_8(v_2)) \subseteq \varphi_8(N_{G[S]}(v_2))$. It implies $|\varphi_8(S) \cap S| \geqslant 8$, contradicting Lemma 20(c).

Thus $|\nabla(u) \cap \nabla(H)| + |\nabla(v) \cap \nabla(H)| \geqslant 3$ for each edge $uv \in \nabla(H)$. Then there is a vertex $u_{10} \in V(G)$ with $|\nabla(u_{10}) \cap \nabla(H)| \geqslant 2$.

Suppose $|S| = 10$. Then $|W| = |Z| = 5$. Let φ_9 be an automorphism of G such that $\varphi_9(v_2) = u_{10}$. Then there is a vertex $u \in \varphi_9(N_{G[S]}(v_2))$ with $|\nabla(u) \cap \nabla(H)| \geqslant 4$, a contradiction.

Thus $11 \leqslant |S| \leqslant 12$. Let R_i be the set of vertices u in S with $d_{G[S]}(u) = i$ for $i = 4, 5$. Then $|R_5| = |R_5 \cap Z| = 4$ if $|S| = 11$, and $|R_5 \cap W| = |R_5 \cap Z| = 3$ if $|S| = 12$.

Suppose that there is a vertex $u_{11} \in V(G)$ with $|\nabla(u_{11}) \cap \nabla(H)| = 3$. For a vertex $v \in S$, let ψ be an automorphism of G such that $\psi(v) = u_{11}$. Then $\psi(S) \cap V(H) \neq \emptyset$ and $\psi(S) \backslash V(H) \neq \emptyset$. As $\delta(G[S]) \geqslant 4$ and $|\nabla(u) \cap \nabla(H)| \leqslant 3$ for each $u \in V(G)$, it follows that $|\psi(S) \cap X| = 4$ and $G[\psi(S) \cap V(H)]$ and $G[\psi(S) \backslash V(H)]$ is isomorphic to $K_{1,4}$ or $K_{2,4}$. It implies $|N_{G[S]}(v) \cap R_4| \geqslant \lfloor \frac{|S|}{6} \rfloor$ and that there are two vertices $v', v'' \in R_4$ with $N_{G[S]}(v') = N_{G[S]}(v'')$. If $|S| = 11$, then $N_{G[S]}(u) \cap R_4 = \emptyset$ for each $u \in W \backslash N_{G[S]}(R_4 \cap Z)$, a contradiction. Thus $|S| = 12$. Then $|N_{G[S]}(u) \cap R_4| \geqslant 2$ for each $u \in S$. So $\delta(G[R_4]) \geqslant 2$. Noting $|R_4| = |R_5| = 6$, we have $12 \geqslant 4|R_4| - \delta(G[R_4])|R_4| \geqslant 4|R_4| - 2|E(R_4)| = |[R_4, R_5]| = 5|R_5| - 2|E(R_5)| \geqslant 30 - 18$, which implies $d_{G[R_4]}(u) = 2$ for each $u \in R_4$. Then $G[R_4]$ is a 6-cycle of G, which contradicts that R_4 has two vertices v' and v'' with $N_{G[S]}(v') = N_{G[S]}(v'')$.

So $|\nabla(u) \cap \nabla(H)| \leqslant 2$ for each $u \in V(G)$. Then $|\nabla(u_{10}) \cap \nabla(H)| = 2$. We can see that there is no automorphism φ of G such that $\varphi(v_2) = u_{10}$, contradicting that G is vertex-transitive.

Subcase 2.2. $13 \leqslant |S| \leqslant 14$.

Claim 4. *For two distinct λ_8-atoms S_1 and S_2 of G with $S_1 \cap S_2 \neq \emptyset$, $G[S_1 \backslash S_2]$ and $G[S_1 \cap S_2]$ are isomorphic to $K_{3,3}$ or $K_{3,4}$.*

By Lemma 20(c), we have $|S_1 \backslash S_2| \leqslant 7$, $|S_1 \cap S_2| \leqslant 7$, $d_G(S_1 \backslash S_2) + d_G(S_2 \backslash S_1) \leqslant 2\lambda_8(G)$ and $d_G(S_1 \cap S_2) + d_G(S_1 \cup S_2) \leqslant 2\lambda_8(G)$. Then $|S_1 \backslash S_2| \geqslant 6$ and $|S_1 \cap S_2| \geqslant 6$. By Lemma

20(a), each of $d_G(S_1\backslash S_2)$, $d_G(S_2\backslash S_1)$, $d_G(S_1 \cap S_2)$ and $d_G(S_1 \cup S_2)$ is not less than $\lambda_5(G)$. Noting $\lambda_5(G) = \lambda_8(G) = 18$, we have $d_G(S_1\backslash S_2) = d_G(S_1 \cap S_2) = 18$. Hence $G[S_1\backslash S_2]$ and $G[S_1 \cap S_2]$ are isomorphic to $K_{3,3}$ or $K_{3,4}$. So Claim 4 holds.

Since $G[S]$ is not a regular graph, there is an automorphism φ_{10} of G such that $\varphi_{10}(S) \neq S$ and $\varphi_{10}(S) \cap S \neq \emptyset$. Then $G[S\backslash\varphi_{10}(S)]$ and $G[S \cap \varphi_{10}(S)]$ are isomorphic to $K_{3,3}$ or $K_{3,4}$ by Claim 4. Set $B = S \cap \varphi_{10}(S)$.

Claim 5. *S has no subset A different from $S\backslash B$ and B such that $G[A]$ is isomorphic to $K_{3,4}$ and $G[S\backslash A]$ is isomorphic to $K_{3,3}$ or $K_{3,4}$.*

Suppose, to the contrary, that S has a subset A satisfying the above condition. Assume $|S| = 13$. As $|W| = |Z| + 1 = 7$, we know $|A \cap W| = 4$. It follows that there is a vertex $v_4 \in S$ with $d_{G[S]}(v_4) = 6$. Choose a vertex $v_5 \in S$ such that $d_{G[S]}(v_5) \geqslant 5$ and $|\{v_4, v_5\} \cap W| = 1$. Let φ_{11} be an automorphism of G such that $\varphi_{11}(v_5) = v_4$. Then $\varphi_{11}(S) \neq S$ and $\varphi_{11}(N_{G[S]}(v_5)) \subseteq N_{G[S]}(v_4)$, contradicting that $G[S\cap\varphi_{11}(S)]$ is isomorphic to $K_{3,3}$ or $K_{3,4}$ by Claim 4. Assume next $|S| = 14$. Then each of $G[S\backslash B]$, $G[B]$, $G[A]$ and $G[S\backslash A]$ is isomorphic to $K_{3,4}$. As $|E(S)| = 33$, we know $d_{G[S]}(B) = 9$. If $|A \cap B| = 1$, then $d_{G[S]}(B\backslash A) = 9 = \frac{1}{2}d_G(B\backslash A)$, contradicting Lemma 20(b). If $|A \cap B| = 6$, then $d_{G[S]}(S\backslash(A\cup B)) = 9 = \frac{1}{2}d_G(S\backslash(A\cup B))$, contradicting Lemma 20(b). If $2 \leqslant |A\cap B| \leqslant 5$, then $9 = d_{G[S]}(B) \geqslant d_{G[A]}(A \cap B) + d_{G[S\backslash A]}(B\backslash A) \geqslant 5 + 5$, a contradiction. Thus Claim 5 holds.

