The number of prefixes of minimal factorisations of an n-cycle

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Abstract

We prove in two different ways that the number of prefixes of length k of minimal factorisations of the n-cycle $(1 \dots n)$ as a product of $n-1$ transpositions is $\binom{n}{k+1} n^{k-1}$. Our first proof is not bijective but makes use of a correspondence between minimal factorisations and Cayley trees. The second proof consists in establishing a bijection between the set which we want to enumerate and the set of parking functions of a certain kind, which can be counted by a standard conjugation argument.

Keywords: Cayley graph of permutations, minimal factorisations, parking functions.

1 Introduction

It is very well known that the n-cycle $(1\,2 \dots n)$ cannot be written as a product of less than $n-1$ transpositions and that there are n^{n-2} distinct ways of writing it as a product of exactly $n-1$ transpositions. The sequence $(n^{n-2})_{n \geqslant 1}$ also counts a variety of other combinatorial objects, including Cayley trees and parking functions, and a wealth of bijections have been described between minimal factorisations, Cayley trees and parking functions (see the works of Dénes [6], Moszkowski [14], Goulden-Pepper [9], Goulden-Yong [8], Stanley [17]).

The enumeration of minimal factorisations of $(1\,2 \dots n)$ as a product of transpositions was also generalised in several directions, including the enumeration of minimal factorisations as a product of cycles of prescribed lengths (Biane [4]), the enumerations of classes of factorisations which differ by exchanging commuting transpositions (Eidswick [7]), the enumeration of minimal factorisations involving only certain types of transpositions, or which involve a fixed number of times a certain integer (Rattan [15], Irving-Rattan [10]).

In the course of the study of the distribution of the eigenvalues of certain random unitary matrices, using the relations between the unitary groups and the symmetric groups (see [12]), we were led to enumerating the sequences of transpositions which appear as initial segments of a minimal factorisation of the n-cycle $(1\,2\ldots n)$. This problem seemed not to have been studied previously, and the way in which we were able to solve it was not particularly enlightening. The object of this paper is to give a simpler and more explicit solution to this enumeration problem. We will also revisit our original proof and give a more self-contained version of it.

Let us give a precise statement of our result. Let $n \geqslant 1$ be an integer. Let $\mathfrak{S}_n$ be the symmetric group of order n. Let $\mathsf{T}_n \subseteq \mathfrak{S}_n$ be the subset which consists of all transpositions. Let $k \in \{0, \ldots, n-1\}$ be an integer. We define

$$\mathscr{F}_{n,k} = \left\{(\tau_1, \ldots, \tau_k) \in \mathsf{T}_n^k : \exists\, (\tau_{k+1}, \ldots, \tau_{n-1}) \in \mathsf{T}_n^{n-1-k}, \tau_1 \ldots \tau_{n-1} = (1\,2\ldots n)\right\}$$

and call any element of $\mathscr{F}_{n,k}$ a k-prefix of a minimal factorisation of $(1\ldots n)$. Our main result is the following.

Theorem 1. *The set $\mathscr{F}_{n,k}$ has $\binom{n}{k+1} n^{k-1}$ elements.*

The cases where $n = 1$, or $k = 0$, or $k = 1$, are trivial. The case $k = n-1$ reproduces the classical result mentioned at the beginning of this introduction. The array $(|\mathscr{F}_{n,k}|)_{n \geqslant 1, 0 \leqslant k \leqslant n-1}$ corresponds to the sequence A033842 in the OEIS.

In Section 2, we recall some basic properties of the geometry of the symmetric group generated by its transpositions. In Section 3, we use classical results on the enumeration of Cayley trees to give a first proof of Theorem 1. We make this section as self-contained as possible by recalling without proof bijections which allow one to prove the results which we use. In Section 4, we investigate in more detail the elements of $\mathscr{F}_{n,k}$, especially those which enjoy a certain monotonicity property. In Section 5, we construct an action of the symmetric group $\mathfrak{S}_k$ on the set $\mathscr{F}_{n,k}$ which allows us to turn any factorisation into a non-decreasing one. In Section 6, we briefly review, as a preparation for our main argument, the classical parking functions and sketch their relation with minimal factorisations, along the lines of the work of Stanley on non-crossing partitions [17]. In Section 7, we define and enumerate the analogues of parking functions which finally enable us to give, in Section 8, a bijective proof of Theorem 1.

2 The geometry of the symmetric group

Let $n \geqslant 1$ be an integer. Recall that $\mathfrak{S}_n$ denotes the symmetric group of order n and $\mathsf{T}_n \subset \mathfrak{S}_n$ the subset which consists of all transpositions. Since T_n is a conjugacy class of $\mathfrak{S}_n$, the Cayley graph of the couple $(\mathfrak{S}_n, \mathsf{T}_n)$ is defined without ambiguity regarding the order of multiplications. The most fundamental property of this graph is expressed by the identity

$$(x_1 \ldots x_p)(y_1 \ldots y_q)(x_p y_q) = (x_1 \ldots x_p y_1 \ldots y_q),$$

valid for any sequence of pairwise distinct integers $x_1, \ldots, x_p, y_1, \ldots, y_q$. According to this identity, two permutations are neighbours in the Cayley graph of $(\mathfrak{S}_n, \mathsf{T}_n)$ if and only if one of them is obtained by merging two cycles of the other. In particular, the total number of cycles of two neighbouring permutations differs by 1.

In fact, the graph distance between any two permutations can be computed by counting cycles. For all $\sigma \in \mathfrak{S}_n$, let us denote by $\ell(\sigma)$ the number of cycles of σ, including trivial cycles. For example, σ is a transposition if and only if $\ell(\sigma) = n - 1$. Then the distance between two permutations σ_1 and σ_2, which we shall denote by $d(\sigma_1, \sigma_2)$, is simply given by $d(\sigma_1, \sigma_2) = n - \ell(\sigma_1 \sigma_2^{-1})$.