Claim 6. *Each vertex in G is contained in exactly two distinct λ_8-atoms of G.*

By the vertex-transitivity of G, it only needs to show that $S' = S$ or $\varphi_{10}(S)$ for a λ_8-atom S' of G with $S' \cap B \neq \emptyset$. Suppose $S' \neq S$ and $S' \neq \varphi_{10}(S)$. By Claims 4 and 5, we have $S'\cap S = B = S'\cap\varphi_{10}(S)$. Then $18 = d_G(B) \geqslant d_{G[S]}(B) + d_{G[\varphi_{10}(S)]}(B) + d_{G[S']}(B) \geqslant 3 \times 9$, a contradiction. Thus Claim 6 holds.

Let D be one of $S\backslash B$ and B such that $G[D]$ is isomorphic to $K_{3,4}$. Choose two vertices v_6 and v_7 in D such that $d_{G[D]}(v_6) = d_{G[D]}(v_7) - 1 = 3$. By Claim 6, there is only one λ_8-atom T of G which is different from S and contains v_6. By Claims 4 and 5, we have $S \cap T = D$. By Claim 6, S and T are also the only λ_8-atoms of G which contain v_7. It implies that there is no automorphism φ of G such that $\varphi(v_6) = v_7$, a contradiction. $\square$

4 Proof of Theorem 2

In this section we complete the proof of Theorem 2.

Proof of Theorem 2. If G is 4-factor-critical, then by Theorem 8 and Theorem 4 we have $k = \lambda(G) \geqslant 5$. So we consider the sufficiency. Suppose $k \geqslant 5$. We will prove that G is 4-factor-critical.

Suppose, to the contrary, that G is not 4-factor-critical. We know by Theorem 1 that G is bicritical. By Lemma 22, there is a subset $X \subseteq V(G)$ with $|X| \geqslant 4$ such that $c_0(G - X) = |X| - 2$ and every component of $G - X$ is factor-critical. Let H_1, H_2, …, H_p, H_{p+1}, …, H_t be the components of $G - X$, where $t = |X| - 2$ and H_1, H_2, …, H_p

are the nontrivial components of $G-X$. We know $p \geqslant 1$ by Lemma 23. For each $i \in [p]$, since H_i is factor-critical, $\delta(H_i) \geqslant 2$. For every subset $J \subseteq [t]$, we have

$$\sum_{i\in J} d_G(H_i) + \lambda(G)(t-|J|) \leqslant \sum_{i=1}^{t} d_G(H_i) \leqslant d_G(X) = k(t+2) - 2|E(X)|,$$

which implies

$$\sum_{i\in J} d_G(H_i) + 2|E(X)| \leqslant k(|J|+2). \tag{1}$$

Hence $|E(X)| \leqslant k$. Set $Y = \bigcup_{j=p+1}^{t} V(H_j)$.

Case 1. $g(G) = 3$.

By Lemma 18, $|E(X)| \geqslant t-p = |X|-2-p$.

Subcase 1.1. *$d_G(A) \geqslant 2k-2$ for all $A \subseteq V(G)$ with $2 \leqslant |A| \leqslant |V(G)|-2$.*

For each $i \in [p]$, we have $d_G(H_i) \geqslant 2k-2$. If k is odd, then $d_G(H_i)$ is odd and hence $d_G(H_i) \geqslant 2k-1$. So $d_G(H_i) \geqslant 2k - \frac{1}{2}(3+(-1)^k)$ for each $i \in [p]$. Now we have

$$(2k - \frac{1}{2}(3+(-1)^k))p + 2(|X|-2-p) \leqslant \sum_{i=1}^{p} d_G(H_i) + 2|E(X)| \leqslant k(p+2), \tag{2}$$

which implies $(k-2-\frac{1}{2}(3+(-1)^k))p + 2(|X|-2-k) \leqslant 0$. Hence $|X| \leqslant k+1$.

Suppose $|X| < k$. Then $p = t = |X|-2$. By Theorem 9, $|X| \geqslant \kappa(G) > \frac{2}{3}k$. Hence we know from (2) that $2k \geqslant (k-\frac{1}{2}(3+(-1)^k))p > (k-\frac{1}{2}(3+(-1)^k))(\frac{2}{3}k-2)$. That is, $k^2-7k+3 < 0$ if k is odd and $k^2-8k+6 < 0$ otherwise. It follows that $k \leqslant 6$. If $k = 6$, then $|X| \geqslant \kappa(G) = k$ by Lemma 11, a contradiction. Thus $k = 5$. Then $\kappa(G) = |X| = 4$. By Lemma 10, $\tau(G) = 2$. It implies that there is an edge $x_0y_0 \in E(G)$ such that $|N_G(x_0) \cap N_G(y_0)| = 4$.

Noting $k = 5$, we know from (2) that $|E(X)| \leqslant 1$. Choose a vertex $u \in X$ with $d_{G[X]}(u) = 0$. Since G is vertex-transitive, there is an automorphism φ_1 of G such that $\varphi_1(x_0) = u$. Assume, without loss of generality, that $\varphi_1(y_0) \in V(H_1)$. Noting $|N_G(x_0) \cap N_G(y_0)| = 4$, we have $N_G(u) \subseteq V(H_1)$. Then $d_G(V(H_1) \cup \{u\}) = d_G(X) - d_G(H_2) - 5 \leqslant 20-9-5 < 2k-2$, a contradiction.

Thus $k \leqslant |X| \leqslant k+1$. Noting $(k-2-\frac{1}{2}(3+(-1)^k))p + 2(|X|-2-k) \leqslant 0$, we have $p \leqslant 2$ and $k \leqslant 8$. Then $|Y| = |X|-2-p \geqslant k-4 \geqslant 1$. For any given vertex v, let q be the number of triangles containing v in G. By the vertex-transitivity of G, each vertex in G is contained in q triangles of G, which implies that each edge in G is contained in at most q triangles of G.

Claim 1. *$E(X)$ is a matching of G.*

Assume $p = 2$ or $|X| = k+1$. Then we know from (2) that $|E(X)| = |X|-2-p = |Y|$. Since there are $q|Y|$ triangles of G containing one vertex in Y, each edge in $E(X)$ is contained in q triangles of G. It implies that $E(X)$ is a matching of G. Next we assume $p = 1$ and $|X| = k$. If two edges in $E(X)$ are adjacent, then $|E(X)| = q \geqslant 2|Y| = 2(k-3)$

and hence $d_G(H_1) + 2|E(X)| \geqslant 2k - 2 + 4(k-3) > 3k$, which contradicts the inequality (1). So Claim 1 holds.

By Claim 1, it follows that each edge incident with a vertex in Y is contained in at most one triangle of G. Then, by the vertex-transitivity of G, each edge in $E(X)$ is contained in at most one triangle of G.

Suppose $|X| = k+1$. From (2), we know $k \leqslant 6$, $p = 1$ and $|E(X)| = |Y| = k-2$. Since G has $q|Y|$ triangles containing one vertex in Y, each edge in $E(X)$ is contained in q triangles of G. Noting that each edge in $E(X)$ is contained in at most one triangle of G, we have $q = 1$. Then $|E(N_G(u))| = 1$ for each $u \in Y$, which implies $|X| \geqslant 2|E(X)| + (k - |E(X)| - 1) = 2k - 3 > k+1$, a contradiction.

Thus $|X| = k$. Then for each $e \in E(X)$ and each $u \in Y$, G has a triangle containing e and u. As each edge in $E(X)$ is contained in at most one triangle of G, it follows that $|Y| = 1$, which implies $p = 2$ and $k = 5$. From (2), we know $d_G(H_1) = d_G(H_2) = 9$ and $|E(X)| = 1$. Assume $|V(H_1)| \leqslant |V(H_2)|$. Let u_1 be the vertex in Y. For a vertex $u_2 \in V(H_1)$ with $N_G(u_2) \cap X \neq \emptyset$, we have $|N_G(u_2) \cap X| \leqslant 3$ as $\delta(H_1) \geqslant 2$. As H_2 is a component of $G - N_G(u_1)$ with maximum cardinality, we have, by the definition of the vertex-transitivity of G, that H_2 also is a component of $G - N_G(u_2)$ with maximum cardinality. Then $N_G(X \backslash N_G(u_2)) \cap V(H_2) = \emptyset$. It implies $d_G(V(H_1) \cup (X \backslash N_G(u_2))) < 8 = 2k - 2$, a contradiction. Hence Subcase 1.1 cannot occur.

Subcase 1.2. *There is a subset $A \subseteq V(G)$ with $2 \leqslant |A| \leqslant |V(G)| - 2$ such that $d_G(A) < 2k - 2$.*

We choose a subset S of $V(G)$ such that $1 < |S| \leqslant \frac{1}{2}|V(G)|$, $d(S)$ is as small as possible, and, subject to these conditions, $|S|$ is as small as possible. Then $d_G(S) \leqslant d_G(A) \leqslant 2k-3$. By Corollary 14, $d_G(S) = |S| \geqslant k$ and $G[S]$ is $(k-1)$-regular. As $2k - 3 < \frac{2}{9}(k+1)^2$, S is an imprimitive block of G by Theorem 13. Thus $G[S]$ is vertex-transitive by Lemma 12. We also know that the orbits $S = S_1, S_2, \ldots, S_{m_1}$ of S under the automorphism group of G form a partition of $V(G)$ and each $G[S_i]$ is $(k-1)$-regular.

Set $I_i = \{j \in \{1, 2, \ldots, m_1\} : S_j \cap V(H_i) \neq \emptyset\}$ for each $i \in [t]$ and set $\mathscr{M} = \{\bigcup_{j \in I_i} S_j : i \in [t]\}$. If any two sets in $\mathscr{M}$ are disjoint, then $2|X| \geqslant 2|\bigcup_{U \in \mathscr{M}} \nabla(U)| \geqslant \sum_{U \in \mathscr{M}} d_G(U) \geqslant |\mathscr{M}| d_G(S)$.

Suppose $|S| = k$. Then each $G[S_i]$ is isomorphic to K_k and hence $G[S_i]$ has common vertices with at most one component of $G - X$. Hence $|\mathscr{M}| = c_0(G - X) = |X| - 2$ and any two sets in $\mathscr{M}$ are disjoint. Then $2|X| \geqslant |\mathscr{M}| d_G(S) = (|X| - 2)k > 2|X|$, a contradiction.

Suppose $|S| = k+1$. As $\delta(H_j) \geqslant 2$ for each $j \in [p]$, we have that for each S_i, $|S_i \backslash X| = |S_i \cap Y| = 2$ or $S_i \backslash X \subseteq V(H_{i'})$ for some $i' \in [t]$. Hence $|\mathscr{M}| \geqslant p + \frac{1}{2}(t - p) = \frac{1}{2}(t+p) \geqslant \frac{1}{2}(t+1) = \frac{1}{2}(|X| - 1)$ and any two sets in $\mathscr{M}$ are disjoint. Then $2|X| \geqslant |\mathscr{M}| d_G(S) \geqslant \frac{1}{2}(|X| - 1)(k+1) > 2|X|$, a contradiction.

Thus $|S| \geqslant k+2$. Noting that $(k-1)|S|$ is even and $k + 2 \leqslant |S| \leqslant 2k - 3$, we have $|S| = k+2$ if $5 \leqslant k \leqslant 6$. For each $i \in [p]$, if $V(H_i) \cap S_j \neq \emptyset$, then $|V(H_i) \cap S_j| \geqslant 2$ as $\delta(H_i) \geqslant 2$.

Claim 2. *For each S_i, there is an element $a_i \in [p]$ such that $V(H_{a_i}) \cap S_i \neq \emptyset$.*

Suppose $S_i \subseteq X \cup Y$. By Lemma 16, $|S_i \cap Y| \leqslant \frac{1}{3}|S_i|$. If $k \geqslant 6$, then $|E(X)| \geqslant |E(S_i \cap X)| = \frac{1}{2}(k-1)(|S_i \cap X| - |S_i \cap Y|) \geqslant \frac{1}{6}(k-1)|S_i| \geqslant \frac{1}{6}(k-1)(k+2) > k$, a contradiction. Thus $k = 5$. Then $|S| = k+2$ and $|S_i \cap Y| \leqslant \lfloor \frac{1}{3}|S_i| \rfloor = 2$. Hence $|E(X)| \geqslant |E(S_i \cap X)| \geqslant \frac{1}{2}(k-1)(|S_i| - 4) = \frac{1}{2}(k-1)(k-2) > k$, a contradiction. So Claim 2 holds.

Claim 3. *$X \backslash S_i \neq \emptyset$ for each S_i.*

Suppose $X \subseteq S_i$. Choose a component H_j of $G - X$ such that $H_j \neq H_{a_i}$. Then $|V(H_j) \cap S_i| = |N_G(V(H_j) \cap S_i) \backslash S_i| \leqslant |V(H_j) \backslash S_i|$. Hence $V(H_j) \backslash S_i \neq \emptyset$. Then there is some $S_{i'} \subseteq V(H_j) \backslash S_i$. Now we know $d_G(V(H_j) \backslash S_i) \geqslant d_G(S) = |S_i|$. On the other hand, we have $d_G(V(H_j) \backslash S_i) \leqslant |S_i \backslash V(H_{a_i})| < |S_i|$, a contradiction. So Claim 3 holds.