The notion of distance on $\mathfrak{S}_n$ allows one to define a partial order on $\mathfrak{S}_n$, by declaring $\sigma_1 \preccurlyeq \sigma_2$ if and only if $d(\mathrm{id}, \sigma_2) = d(\mathrm{id}, \sigma_1) + d(\sigma_1, \sigma_2)$. The trivial permutation is the minimum of $\mathfrak{S}_n$ with respect to this order, and the n-cycles are its maximal elements. Observe that the set of k-prefixes of minimal factorisations of $(1 \ldots n)$ can be rewritten as

$$\begin{aligned}\mathscr{F}_{n,k} &= \left\{(\tau_1, \ldots, \tau_k) \in (\mathsf{T}_n)^k : d(\mathrm{id}, \tau_1 \ldots \tau_k) = k,\ \tau_1 \ldots \tau_k \preccurlyeq (1 \ldots n)\right\} \\ &= \left\{(\tau_1, \ldots, \tau_k) \in (\mathsf{T}_n)^k : \mathrm{id} \preccurlyeq \tau_1 \preccurlyeq \tau_1\tau_2 \preccurlyeq \ldots \preccurlyeq \tau_1 \ldots \tau_k \preccurlyeq (1 \ldots n)\right\}.\end{aligned}$$

The last description suggests to think of the elements of $\mathscr{F}_{n,k}$ as paths in the Cayley graph of the symmetric group and this is a point of view which we shall indeed adopt later.

The following very useful characterisation of the set of permutations which are smaller than the n-cycle $(1 \ldots n)$ is a consequence of [5, Theorem 1].

Proposition 2. *Let $\sigma \in \mathfrak{S}_n$ be a permutation. The relation $\sigma \preccurlyeq (1 \ldots n)$ holds if and only if the following two conditions hold:*

1. *Each cycle of σ has the cyclic order induced by $(1 \ldots n)$.*
2. *The partition of $\{1, \ldots, n\}$ by the orbits of σ is non-crossing with respect to the cyclic order defined by $(1 \ldots n)$.*

The first condition is equivalent to the following: each cycle of σ can be written $(i_1 \ldots i_r)$ with $i_1 < \ldots < i_r$. The second condition means that there exists no subset $\{i, j, k, l\}$ of $\{1, \ldots, n\}$ with $i < j < k < l$ such that i and k belong to a cycle of σ and j and l belong to another cycle of σ.

3 Minimal factorisations and Cayley trees

A Cayley tree of size n is a connected graph with $n - 1$ edges and n labelled vertices. Unless otherwise stated, the vertices of a Cayley tree of size n will be labelled by the integers $\{1, \ldots, n\}$.

Theorem 3 (Cayley [1])**.** *There are n^{n-2} Cayley trees of size n.*

Proof. A simple bijective proof of this fact is given by considering the Prüfer code of a tree. This code is a sequence of $n-2$ elements of $\{1,\dots,n\}$, obtained by repeating $n-2$ times the following operation: identify the leaf of the tree with the smallest label, write the label of the vertex to which it is attached, and erase the leaf. The observation that each integer appears once less in the code of a tree than the degree of the corresponding vertex in the tree leads to a short proof of the fact that the correspondence between a Cayley tree and its Prüfer code is bijective. □

Let us now recall Dénes' argument for counting the minimal factorisations of $(1\dots n)$.

Theorem 4 (Dénes [6])**.** *There are n^{n-2} minimal factorisations of $(1\dots n)$.*

Proof. Dénes counts minimal factorisations not only of $(1\dots n)$, but of an arbitrary n-cycle. Let $((i_1\,j_1),\dots,(i_{n-1}\,j_{n-1}))$ be such a factorisation. Construct a graph with vertices $\{1,\dots,n\}$ by successively adding the edges $\{i_1,j_1\},\dots,\{i_{n-1},j_{n-1}\}$, respectively labelled $1,\dots,n-1$. Since for all $l\geqslant 1$ the transposition $(i_l\,j_l)$ exchanges two points which are not in the same cycle of $(i_1\,j_1)\dots(i_{l-1}\,j_{l-1})$, the corresponding edge joins two vertices which were not yet in the same connected component of the graph being constructed. Hence, the final graph is connected, hence a tree. Moreover, this tree together with the labelling of its edges allows one to reconstruct the factorisation. Finally, Cayley trees with vertex set $\{1,\dots,n\}$ and edges labelled by $\{1,\dots,n-1\}$ are in one-to-one correspondence with minimal factorisation of an n-cycle. Since there are $(n-1)!n^{n-2}$ Cayley trees with labelled edges and $(n-1)!$ n-cycles, there are exactly n^{n-2} minimal factorisations of any given n-cycle. □

Let us finally recall the number of non-crossing partitions of $\{1,\dots,n\}$ whose block sizes agree with a prescribed partition of the integer n. We will denote by $1^{s_1}2^{s_2}\dots n^{s_n}$ the partition which for all $i\in\{1,\dots,n\}$ has s_i parts equal to i. In the rest of this section, we shall consider partitions of $n=s_1+2s_2+\dots+ns_n$ with $n-k=s_1+\dots+s_n$ blocks. Let us note for later reference that for such a partition, $0s_1+1s_2+\dots+(n-1)s_n=k$.

Theorem 5 (Kreweras [11])**.** *The number of non-crossing partitions of $\{1,\dots,n\}$ with $n-k$ blocks and the block sizes of which determine the partition $1^{s_1}2^{s_2}\dots n^{s_n}$ of n, is $\frac{n!}{(k+1)!s_1!\dots s_n!}$.*

The original proof of this result is due to Kreweras, but we sketch a bijective argument due to Liaw, Yeh, Hwang and Chang [13].

Proof. Set $\lambda=(\lambda_1,\dots,\lambda_{n-k})=1^{s_1}2^{s_2}\dots n^{s_n}$. Let $\pi=(\pi_1,\dots,\pi_{n-k})$ be a non-crossing partition of $\{1,\dots,n\}$ whose blocks are labelled from 1 to $n-k$ and have sizes $\lambda_1,\dots,\lambda_{n-k}$. Set $m=\max\pi_{n-k}$ and, for each $i\in\{1,\dots,n-k-1\}$, let p_i be the first element of π_i which one meets after starting from m and moving clockwise. Then the mapping which to π associates $(p_1,\dots,p_{n-k-1})$ is a bijection from the set of non-crossing partitions of $\{1,\dots,n\}$ whose blocks are labelled from 1 to $n-k$ and have respective sizes $\lambda_1,\dots,\lambda_k$ to the set of arrangements of $n-k-1$ distinct elements of $\{1,\dots,n\}$.

There are thus $\frac{n!}{(k+1)!}$ such partitions with labelled blocks. The final result follows from the fact that there are $s_1! \dots s_n!$ disctinct ways of labellings the blocks of π such that they have respective sizes $\lambda_1, \dots, \lambda_{n-k}$. $\square$

These results allow us to count k-prefixes of minimal factorisations of $(1 \dots n)$.

Proof of Theorem 1. Let us count the prefixes $(\tau_1, \dots, \tau_k)$ according to the cycle structure of the permutation $\tau_1 \dots \tau_k$. This permutation has $n-k$ cycles whose sizes can determine any partition of n in $n-k$ parts. If $1^{s_1} \dots n^{s_n}$ is such a partition, then a minimal factorisation of a permutation with this cycle structure is obtained by choosing a minimal factorisation of each of its cycles, and shuffling these minimal factorisations. Thus,

$$|\mathscr{F}_{n,k}| = \sum_{\substack{s_1+\dots+s_n=n-k \\ 1s_1+\dots+ns_n=n}} \frac{n!}{(k+1)!s_1!\dots s_n!} 1^{(1-2)s_1} \dots n^{(n-2)s_n} \frac{k!}{0!^{s_1} \dots (n-1)!^{s_n}},$$

where the fisrt term counts, according to Proposition 2 and Theorem 5, the number of permutations in the chosen conjugacy class which are dominated by $(1 \dots n)$, the second, according to Theorem 4, the number of choices for the minimal factorisations of each cycle and the third the number of shufflings of these factorisations.

In the next step of the computation, we seem to lose any sense of bijectivity: we replace the sum over the partitions of the integer n with $n-k$ parts by a sum over all partitions of the set $\{1, \dots, n\}$ with $n-k$ blocks, be they crossing or not. This requires that we include a factor which compensates for the repetitions that this change of index introduces. We find

$$\begin{aligned} |\mathscr{F}_{n,k}| &= \sum_{\substack{\text{partitions of } \{1,\dots,n\} \\ \text{with } n-k \text{ blocks,} \\ s_i \text{ of size } i,\, i=1\dots n}} \frac{1!^{s_1}s_1!\dots n!^{s_n}s_n!}{n!} \frac{n!}{(k+1)!s_1!\dots s_n!} 1^{(1-2)s_1} \dots n^{(n-2)s_n} \frac{k!}{0!^{s_1} \dots (n-1)!^{s_n}} \\ &= \frac{1}{k+1} \sum_{\substack{\text{partitions of } \{1,\dots,n\} \\ \text{with } n-k \text{ blocks}}} 1^{(1-1)s_1} \dots n^{(n-1)s_n}, \end{aligned}$$

where $s_1, \dots, s_n$ are respectively the numbers of blocks of size $1, \dots, n$ of the partition considered.

It now appears, according to Theorem 3, that the sum counts the forests with vertex set $\{1, \dots, n\}$ and with $n-k$ connected components, in every connected component of which a distinguished vertex has been chosen. Adding a vertex labelled 0 to such a forest and joining the distinguished vertices to this new vertex produces a Cayley tree with vertex set $\{0, \dots, n\}$ in which 0 has degree $n-k$. This correspondence between forests and trees is bijective, and the Cayley trees with vertex set $\{0, \dots, n\}$ in which 0 has degree $n-k$ are exactly those whose Prüfer code involves $n-k-1$ times the letter 0. There are $\binom{n-1}{k}n^k$ such Prüfer codes, hence such trees. Finally,

$$|\mathscr{F}_{n,k}| = \frac{1}{k+1}\binom{n-1}{k}n^k = \binom{n}{k+1}n^{k-1},$$

as expected. $\square$

It is interesting that this proof eventually shows that $(k+1)|\mathscr{F}_{n,k}|$ is equal to the number of Cayley trees on $\{0,\ldots,n\}$ in which 0 has $n-k$ neighbours. We were not able to find a bijective proof of this fact.

4 Non-decreasing minimal factorisations

We now embark on a bijective and much more concrete proof of Theorem 1. Let us introduce some notation. Let us first agree that writing a transposition under the form $(i\,j)$ implies that the inequality $i<j$ holds. Then, when considering an element $\gamma = (\tau_1,\ldots,\tau_k)$ of $\mathscr{F}_{n,k}$, we shall write $\gamma_0 = \mathrm{id}$ and, for all $l\in\{1,\ldots,k\}$, $\gamma_l = \tau_1\ldots\tau_l$. For all $x\in\{1,\ldots,n\}$ and all $l\in\{0,\ldots,k\}$, we shall also denote by $C_l(x)$ the cycle of γ_l containing x, which in fact we see simply as a subset of $\{1,\ldots,n\}$, namely the orbit of x under γ_l. Finally, for all permutation σ, we write $|\sigma| = d(\mathrm{id},\sigma)$.

Lemma 6. *Let $\gamma = ((i_1\,j_1),\ldots,(i_k\;j_k))$ be an element of $\mathscr{F}_{n,k}$. Choose $l\in\{1,\ldots,k\}$. The following properties hold.*

1. *$i_l < \min C_{l-1}(j_l)$ and i_l is the largest element of $C_{l-1}(i_l)$ with this property.*
2. *$j_l = \max C_{l-1}(j_l)$.*
3. *$i_l < \min C_{l-1}(i_l+1)$.*

Proof. 1 and 2. Since $|\gamma_l| = |\gamma_{l-1}|+1$, i_l and j_l belong to distinct cycles of γ_{l-1} and to the same cycle of γ_l. The cycle of γ_l which contains i_l and j_l has the cyclic order induced by $(1\ldots n)$, so that it is of the form $(x_1<\ldots<x_r<i_l<y_1<\ldots<y_s<j_l<z_1<\ldots<z_t)$. The cycles of γ_{l-1} which contain i_l and j_l are thus respectively $(x_1\ldots x_r\; i_l\; z_1\ldots z_t)$ and $(y_1\ldots y_s\; j_l)$. The first two assertions follow. See Figure 1 for an illustration.