Claim 4. *For each $i \in [p]$, we have $d_G(H_i) \geqslant 2k-2$ if there is some S_j such that $S_j \cap V(H_i) \neq \emptyset$ and $S_j \backslash V(H_i) \neq \emptyset$.*

Suppose $S_j \cap V(H_i) \neq \emptyset$ and $S_j \backslash V(H_i) \neq \emptyset$. By Claim 3, $\overline{X \backslash S_j} \neq \emptyset$. Suppose $|\overline{V(H_i) \cup S_j}| = 1$. Then $\overline{V(H_i) \cup S_j} = X \backslash S_j$, which implies $|\overline{V(H_i) \cup X}| = 1$. Hence $t = 2$ and $p = 1$, implying $t = |X| - 2 \geqslant k - 2 > 2$, a contradiction. Thus $|\overline{V(H_i) \cup S_j}| \geqslant 2$. Then $|S_j| = d_G(S) \leqslant d_G(V(H_i) \cup S_j) \leqslant |[V(H_i), \overline{V(H_i) \cup S_j}]| + |S_j \backslash V(H_i)|$, which implies $|[V(H_i), \overline{V(H_i) \cup S_j}]| \geqslant |S_j \cap V(H_i)|$. Hence $d_G(H_i) \geqslant d_{G[S_j]}(S_j \cap V(H_i)) + |[V(H_i), \overline{V(H_i) \cap S_j}]| \geqslant d_{G[S_j]}(S_j \cap V(H_i)) + |S_j \cap V(H_i)|$. If $|S_j \backslash V(H_i)| \geqslant 2$, then $d_{G[S_j]}(S_j \cap V(H_i)) \geqslant 2k-4$ by Corollary 15, which implies $d_G(H_i) \geqslant 2k-4+|S_j \cap V(H_i)| \geqslant 2k-2$. If $|S_j \backslash V(H_i)| = 1$, then $d_G(H_i) \geqslant k - 1 + |S_j \cap V(H_i)| \geqslant 2k$. Claim 4 holds.

Claim 5. *$S_i \subseteq V(H_{a_i}) \cup X$ for each S_i.*

Suppose, to the contrary, that $G - X$ has a component H_b with $V(H_b) \cap (S_i \backslash V(H_{a_i})) \neq \emptyset$. Let θ be an integer such that $\theta = 1$ if $|V(H_b) = 1$ and $\theta = 0$ otherwise. As $X \backslash S_i \neq \emptyset$ by Claim 2, there is some S_j with $S_j \cap (X \backslash S_i) \neq \emptyset$. Set $J = \{a_i, b\} \cup \{a_j\}$. For each $i' \in [p]$, we have $d_G(H_{i'}) \geqslant d_G(S) \geqslant k+2$ and furthermore $d_G(H_{i'}) \geqslant 2k-2$ by Claim 4 if $i' \in [p] \cap J$. If $|J| = 2$, then, noting that $d_{G[S_i]}(V(H_{a_j}) \cap S_i) \geqslant 2k-4$ by Corollary 15 and $\lambda(G[S_j]) = k-1$ by Theorem 8, we have $d_G(H_{a_j}) \geqslant d_{G[S_i]}(V(H_{a_j}) \cap S_i) + d_{G[S_j]}(V(H_{a_j}) \cap S_j) \geqslant 2k-4+k-1 = 3k-5$.

Assume $5 \leqslant k \leqslant 6$. We know that $|S| = k+2$ and $G[S_i \cap V(H_{i'})]$ is isomorphic to K_2 for each $i' \in \{a_i, b\} \cap [p]$. Hence $S_i \subseteq V(H_{a_j}) \cup V(H_b) \cup X$. If $\theta = 1$, then $|E(G[S_i \cap X])| = \frac{1}{2}((k-1)|S_i \cap X| - (k-1) - (2k-4)) = \frac{1}{2}(k^2 - 5k + 6) \geqslant 3$. If $\theta = 0$, then $k = 6$ as $G[S_i]$ is vertex-transitive, which implies $|E(S_i \cap X)| = 2$. Now we have

$$
\begin{aligned}
&\sum_{i' \in J} d_G(H_{i'}) + 2|E(X)| \\
\geqslant &(3k-5)(3-|J|) + 2(2k-2)(|J|-2) + \theta k + (1-\theta)(2k-2) + 2|E(S_i \cap X)| \\
= &k(|J|+2) + |J| + k - 9 - \theta(k-2) + 2|E(S_i \cap X)| > k(|J|+2),
\end{aligned}
$$

which contradicts the inequality (1).

Assume $k \geqslant 7$. If $\theta = 1$, then $t = |X| - 2 \geqslant k - 2 \geqslant 5$. If $\theta = 0$, then $t = |X| - 2 \geqslant$

$\lceil \frac{2k}{3} \rceil - 2 \geqslant 3$ by Theorem 9. Now we have

$$\begin{aligned}&\sum_{i' \in [t]} d_G(H_{i'}) + 2|E(X)| \\ \geqslant&(3k-5)(3-|J|) + 2(2k-2)(|J|-2) + \theta(p-|J|+1)(k+2) + \\ &(1-\theta)(2k-2+(p-|J|)(k+2)) + (t-p)k + 2(t-p) \\ =&k(t+2) + 2t + \theta(k+2) + (1-\theta)(2k-2) - |J| - k - 7 > k(t+2),\end{aligned}$$

which contradicts the inequality (1). So Claim 5 holds.

By Claims 2 and 5, it follows that $|\mathscr{M}| = p = t$ and any two sets in $\mathscr{M}$ are disjoint. Then $2|X| \geqslant |\mathscr{M}| d_G(S) \geqslant (|X|-2)(k+2) > 2|X|$, a contradiction.

Case 2. $g(G) \geqslant 4$.

For each $j \in [p]$, we know from (1) that $d_G(H_j) \leqslant 3k$. Let F_j be a component of $G[\overline{V(H_j)}]$ which contains a vertex in $V(G) \backslash (V(H_j) \cup X)$. Then $\nabla(F_j)$ is a 5-restricted edge-cut of G. Hence $\lambda_5(G) \leqslant d_G(F_j) \leqslant d_G(H_j) \leqslant 3k$. As it follows from Corollary 15 that $\lambda_4(G) \geqslant 2k-2$, we have $2k-2 \leqslant \lambda_4(G) \leqslant \lambda_5(G) \leqslant 3k$.

Claim 6. *If $\lambda_5(G) \geqslant 4k-8$ and $k \leqslant 6$, then $p=1$, $|V(H_1)| \geqslant 7$, $\lambda_7(G) \leqslant 3k$ and furthermore, $\lambda_8(G) \leqslant 3k$ if $\lambda_5(G) > 4k-8$.*

Suppose $\lambda_5(G) \geqslant 4k-8$ and $k \leqslant 6$. Then $p=1$ by Lemma 23. We claim that $G[\overline{V(H_1)}]$ is connected. Otherwise, $d_G(H_1) \geqslant \lambda(G) + d_G(F_1) \geqslant k + \lambda_5(G) > 3k$, a contradiction. Suppose $|V(H_1)| = 5$. As $g(G) \geqslant 4$ and H_1 is factor-critical, H_1 is a 5-cycle of G. It follows that $k=5$, $E(X) = \emptyset$ and $|X| \geqslant 8$. Then $g_0(G) \geqslant 7$ by Lemma 17, a contradiction. Thus $|V(H_1)| \geqslant 7$. Then $\nabla(H_1)$ is a 7-restricted edge-cut of G and $\lambda_7(G) \leqslant d_G(V(H_1)) \leqslant 3k$. If $\lambda_5(G) > 4k-8$, then $|X| \geqslant 7$ and $|V(H_1)| \geqslant 9$ by Lemma 23, which implies $\lambda_8(G) \leqslant d_G(H_1) \leqslant 3k$. So Claim 6 holds.