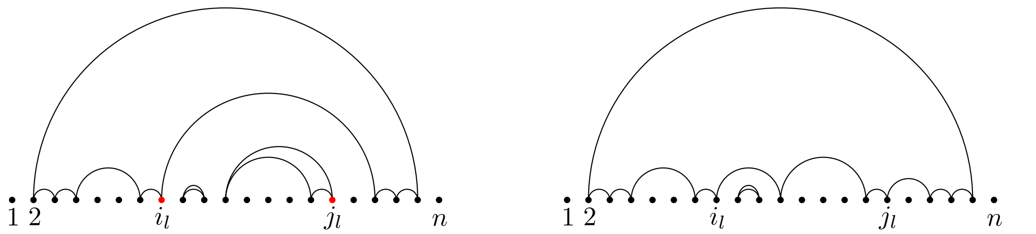

Figure 1: On this picture, we see the cycles of γ_{l-1} and γ_l which contain i_l, i_l+1 and j_l.

3. A glance at the left part of Figure 1 should help to see why this assertion is true. Let us prove it. If $i_l+1\in C_{l-1}(j_l)$, then the third assertion follows from the first part of the first assertion. Let us now assume that $i_l+1\notin C_{l-1}(j_l)$. In this case, and since by the second part of the first assertion we know that $i_l+1\notin C_{l-1}(i_l)$, we have the equality $C_{l-1}(i_l+1) = C_l(i_l+1)$. Suppose there was an element x in $C_{l-1}(i_l+1)$ such that $x<i_l$.

Then the quadruplet $x < i_l < i_l + 1 < j_l$ would violate the non-crossing condition on the cycles of γ_l imposed by the condition $\gamma_l \preccurlyeq (1 \dots n)$. This concludes the proof of the third assertion. $\square$

Let us now make an observation of monotonicity (see also [8, Theorem 2.2]).

Lemma 7. *Consider $\gamma = ((i_1\, j_1), \dots, (i_k\, j_k)) \in \mathscr{F}_{n,k}$ and $l, m \in \{1, \dots, k\}$ with $l < m$.*

1. *If $i_l = i_m$, then $j_l > j_m$.*

2. *If $j_l = j_m$, then $i_l > i_m$.*

3. *If $i_1 \leqslant \dots \leqslant i_k$, then $j_1, \dots, j_k$ are pairwise distinct.*

Proof. The multiplication of γ_{m-1} by $(i_m\, j_m)$ inserts the cycle of j_m in γ_{m-1} in the cycle of i in γ_{m-1}, immediately after i. Hence, i, j_m, j_l are in this cyclic order in their common cycle of γ_m. Since $\gamma_m \preccurlyeq (1 \dots n)$ and $i < j_l$, this implies $i < j_m < j_l$.

The second assertion follows from the first and the existence of a simple involution of $\mathscr{F}_{n,k}$, which we describe in Lemma 8 below.

The third assertion follows immediately from the second. $\square$

Lemma 8. *Let $((i_1\, j_1), \dots, (i_k\ j_k))$ be an element of $\mathscr{F}_{n,k}$. Then the chain of transpositions $((n + 1 - j_k \quad n + 1 - i_k), \dots, (n + 1 - j_1 \quad n + 1 - i_1))$ is also an element of $\mathscr{F}_{n,k}$.*

Proof. Let $\varphi \in \mathfrak{S}_n$ be the involution which exchanges i and $n+1-i$ for all $i \in \{1, \dots, n\}$. The point is the identity $\varphi(1 \dots n)^{-1}\varphi^{-1} = (1 \dots n)$. Let $(\tau_1, \dots, \tau_k)$ be an element of $\mathscr{F}_{n,k}$. Then on one hand $|\varphi\tau_k \dots \tau_1\varphi^{-1}| = |\tau_k \dots \tau_1| = |\tau_1 \dots \tau_k| = k$. On the other hand, we have the equality $|(1 \dots n)^{-1}\varphi\tau_k \dots \tau_1\varphi^{-1}| = |\varphi^{-1}(1 \dots n)^{-1}\varphi\tau_k \dots \tau_1| = |(1 \dots n)(\tau_1 \dots \tau_k)^{-1}| = n - 1 - k$. Hence, $\varphi\tau_k \dots \tau_1\varphi^{-1} \preccurlyeq (1 \dots n)$. Finally, the chain of transpositions $(\varphi\tau_k\varphi^{-1}, \dots, \varphi\tau_1\varphi^{-1})$ belongs to $\mathscr{F}_{n,k}$. $\square$

The key of our argument is that the elements of $\mathscr{F}_{n,k}$ for which the sequence $(i_1, \dots, i_k)$ is non-decreasing are easy to describe and to characterise. We call them *non-decreasing prefixes* and we denote by $\mathscr{F}^{\uparrow}_{n,k}$ the subset of $\mathscr{F}_{n,k}$ consisting of non-decreasing prefixes.

By the *support* of an element γ of $\mathfrak{S}_n$, we mean the set $\{i \in \{1, \dots, n\} : \gamma(i) \neq i\}$.