By Claim 6, we can discuss Case 2 in the following two subcases.

Subcase 2.1. $k=5$, $\lambda_5(G) = 12$ *and* $\lambda_7(G) \geqslant 13$.

We have $\lambda_4(G) = 12$. As $\lambda_7(G)$ exists, $|V(G)| \geqslant 14$. Then, by Lemma 20(a), $d_G(A) \geqslant \lambda_7(G)$ for each subset $A \subseteq V(G)$ with $|A| = 7$, which implies that G has no subgraphs isomorphic to $K_{3,4}$. By the definition of the vertex-transitivity of G, we can obtain that G has no subgraphs isomorphic to $K_{2,5}$. By Claim 6, $p=1$ and $|V(H_1)| \geqslant 7$. Hence $|X| \geqslant 6$ and $|V(G)| \geqslant 16$. By Lemma 20(a), $d_G(V(H_1) \cup A) \geqslant \lambda_7(G)$ for each subset $A \subseteq X$ with $|A| \leqslant 1$, which implies $d_G(H_1) \geqslant 13$ and $|N_G(u) \cap V(H_1)| \leqslant 3$ for each $u \in X$. Noting $\delta(H_1) \geqslant 2$, we have $|\nabla(u) \cap \nabla(H_1)| \leqslant 3$ for each $u \in V(G)$.

Claim 7. *There is no subset $A \subseteq V(G)$ with $|A| \leqslant 3$ such that $A \cap V(H_1) \neq \emptyset$, $|\nabla(A) \cap \nabla(H_1)| = 3|A|$ and $d_G((V(H_1) \cup A) \backslash (V(H_1) \cap A)) \leqslant 12$.*

Suppose, to the contrary, that such subset A of $V(G)$ exists. Set $B = (V(H_1) \cup A) \backslash (V(H_1) \cap A)$. Then $|B| \geqslant 4$ and $|\overline{B}| \geqslant 7$. By Lemma 20(a), we have $d_G(B) \geqslant \lambda_4(G)$ and furthermore, $d_G(B) \geqslant \lambda_7(G)$ if $|B| \geqslant 7$. As $d_G(B) \leqslant 12$, we know $|B| \leqslant 6$ and $d_G(B) = 12$. It implies that $E(V(H_1) \cap A) = \emptyset$ and $G[B]$ is isomorphic to $k_{2,2}$ or $K_{3,3}$.

Hence $G[V(H_1) \cup A]$ is bipartite. Then H_1 is bipartite, contradicting the fact that H_1 is factor-critical. So Claim 7 holds.

As $\lambda_5(G) = 12 < \lambda_7(G)$ and $k = 5$, each λ_5-atom of G induces a subgraph which is isomorphic to $K_{3,3}$. Let $T_1, T_2, \ldots, T_{m_2}$ be all the subsets of $V(G)$, which induce subgraphs isomorphic to $K_{3,3}$. Let R_i be the set of vertices in X with i neighbors in $V(H_1)$ for $1 \leqslant i \leqslant 3$ and let Q be the set of vertices in $V(H_1)$ with 3 neighbors in X.

Subcase 2.1.1. *There are two distinct T_i and T_j with $T_i \cap T_j \neq \emptyset$.*

Noting that G has no subgraphs isomorphic to $K_{3,4}$ or $K_{2,5}$, we have $|T_i \cap T_j| = 2$ or 4. If $|T_i \cap T_j| = 4$, then $d_G(T_i \cap T_j) \leqslant 12 < \lambda_7(G)$, which contradicts Lemma 20(a). Thus $|T_i \cap T_j| = 2$. Assume $T_i \cap T_j = \{v_1, v_2\}$.

Claim 8. *For each $u \in X$ with $d_{G[X]}(u) = 0$ and $N_G(u) \cap V(H_1) \neq \emptyset$, we have $N_G(u) \cap V(H_1) \subseteq Q$ if $u \in R_1 \cup R_2$, and $|N_G(u) \cap V(H_1) \cap Q| \geqslant 1$ if $u \in R_3$.*

Since G is vertex-transitive, there is an automorphism φ_2 of G such that $\varphi_2(v_1) = u$. If $u \in R_1 \cup R_2$, then $\varphi_2(N_G(v_2)) \subseteq X$, which implies $N_G(u) \cap V(H_1) \subseteq Q$. If $u \in R_3$, then $|\varphi_2(N_G(v_2)) \cap X| \geqslant 3$, which implies $|N_G(u) \cap V(H_1) \cap Q| \geqslant 1$. So Claim 8 holds.

Assume $E(X) \neq \emptyset$. Then $|E(X)| = 1$ and $\sum_{i=1}^{3} i|R_i| = d_G(H_1) = 13$, which implies $\sum_{i=1}^{3} |R_i| \geqslant 5$. By Claim 8, $Q \neq \emptyset$. We have $d_G(V(H_1)\backslash\{u\}) \leqslant 12$ for each $u \in Q$, contradicting Claim 7.

Thus $E(X) = \emptyset$. As $d_G(V(H_1) \cup A) \geqslant \lambda_4(G)$ for each subset $A \subseteq X$ with $|A| = 4$ by Lemma 20(a), we have $|R_3| \leqslant 3$. By Claim 8, $|\nabla(Q) \cap \nabla(H_1)| \geqslant |R_3| + 2|R_2| + |R_1| = 15 - 2|R_3| \geqslant 9$, which implies $|Q| \geqslant 3$. Choose a subset $Q' \subseteq Q$ with $|Q'| = 3$. Then $d_G(V(H_1)\backslash Q') \leqslant 12$, contradicting Claim 7. Hence Subcase 2.1.1 cannot occur.

Subcase 2.1.2. *Any two distinct T_i and T_j are disjoint.*

By the vertex-transitivity of G, each vertex in G is contained in a λ_5-atom of G. Hence $T_1, T_2, \ldots, T_{m_2}$ form a partition of $V(G)$.

Assume $E(X) \neq \emptyset$. As $c_0(G - X) = |X| - 2$ and $|E(X)| = 1$, it follows that there is some T_i such that $T_i \cap X \neq \emptyset$, $T_i \cap V(H_1) \neq \emptyset$ and $E(T_i) \cap E(X) = \emptyset$. Then there is a vertex $u_1 \in T_i \cap (R_3 \cup Q)$. By Claim 7, it follows that $u_1 \in X$. We have $d_G(V(H_1) \cup \{u_1\}) = 12 < \lambda_7(G)$, contradicting Lemma 20(a).

Thus $E(X) = \emptyset$. Set $\mathscr{B}_1 = \{T_j : |T_j \cap X| = 3, j \in [m_2]\}$ and $\mathscr{B}_2 = \{T_j : |T_j \cap X| < 3, j \in [m_2]\}$. Let $D = (\bigcup_{A \in \mathscr{B}_1} A \cap V(H_1)) \cup (\bigcup_{A \in \mathscr{B}_2} A \cap X)$. Noting $c_0(G - X) = |X| - 2$ and $p = 1$, we have $|D| = 3$. By Claim 7, we have $D \subseteq X$. If $|X| \geqslant 7$, then $d_G(H_1 + D) = 12 < \lambda_7(G)$, which contradicts Lemma 20(a). Thus $|X| = 6$. As G has no subgraphs isomorphic to $K_{2,5}$, we know that $|R_2| = |R_3| = 3$ and $G[Y \cup R_2]$ is isomorphic to $K_{3,3}$. Choose a vertex $u_3 \in R_3$ and a vertex $u_4 \in Y$. Let φ_3 be an automorphism of G such that $\varphi_3(u_4) = u_3$. Noting $\varphi_3(Y \cup R_2) \cap (Y \cup R_2) = \emptyset$, we have $\varphi_3(Y) = R_3$ and $\varphi_3(R_2) \subseteq V(H_1)$. It implies $D \subseteq V(H_1)$ by the choice of D, a contradiction. Hence Subcase 2.1 cannot occur.

Subcase 2.2. $k \neq 5$, $\lambda_5(G) \neq 12$ *or* $\lambda_5(G) = \lambda_7(G) = 12$.

Let S' be a λ_s-atom of G, where

$$s = \begin{cases} 4, \text{ if } k \leqslant 6 \text{ and } \lambda_5(G) < 4k-8; \\ 7, \text{ if } k = 5 \text{ and } \lambda_5(G) = \lambda_7(G) = 12; \\ 6, \text{ if } k = 5 \text{ and } \lambda_5(G) = \lambda_6(G) = 13; \\ 7, \text{ if } k = 5, \lambda_5(G) = 13 \text{ and } \lambda_6(G) = \lambda_7(G) = 14; \\ 8, \text{ if } k = 5, \lambda_5(G) = 13, \lambda_6(G) \geqslant 14 \text{ and } \lambda_8(G) = 15; \\ 5, \text{ if } k = 5 \text{ and } \lambda_5(G) = 14; \\ 6, \text{ if } k = 5 \text{ and } \lambda_5(G) = \lambda_6(G) = 15; \\ 5, \text{ if } k = 6 \text{ and } \lambda_5(G) = 4k-8; \\ 8, \text{ if } k = 6 \text{ and } \lambda_5(G) = 18; \\ 5, \text{ if } k \geqslant 7. \end{cases}$$

Claim 9. *S' is an imprimitive block of G such that $|S'| > \frac{1}{2}\lambda_s(G)$ if $k \leqslant 6$ and $|S'| > \frac{1}{3}\lambda_s(G)$ otherwise.*

If $k = 5$ and $\lambda_5(G) = \lambda_7(G) = 12$, then, by Lemma 24, Claim 9 holds. So we assume $k > 5$ or $\lambda_5(G) \neq 12$. By Lemma 6, $\frac{1}{2}|S'|^2 \geqslant 2|E(S')| = k|S| - \lambda_s(G)$. If $5 \leqslant k \leqslant 6$ and $\lambda_5(G) < 4k-8$, then $\frac{1}{2}|S'|^2 \geqslant k|S'| - \lambda_s(G) > k|S'| - 4k + 8$, which implies $|S'| > 2k-4 \geqslant \max\{2(s-1), \frac{1}{2}\lambda_s(G)\}$. If $5 \leqslant k \leqslant 6$ and $\lambda_5(G) \geqslant 4k-8$, then $|S'| > 2(s-1)$ and $2|S'| > \lambda_s(G)$ by Lemmas 21 and 25-29. If $k \geqslant 7$, then $\frac{1}{2}|S'|^2 \geqslant k|S'| - \lambda_s(G) \geqslant k|S'| - 3k$ and hence $|S'| > k+2 > \max\{2(s-1), \frac{1}{3}\lambda_s(G)\}$. Suppose S' is not an imprimitive block of G. Then there is an automorphism φ of G such that $\varphi(S') \neq S'$ and $\varphi(S') \cap S' \neq \emptyset$. By Lemma 20(c), $|S'| = |S' \cap \varphi(S')| + |S' \backslash \varphi(S')| \leqslant 2(s-1)$, a contradiction. So Claim 9 holds.

By Claim 9 and Lemma 12, $G[S']$ is vertex-transitive and hence it is $(k-1)$-regular if $k \leqslant 6$ and is $(k-1)$-regular or $(k-2)$-regular otherwise. From Claim 9, we also know that the orbits $S' = S'_1, S'_2, \ldots, S'_{m_3}$ of S' under the automorphism group of G form a partition of $V(G)$.

Claim 10. *$G[S']$ is $(k-1)$-regular.*

Suppose that $G[S']$ is $(k-2)$-regular. Then $k \geqslant 7$, $s = 5$ and $2|S'| = \lambda_s(G) \leqslant 3k$, which implies $|S'| \leqslant \frac{3}{2}k$. By Lemma 6, $\frac{1}{4}|S'|^2 \geqslant |E(S')| = \frac{1}{2}(k-2)|S'|$, which implies $|S'| \geqslant 2(k-2)$. Now $2(k-2) \leqslant |S'| \leqslant \frac{3}{2}k$, which implies $k \leqslant 8$ and $|S'| = 2(k-2)$. Hence $G[S']$ is isomorphic to $K_{k-2,k-2}$. For each $i \in [p]$, noting $3k \geqslant d_G(H_i) \geqslant \lambda_s(G) = 4(k-2)$ and that $d_G(H_i)$ has the same parity with k, we have $d_G(H_i) = 3k$. Hence $p = 1$, $E(X) = \emptyset$, $|V(H_1)| > 5$ and $|X| \geqslant k$. As $c_0(G-X) = |X| - 2$, there is some S'_i with $S'_i \cap X \neq \emptyset$ and $S'_i \cap V(H_1) \neq \emptyset$. Then there is a vertex $u \in S'_i$ with $|\nabla(u) \cap \nabla(H_1)| \geqslant k-2$. Then $d_G(V(H_1) \cup \{u\}) \leqslant d_G(H_1) - (k-4) = 2k+4 < 4(k-2) = \lambda_s(G)$ if $u \in X$ and $d_G(V(H_1) \backslash \{u\}) < \lambda_s(G)$ otherwise, contradicting Lemma 20(a). So Claim 10 holds.

As $\delta(H_i) \geqslant 2$ for each $i \in [p]$, it follows from Claim 10 that $\delta(G[V(H_j) \cap S'_i]) \geqslant 1$ if $V(H_j) \cap S'_i \neq \emptyset$.