Lemma 9. *Let $\gamma = ((i_1\, j_1), \dots, (i_k\ j_k))$ be an element of $\mathscr{F}_{n,k}$ such that $j_1, \dots, j_k$ are pairwise distinct. The following properties hold.*

1. *For all $l \in \{1, \dots, k\}$, j_l is a fixed point of γ_{l-1} and γ_l is obtained from γ_{l-1} by inserting j_l into the cycle of i_l immediately after i_l.*

2. *For all $m \in \{1, \dots, k\}$, the support of γ_m is $\bigcup_{l=1}^{m}(\{i_l\} \cup \{j_l\})$.*

Proof. For all $l \in \{1, \dots, k\}$, we have $j_l > i_l \geqslant \dots \geqslant i_1$ and, by assumption, $j_l \notin \{j_1, \dots, j_{l-1}\}$, so that j_l is a fixed point of γ_{l-1}. Hence, $\gamma_l(i_l) = j_l$, and we have $\gamma_l(j_l) = \gamma_{l-1}(i_l)$. This is exactly the first assertion.

The second assertion follows immediately by induction on k. $\square$

We can now characterise the elements of $\mathscr{F}^{\uparrow}_{n,k}$.

Proposition 10. *Consider* $((i_1\,j_1),\ldots,(i_k\,j_k)) \in (\mathsf{T}_n)^k$. *The following properties are equivalent.*

1. $((i_1\,j_1),\ldots,(i_k\,j_k)) \in \mathscr{F}^{\uparrow}_{n,k}$.

2. $i_1 \leqslant \ldots \leqslant i_k$ *and for all* $l,m \in \{1,\ldots,k\}$ *such that* $l < m$*, one either has* $j_l \leqslant i_m$ *or* $j_l > j_m$.

Proof. $1 \Rightarrow 2$. Let us prove that the first property implies the second. For this, let us choose $\gamma = ((i_1\,j_1),\ldots,(i_k\,j_k)) \in \mathscr{F}^{\uparrow}_{n,k}$ and l,m with $1 \leqslant l < m \leqslant k$. It follows from the third assertion of Lemma 7 that $j_l \neq j_m$. Let us assume by contradiction that $i_m < j_l < j_m$. Then, by Lemma 7, $i_l < i_m$. Hence, $i_l < i_m < j_l < j_m$. We will find a contradiction with the non-crossing property of the cycles of γ_m.

To start with, i_l and j_l belong to the same cycle of γ_{m-1}. We claim that neither i_m nor j_m belong to this cycle, so that the cycles of γ_m which contain i_l and j_l on one hand and i_m and j_m on the other hand are distinct, which yields the expected contradiction.

For j_m, our claim follows from the first assertion of Lemma 9, according to which $C_{m-1}(j_m) = \{j_m\}$. For i_m, let us assume that i_m belongs to $C_{m-1}(i_l) = C_{m-1}(j_l)$. Then j_l would belong to $C_{m-1}(i_m)$ and satisfy both $j_l > i_m$ and $j_l < \{j_m\} = C_{m-1}(j_m)$, in contradiction with the first assertion of Lemma 6.

$2 \Rightarrow 1$. Let us now prove that the second property implies the first. To start with, note that the second property implies that $j_1,\ldots,j_k$ are pairwise distinct. Indeed, if there exists $l < m$ such that $j_l = j_m$, then $j_m = j_l \leqslant i_m \leqslant j_m$, in contradiction with our agreement that $i_m < j_m$.

Moreover, note that for the same reason, the equality $i_l = i_m$ for $l < m$ implies $j_l > j_m$.

We now proceed by induction on k. If $k = 1$, then the result is true because $\mathscr{F}^{\uparrow}_{n,1} = \mathsf{T}_n$. Let us assume that the result holds for paths of length up to $k-1$ and let us consider a path $\gamma = ((i_1\,j_1),\ldots,(i_k\,j_k)) \in (\mathsf{T}_n)^k$ such that the second property holds. By induction, γ_{k-1} is a product of $n-k+1$ cycles with the cyclic order induced by $(1\ldots n)$ and whose cycles form a non-crossing partition of $\{1,\ldots,n\}$.

By the second assertion of Lemma 9, the support of γ_{k-1} is the set $\bigcup_{l=1}^{k-1}(\{i_l\}\cup\{j_l\})$. On one hand, for all $l \in \{1,\ldots,k-1\}$, we have $i_l \leqslant i_{k-1} < j_{k-1}$. On the other hand, we observed that $j_1,\ldots,j_k$ are pairwise distinct. Thus, j_k is a fixed point of γ_{k-1}, so that γ_k is a product of $n-k$ cycles.

Let us prove that the cyclic order of the cycle of γ_k containing j_k is the order induced by $(1\ldots n)$. We certainly have $i_k < j_k$ and we claim that $i_k < j_k < \gamma_{k-1}(i_k)$ in the cyclic order of $(1\ldots n)$, which means exactly that $\gamma_{k-1}(i_k) \leqslant i_k$ or $\gamma_{k-1}(i_k) > j_k$. But $\gamma_{k-1}(i_k)$ is either i_l for some $l \in \{1,\ldots,k-1\}$, in which case $\gamma_{k-1}(i_k) \leqslant i_k$, or $\gamma_{k-1}(i_k)$ is j_l for some $l \in \{1,\ldots,k-1\}$, in which case $\gamma_{k-1}(i_k) \leqslant i_k$ or $\gamma_{k-1}(i_k) > j_k$, by the main assumption.

Let us finally prove that the cycles of γ_k form a non-crossing partition. The only way this could not be true is if some cycle contained two elements x and y such that $i_k < x < j_k < y < \gamma_{k-1}(i_k)$ in the cyclic order. But, by the main assumption again, any

x such that $i_k < x < j_k$ does neither belong to $\{i_1, i_2, \dots, i_k\}$ nor to $\{j_1, j_2, \dots, j_k\}$ and hence is a fixed point of γ_k. $\square$

This proposition allows us to prove that a non-decreasing factorisation $\gamma \in \mathscr{F}^{\uparrow}_{n,k}$ is completely determined by the sequence $(i_1, \dots, i_k)$ and the support of γ_k.

Corollary 11. *Let $\gamma = ((i_1\, j_1), \dots, (i_k\, j_k))$ be an element of $\mathscr{F}^{\uparrow}_{n,k}$. For all $m \in \{1, \dots, k\}$, j_m is the minimum of the intersection of $\{i_m + 1, \dots, n\}$ with the support of γ_m.*

Moreover, if $\gamma' = ((i_1\, j'_1), \dots, (i_k\, j'_k))$ is another element of $\mathscr{F}^{\uparrow}_{n,k}$ such that γ'_k and γ_k have the same support, then $\gamma' = \gamma$.