Claim 11. *For each S'_i, $S'_i \backslash (X \cup Y) \neq \emptyset$ or $|S'_i \cap X| = |S'_i \cap Y|$.*

Suppose $|S'_i \cap X| > |S'_i \cap Y|$ for some $S'_i \subseteq X \cup Y$. If $G[S'_i]$ is bipartite, then $|S'_i \cap Y| \leqslant |S'_i \cap X| - 2$. If $G[S'_i]$ is non-bipartite, then $|S'_i \cap Y| \leqslant \alpha(G[S'_i]) \leqslant \frac{1}{2}|S'_i| - \frac{k-1}{4}$ by Lemma 16, which implies $|S'_i \cap Y| \leqslant |S'_i \cap X| - \frac{k-1}{2} \leqslant |S_i \cap X| - 2$. Thus $|E(S'_i \cap X])| = \frac{1}{2}(k-1)(|S'_i \cap X| - |S'_i \cap Y|) \geqslant k - 1$. Noting $d_G(H_1) \geqslant \lambda_5(G) \geqslant 2k - 2$, we have $d_G(H_1) + 2|E(X)| \geqslant 2k - 2 + 2(k-1) > 3k$, a contradiction. So Claim 11 holds.

Subcase 2.2.1. $|S'| \leqslant 2k - 1$.

Claim 12. *If $S'_i \cap V(H_j) \neq \emptyset$ for some $j \in [p]$, then $S'_i \subseteq V(H_j) \cup X$.*

Suppose $S'_i \cap V(H_j) \neq \emptyset$ for some $j \in [p]$ and $S'_i \cap V(H_{j'}) \neq \emptyset$ for some $j' \in [t] \backslash \{j\}$. As $\delta(G[S'_i \cap V(H_j)]) \geqslant 1$, there is an edge $x_1 y_1 \in E(S'_i \cap V(H_j))$. Then $|S'_i \cap (V(H_j) \cup X)| \geqslant |N_{G[S'_i]}(x_1) \cup N_{G[S'_i]}(y_1)| = 2k - 2$. It implies $|S'_i \cap V(H_{j'})| = 1$ and $|S'_i| = 2k - 1$. Then $|V(H_{j'})| = 1$ and $|X| \geqslant |N_G(V(H_{j'}))| = k$. Hence $|\overline{V(H_j) \cup S'_i}| \geqslant |N_G(V(H_{j'}))\backslash S'_i| + (c_0(G - X) - 2) \geqslant 1 + k - 4 \geqslant 2$. By Corollary 15, we have

$$\begin{aligned}
2k - 2 &\leqslant d_G(V(H_j) \cup S'_i) \\
&\leqslant d_G(H_j) - d_{G[S'_i]}(S'_i \cap V(H_j)) + |S'_i \backslash V(H_j)| \\
&= d_G(H_j) - ((k-1)|S'_i \cap X| - 2|E(S'_i \cap X)| - (k-1)) + |S'_i \cap X| + 1 \\
&= d_G(H_j) + 2|E(S'_i \cap X)| - (k-2)|S'_i \cap X| + k \\
&\leqslant 3k - (k-2)(k-1) + k = -k^2 + 7k - 2,
\end{aligned}$$

which implies $k = 5$. It is easy to verify that there is no triangle-free non-bipartite 4-regular graph of order 9, which implies $|S'| \neq 9 = 2k - 1$, a contradiction. So Claim 12 holds.

Set $I'_i = \{j \in [m_3] : S'_j \cap V(H_i) \neq \emptyset\}$ for each $i \in [t]$ and $\mathscr{M}' = \{\bigcup_{j \in I'_i} S'_j : i \in [t]\}$. Then any two sets in $\mathscr{M}'$ are disjoint by Claim 12. By Lemma 20(a), $d_G(U) \geqslant \lambda_s(G)$ for each $U \in \mathscr{M}'$. Then, by Claim 11, we have

$$\begin{aligned}
&2(p + 2 + (k-1)(|\mathscr{M}'| - p)) \\
=&2|X| \geqslant 2|\bigcup_{U \in \mathscr{M}'} \nabla(U)| = \sum_{U \in \mathscr{M}'} d_G(U) \geqslant |\mathscr{M}'|\lambda_s(G) \geqslant |\mathscr{M}'|(2k-2),
\end{aligned}$$

which implies $p \leqslant \frac{2}{k-2} < 1$, a contradiction. Hence Subcase 2.2.1 cannot occur.

Subcase 2.2.2 $|S'| \geqslant 2k$.

We have $\lambda_s(G) = |S'| \geqslant 2k$. If $s = 4$, then $\lambda_5(G) \geqslant \lambda_s(G) \geqslant 2k$. If $s \geqslant 5$, then $\lambda_5(G) \geqslant 2k$ by the choice of s. Then $2kp \leqslant p\lambda_5(G) \leqslant \sum_{i=1}^{p} d_G(H_i) + 2|E(X)| \leqslant k(2+p)$, which implies $p \leqslant 2$.

Let

$$\mathscr{N} = \{S'_i : S'_i \cap X \neq \emptyset \text{ and } S'_i \backslash (X \cup Y) \neq \emptyset, i \in [m_3]\}.$$

By Claim 11, $\sum_{A \in \mathscr{N}}(|A \cap X| - |A \cap Y|) = \sum_{i=1}^{m_3}(|S'_i \cap X| - |S'_i \cap Y|) = |X| - |Y| = p + 2$. Noting $|A \cap X| > |A \cap Y|$ for each $A \in \mathscr{N}$, we have $1 \leqslant |\mathscr{N}| \leqslant p + 2$. Choose a set $S'_{j_1} \in \mathscr{N}$. Without loss of generality, we assume $S'_{j_1} \cap V(H_1) \neq \emptyset$.

Suppose $p = 2$. Then $E(X) = \emptyset$ and $2k = \lambda_5(G) = d_G(H_1) = d_G(H_2)$. Hence $\lambda_4(G) = \lambda_5(G) = 2k = |S'|$. For each $u \in V(G)$ and each $i \in [p]$, we have $d_G(V(H_i) \cup \{u\}) \geqslant \lambda_4(G)$ and $d_G(V(H_i) \backslash \{u\}) \geqslant \lambda_4(G)$ by Lemma 20(a), which implies $|\nabla(u) \cap \nabla(H_i)| \leqslant k - 3$. Hence $|S'_{j_1} \backslash V(H_1)| \geqslant 2$ and $\delta(G[S'_{j_1} \cap V(H_1)]) \geqslant 2$, which implies $|S'_{j_1} \cap V(H_1)| \geqslant 4$. Choose an edge $x_2y_2 \in E(S'_{j_1} \cap V(H_1))$. Then $|S'_{j_1} \backslash (V(H_1) \cup X)| \leqslant |S'_{j_1} \backslash (N_{G[S'_{j_1}]}(x_2) \cup N_{G[S'_{j_1}]}(y_2))| = 2$. It follows that $S'_{j_1} \cap V(H_2) = \emptyset$. Noting that $d_{G[S'_{j_1}]}(S'_{j_1} \cap V(H_1)) \geqslant 2k-4$ by Corollary 15, we have $|S'_{j_1} \cap X| \geqslant |S'_{j_1} \cap Y| + 2$. Now

$$\begin{aligned} d_G(V(H_1) \cup S'_{j_1}) &\leqslant d_G(H_1) - d_{G[S'_{j_1}]}(V(H_1) \cap S'_{j_1}) + |S'_{j_1} \backslash V(H_1)| \\ &= 2k - (k-1)(|S'_{j_1} \cap X| - |S'_{j_1} \cap Y|) + |S'_{j_1} \backslash V(H_1)| \\ &\leqslant 2k - 2(k-1) + 2k - 4 < 2k = \lambda_4(G), \end{aligned}$$

contradicting Lemma 20(a).

Thus $p = 1$. Suppose $|\mathscr{N}| = 1$. Then $|S'_{j_1} \cap X| = |S'_{j_1} \cap Y| + 3$ and there is some $S'_j \subseteq \overline{V(H_1) \cup S'_{j_1}}$. We know by Claim 11 that $G[S']$ is bipartite. Hence there is some $S'_{j'} \subseteq V(H_1) \backslash S'_{j_1}$. By Lemma 20(a), we have

$$\begin{aligned} |S'| = \lambda_s(G) \leqslant d_G(V(H_1) \cup S'_{j_1}) &\leqslant d_G(H_1) - d_{G[S'_{j_1}]}(S'_{j_1} \backslash V(H_1)) + |S'_{j_1} \backslash V(H_1)| \\ &= d_G(H_1) + 2|E(S'_{j_1} \cap X)| - 3(k-1) + |S'_{j_1} \backslash V(H_1)| \\ &\leqslant 3k - 3(k-1) + |S'_{j_1} \backslash V(H_1)|. \end{aligned}$$

Similarly, we can obtain $|S'| \leqslant d_G(H_1 - S'_{j_1}) \leqslant 3 + |S'_{j_1} \cap V(H_1)|$. Then $2|S'| \leqslant 6 + |S'_{j_1}|$, which implies $|S'| \leqslant 6 < 2k$, a contradiction.

Thus $|\mathscr{N}| \geqslant 2$. For each $S'_i \in \mathscr{N}$, noting $|S'_i \cap V(H_1)| \geqslant 2$, if $|S'_i \backslash V(H_1)| \geqslant 2$, then, by Corollary 15, we have $d_{G[S'_i]}(S'_i \cap V(H_1)) \geqslant 2k - 4$, which implies that $|S'_i \cap X| = 1$ if $|S'_i \cap X| = |S'_i \cap Y| + 1$. If $|\mathscr{N}| = 3$, then $|S'_i \cap X| = 1$ for each $S'_i \in \mathscr{N}$ and hence $d_G(V(H_1) \cup (\bigcup_{S'_i \in \mathscr{N}} S'_i)) \leqslant d_G(H_1) - 3(k-2) \leqslant 6 < \lambda_s(G)$, which contradicts Lemma 20(a). Thus $|\mathscr{N}| = 2$. Assume $\mathscr{N} = \{S'_{j_1}, S'_{j_2}\}$ and $|S'_{j_1} \cap X| = 1$. We know that there is some $S'_j \subseteq V(G) \backslash (V(H_1) \cup S'_{j_1} \cup S'_{j_2})$. By Lemma 20(a),

$$\begin{aligned} |S| = \lambda_s(G) &\leqslant d_G(V(H_1) \cup S'_{j_1} \cup S'_{j_2}) \\ &\leqslant d_G(H_1) - d_{G[S'_{j_2}]}(S'_{j_2} \cap V(H_1)) + |S'_{j_2} \backslash V(H_1)| - (k-2) \\ &= d_G(H_1) + 2|E(S'_{j_2} \cap X)| - 2(k-1) + |S'_{j_2} \backslash V(H_1)| - (k-2) \\ &\leqslant 3k - 3k + 4 + |S'_{j_2} \backslash V(H_1)|. \end{aligned}$$

Similarly, we can obtain $|S'| \leqslant d_G((V(H_1) \cup S'_{j_1}) \backslash S'_{j_2}) \leqslant 4 + |S'_{j_2} \cap V(H_1)|$. Then $2|S'| \leqslant 8 + |S'_{j_2}|$, which implies $|S'| \leqslant 8 < 2k$, a contradiction. □

References

[1] B. Andrásfai, P. Erdős, and V. T. Sós, On the connection between chromatic number, maximal clique and minimal degree of a graph, *Discrete Math.*, 8:205–218, 1974.

[2] N. Biggs, *Algebraic Graph Theory*, pages 36–38. Cambridge University Press, 1993.

[3] O. Chan, C.C. Chen, and Q. Yu, On 2-extendable obelian Cayley graphs, *Discrete Math.*, 146:19–32, 1995.

[4] C.C. Chen, J. Liu, and Q. Yu, On the classification of 2-extendable Cayley graphs on dihedral groups, *Australas. J. Combin.*, 6:209–219, 1992.

[5] R. Diestel, *Graph Theory*, page 41. Springer, 2006.

[6] J. Fàbrega, and M.A. Fiol, Extraconnectivity of graphs with large girth, *Discrete Math.*, 127:163–170, 1994.

[7] O. Favaron, On k-factor-critical graphs, *Discuss. Math. Graph Theory*, 16:41–51, 1996.

[8] O. Favaron, Extendability and factor-criticality, *Discrete Math.*, 213:115–122, 2000.

[9] T. Gallai, Neuer Beweis eines Tutte-schen Satzes, *Magyar Tud. Akad. Matem. Kut. Int. Közl.*, 8:135–139, 1963.

[10] J. Heuvel, and B. Jacskon, On the edge connectivity, hamiltonicity, and toughness of vertex transitive graphs, *J. Combin. Theory Ser. B*, 77:138–149, 1999.

[11] A. Holtkamp, D. Meierling, and L.P. Montejano, k-restricted edge-connectivity in triangle-free graphs, *Discrete Appl. Math.*, 160:1345–1355, 2012.

[12] L. Lovász, On the structure of factorizable graphs, *Acta Math. Acad. Sci. Hungar.*, 23:179–195 1972.

[13] L. Lovász, and M.D. Plummer, *Matching Theory*, page 207. North-Holland, 1986.

[14] W. Mader, Minimale n-fach kantenzusammenhängenden Graphen, *Math. Ann.*, 191:21–28, 1971.

[15] W. Mantel, Problem 28, *Wiskundige Opgaven*, 10:60-61, 1907.

[16] Š. Miklavič, and P. Šparl, On extendability of Cayley graphs, *Filomat*, 23:93–101, 2009.

[17] M.D. Plummer, On n-extendable graphs, *Discrete Math.*, 31:201–210, 1980.

[18] R. Tindell, Connectivity of Cayley graphs, In *D.Z. Du, D.F. Hsu (Eds.). Combinatorial Network Theory*, pages 41–64. Kluwer, 1996.

[19] M.E. Watkins, Connectivity of transitive graphs, *J. Combin. Theory*, 8:23–29, 1970.

[20] M. Yang, Z. Zhang, C. Qin, and X. Guo, On super 2-restricted and 3-restricted edge-connected vertex transitive graphs, *Discrete Math.*, 311:2683–2689, 2011.

[21] Q. Yu, Characterizations of various matching extensions in graphs, *Australas. J. Combin.*, 7:55–64, 1993.

[22] H. Zhang, and W. Sun, 3-Factor-criticality of vertex-transitive graphs, *J. Graph Theory*, 81:262–271, 2016.