Proof. The support of γ_m is $\bigcup_{l=1}^{m} \{i_l\} \cup \{j_1, \dots, j_m\}$. For all $l < m$, we have $i_l \leqslant i_m$ and, by Proposition 10, $j_l \leqslant i_m$ or $j_l > j_m$. The first assertion follows.

Let us prove the second assertion by induction on k. The result is true for $k = 0$. Let us assume that is has been proved for paths of length up to $k - 1$. By the first assertion, $j'_k = j_k$. Hence, $\delta = ((i_1\, j_1), \dots, (i_{k-1}\ j_{k-1}))$ and $\delta' = ((i_1\, j'_1), \dots, (i_{k-1}\ j'_{k-1}))$ are two elements of $\mathscr{F}_{n,k-1}$ such that δ'_{k-1} and δ_{k-1} have the same support. By induction, they are equal. $\square$

5 Permutation of factorisations

In this section, we describe an action of the symmetric group $\mathfrak{S}_k$ on $\mathscr{F}_{n,k}$ such that every orbit contains exaclty one non-decreasing prefix. More precisely, let us consider the projection $\mathcal{I} : \mathscr{F}_{n,k} \to \{1, \dots, n-1\}^k$ which sends the chain $((i_1\, j_1), \dots, (i_k\, j_k))$ to the sequence $(i_1, \dots, i_k)$. The group $\mathfrak{S}_k$ acts naturally on $\{1, \dots, n-1\}^k$ by the formula $\sigma \cdot (i_1, \dots, i_k) = (i_{\sigma^{-1}(1)}, \dots, i_{\sigma^{-1}(k)})$ and we will endow $\mathscr{F}_{n,k}$ with an action of $\mathfrak{S}_k$ such that $\mathcal{I}$ is an equivariant mapping, which moreover preserves the stabilisers. This last condition is equivalent to the fact that the restriction of $\mathcal{I}$ to each orbit of $\mathfrak{S}_k$ in $\mathscr{F}_{n,k}$ is an injection. Hence, the action of $\mathfrak{S}_k$ on $\{1, \dots, n-1\}^k$ can be lifted uniquely to $\mathscr{F}_{n,k}$ through $\mathcal{I}$.

In order to define the action of $\mathfrak{S}_k$ on $\mathscr{F}_{n,k}$, we will use the classical action of the braid group B_k on the product of k copies of an arbitrary group G (see for example [2]). If $\beta_1, \dots, \beta_{k-1}$ are the usual generators of B_k, this action is given by the formula

$$\beta_l \cdot (g_1, \dots, g_k) = (g_1, \dots, g_{l+1}, g_{l+1}^{-1} g_l g_{l+1}, \dots, g_k),$$

valid for all $(g_1, \dots, g_k) \in G^k$ and all $l \in \{1, \dots, k-1\}$. Observe that if $T \subset G$ is a conjugacy class, then T^k is stable under this action. Moreover, the product map $(g_1, \dots, g_n) \mapsto g_1 \dots g_n$ is invariant under this action.

Let us denote by $\sigma_1 = (1\,2), \dots, \sigma_{k-1} = (k-1\; k)$ the Coxeter generators of $\mathfrak{S}_k$, so that the natural projection $B_k \to \mathfrak{S}_k$ sends β_l to σ_l for all $l \in \{1, \dots, k-1\}$. Consider $\gamma = ((i_1\, j_1), \dots, (i_k\, j_k))$ in $\mathscr{F}_{n,k}$ and $l \in \{1, \dots, k-1\}$. Set

$$\sigma_l \cdot \gamma = \begin{cases} \gamma & \text{if } i_l = i_{l+1}, \\ \beta_l \cdot \gamma & \text{if } i_l < i_{l+1}, \\ \beta_l^{-1} \cdot \gamma & \text{if } i_l > i_{l+1}. \end{cases} \tag{1}$$

Since the action of the braid group preserves the ordered product of the components, $\sigma_l \cdot \gamma$ belongs to $\mathscr{F}_{n,k}$.

Practically, $\sigma_l \cdot \gamma$ is obtained from γ by doing nothing if $i_l = i_{l+1}$, and otherwise, by swapping the l-th and $(l+1)$-th transpositions of γ and conjugating the one with the smallest i by the other. In this way, the transposition with the largest i is not modified, and only the j of the other is affected. For example, if $k = 2$,

$$\begin{aligned}
\sigma_1 \cdot ((1\,3),(1\,2)) &= ((1\,3),(1\,2)),\\
\sigma_1 \cdot ((1\,2),(2\,3)) &= ((2\,3),(1\,3)),\\
\sigma_1 \cdot ((2\,3),(1\,3)) &= ((1\,2),(2\,3)).
\end{aligned}$$

Proposition 12. *The action of the Coxeter generators of $\mathfrak{S}_k$ on $\mathscr{F}_{n,k}$ defined by* (1) *extends to an action of $\mathfrak{S}_k$.*

Moreover, the mapping $\mathcal{I} : \mathscr{F}_{n,k} \to \{1,\ldots,n-1\}^k$ is equivariant and preserves the stabilisers. This means that for all $\gamma \in \mathscr{F}_{n,k}$ and all $\pi \in \mathfrak{S}_k$, one has $\pi \cdot \mathcal{I}(\gamma) = \mathcal{I}(\gamma)$ if and only if $\pi \cdot \gamma = \gamma$.

Proof. We must prove that the operations which we have defined satisfy the Coxeter relations $\sigma_l^2 = \mathrm{id}$ for $l \in \{1,\ldots,k-1\}$, $(\sigma_l\sigma_m)^2 = \mathrm{id}$ for $l, m \in \{1,\ldots,k-1\}$ with $|l-m| \geqslant 2$, and $(\sigma_l\sigma_{l+1})^3 = \mathrm{id}$ for $l \in \{1,\ldots,n-2\}$.

To prove the first relation, it suffices to observe that $\sigma_l \cdot (\sigma_l \cdot \gamma)$ is either γ or $\beta_l\beta_l^{-1} \cdot \gamma$ or $\beta_l^{-1}\beta_l \cdot \gamma$, hence in any case γ. The second relation is equivalent to $\sigma_l \cdot (\sigma_m \cdot \gamma) = \sigma_m \cdot (\sigma_l \cdot \gamma)$ and it clearly holds for $|l-m| \geqslant 2$. In order to prove the third relation, there are ten cases to consider, corresponding to the possible relative positions of i_l, i_{l+1} and i_{l+2}. In the six cases where i_l, i_{l+1}, i_{l+2} are pairwise distinct, the relation $\beta_l\beta_{l+1}\beta_l = \beta_{l+1}\beta_l\beta_{l+1}$ implies the relation on $(\sigma_l\sigma_{l+1})^3 = \mathrm{id}$. The other cases are simpler.

The complete verification is probably best done by the reader himself, but Figure 2 below shows, when $k = 3$, the effect of σ_1 and σ_2 on γ in all the possible cases regarding the respective positions of i_1, i_2, i_3.

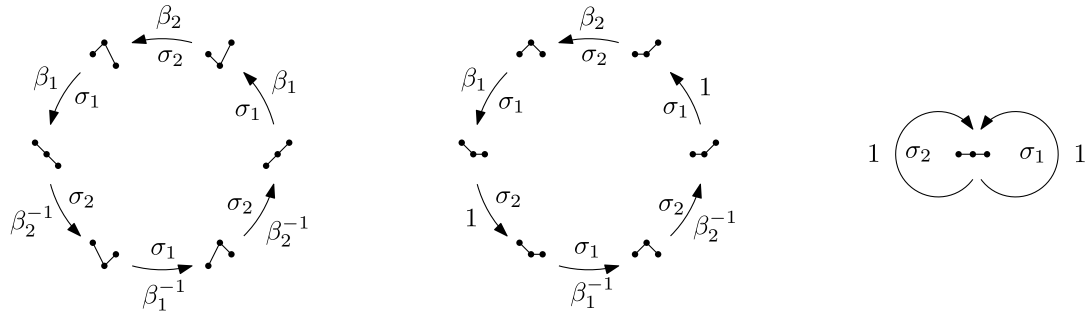

Figure 2: Verification of the Coxeter relations for the action of $\mathfrak{S}_k$ on $\mathscr{F}_{n,k}$. The top arrow of the middle circle indicates, for example, that if $i_1 = i_2 < i_3$, then applying σ_2 to γ corresponds, by definition, to applying β_2, and produces an triple of transpositions such that $i_1 = i_3 < i_2$.

We have thus an action of the symmetric group $\mathfrak{S}_k$ on $\mathscr{F}_{n,k}$. It is a straightforward consequence of its definition that the mapping I is equivariant under this action and the natural action on $\{1,\dots,n-1\}^k$. If $\gamma \in \mathscr{F}_{n,k}$ and $\pi \in \mathfrak{S}_k$ satisfy $\pi\cdot\gamma=\gamma$, then $\pi\cdot\mathcal{I}(\gamma)=\mathcal{I}(\pi\cdot\gamma)=\mathcal{I}(\gamma)$. Finally, let us prove that $\pi\cdot\mathcal{I}(\gamma)=\mathcal{I}(\gamma)$ implies $\pi\cdot\gamma=\gamma$. Let us choose $\gamma\in\mathscr{F}_{n,k}$. A permutation π stabilises $\mathcal{I}(\gamma)$ if and only if its cycles are contained in the level sets of the mapping $1\mapsto i_1,\dots,k\mapsto i_k$. Thus, the stabiliser of $\mathcal{I}(\gamma)$ is generated by the transpositions which it contains, and we may restrict ourselves to the case where π is a transposition $(l\,m)$ with $i_l=i_m$. We have $(l\,m)=\sigma_l\dots\sigma_{m-2}\sigma_{m-1}\sigma_{m-2}\dots\sigma_l$ and $\sigma_{m-2}\dots\sigma_l=(m-1\dots l)$. Since $\mathcal{I}$ is equivariant, the transpositions which are at the positions $m-1$ and m in the chain $\sigma_{m-2}\dots\sigma_l\cdot\gamma$ have respectively i_l and i_m as their smallest element. Since $i_l=i_m$, we find

$$(l\,m)\cdot\gamma=\sigma_l\dots\sigma_{m-2}\sigma_{m-1}\sigma_{m-2}\dots\sigma_l\cdot\gamma=\sigma_l\dots\sigma_{m-2}\sigma_{m-2}\dots\sigma_l\cdot\gamma=\gamma,$$

as expected. □

Corollary 13. *Let $\gamma=((i_1\,j_1),\dots,(i_k\,j_k))$ be an element of $\mathscr{F}_{n,k}$. The support of $\gamma_k=(i_1\,j_1)\dots(i_k\,j_k)$ is the set $\bigcup_{l=1}^k(\{i_l\}\cup\{j_l\})$.*

Proof. The action of $\mathfrak{S}_k$ on $\mathscr{F}_{n,k}$ preserves both the support of γ_k and the set to which we wish to show that it is equal. Since every orbit contains a non-decreasing chain, that is, a chain for which the sequence $(i_1,\dots,i_n)$ is non-decreasing, we may assume that the element γ which we are considering has this property, and apply the second assertion of Lemma 9 with $m=k$. □

In the context of minimal factorisations of a cycle, the natural action of the braid group is called the Hurwitz action and it is known to be transitive (see for example [16]). The action which we have defined here is germane to this action but different, as it is an action of the symmetric group. In [4], P. Biane defined another similar action of the symmetric group on minimal factorisations of a cycle as a product of cycles. The proof of Proposition 12 is inspired by his work.

6 Parking functions

Let $n\geqslant 1$ be an integer. For each sequence $I=(i_1,\dots,i_{n-1})$ of positive integers, let us call *spread* of I the function $s_I:\{1,\dots,n-1\}\to\mathbb{N}$ defined by backwards induction by setting $s_I(n-1)=i_{n-1}+1$ and, for all $k\leqslant n-2$,

$$s_I(k)=\min\{t\in\mathbb{N}:t\geqslant i_k+1,t\neq s_I(k+1),\dots,t\neq s_I(n-1)\}.$$

For example, $s_{(3,5,2,1,2,1)}$ sends 6 to 2, 5 to 3, 4 to 4, 3 to 5, 2 to 6 and 1 to 7.

Imagine a linear bike shed with infinitely many parking spaces numbered $1,2,\dots$ in the natural order. If $n-1$ bikes arrive successively in this shed, immediately after parking spaces $i_{n-1},\dots,i_1$ respectively, start exploring the shed in the direction of increasing labels, and if each bike parks in the first available space it finds, then for each $l\in\{1,\dots,n-1\}$, the l-th bike parks in space $s_I(l)$.

Apart from any ecological consideration, this scheme differs slightly from the usual one in that the bikes arrive in the reversed order of the sequence I and the l-th bike starts exploring the shed from space $i_l + 1$ instead of i_l. None of these modifications are important.

The sequence I is called a *parking function* if s_I is a bijection from $\{1, \ldots, n-1\}$ to $\{2, \ldots n\}$. It is well known that this is the case if and only if the non-decreasing reordering $(a_1 \leqslant \ldots \leqslant a_{n-1})$ of I satisfies $a_l \leqslant l$ for all $l \in \{1, \ldots, n-1\}$.

The main connection between parking functions and minimal factorisations is revealed by the following observation, first made by Stanley [17]. In what follows, we stick to the convention that writing a transposition under the form $(i\,j)$ implies that the inequality $i < j$ holds.

Proposition 14. *If $(i_1\ \ j_1) \ldots (i_{n-1}\ \ j_{n-1}) = (1 \ldots n)$, then $(i_1, \ldots, i_{n-1})$ is a parking function.*

Proof. Consider the permutations $\gamma_0 = \mathrm{id}$, $\gamma_1 = (i_1\ j_1), \ldots, \gamma_{n-1} = (i_1\ j_1) \ldots (i_{n-1}\ j_{n-1})$. Since multiplying a permutation by a transposition can only join two of its cycles or split one of its cycles in two, and since the chain $\gamma_0, \ldots, \gamma_{n-1}$ joins in $n-1$ steps a permutation with n cycles to a permutation with 1 cycle, it must be the case that for all $l \in \{1, \ldots, n-1\}$, the integers i_l and j_l belong to distinct cycles of γ_{l-1} and to the same cycle of γ_l.

Consider an integer $i \in \{1, \ldots, n-1\}$. For each l such that $i_l = i$, we have $j_l \notin C_{l-1}(i)$ but $j_l \in C_l(i)$. Since $j_l \in \{i+1, \ldots, n\}$, there can be no more than $n - i$ such integers l. It follows easily that $(i_1, \ldots, i_n)$ is a parking function. $\square$

Stanley actually proved that this correspondence between factorisations and parking functions is a bijection. We shall prove this fact later in a more general setting. Let us give a first indication of the way in which the factorisation can be reconstructed from the parking function.

Proposition 15. *Let $I = (i_1, \ldots, i_n)$ be a non-decreasing parking function. Then the equality $(i_1\ s_I(1)) \ldots (i_{n-1}\ s_I(n-1)) = (1 \ldots n)$ holds.*

This is also a result of which we will prove a more general version. In order to be able to reconstruct a factorisation from an arbitrary parking function, we use the action of $\mathfrak{S}_{n-1}$ on $\mathscr{F}_{n,n-1}$ defined in the previous section. Let us explain this now, also without proof, as we will prove a more general result in the next section. Recall that for any set X, the group $\mathfrak{S}_{n-1}$ acts on X^{n-1} by the formula $\sigma \cdot (x_1, \ldots, x_{n-1}) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n-1)})$.

Proposition 16. *Let $I = (i_1, \ldots, i_{n-1})$ be a parking function. Let $s_I : \{1, \ldots, n-1\} \to \{2, \ldots, n\}$ be the spread of I. Let $\sigma \in \mathfrak{S}_{n-1}$ be any permutation such that $\sigma \cdot I$ is non-decreasing. The unique minimal factorisation $\gamma \in \mathscr{F}_{n,n-1}$ such that $\mathcal{I}(\gamma) = I$ is*

$$\gamma = \sigma^{-1} \cdot ((i_{\sigma^{-1}(1)}\ s_{\sigma \cdot I}(1)), \ldots, (i_{\sigma^{-1}(n-1)}\ s_{\sigma \cdot I}(n-1))).$$

For example, take $n = 7$ and start with the parking function $(3, 5, 2, 1, 2, 1)$ of length 6. Sort this sequence and compute the spread of the resulting sequence. This yields the minimal factorisation $(1\,7)(1\,2)(2\,5)(2\,3)(3\,4)(5\,6)$ of $(1\dots7)$. Finally, permute the transpositions of this factorisation to find a factorisation whose image by I is $(3, 5, 2, 1, 2, 1)$. For this, exchange repeatedly neighbouring transpositions using the definition of the action of $\mathfrak{S}_6$. Possible intermediate steps are

$$\begin{array}{l}(3\,4)(1\,7)(1\,2)(2\,5)(2\,4)(5\,6)\\(3\,4)(5\,6)(1\,7)(1\,2)(2\,6)(2\,4)\\(3\,4)(5\,6)(2\,6)(1\,7)(1\,6)(2\,4)\\(3\,4)(5\,6)(2\,6)(1\,7)(2\,4)(1\,6)\end{array}$$

Each line is an element of $\mathscr{F}_{7,6}$, and the last is the unique one which corresponds to the parking function $(3, 5, 2, 1, 2, 1)$.

7 Restricted parking functions

Let k be a positive integer. Let $I = (i_1, \dots, i_k)$ be a sequence of positive integers. Let J be a subset of $\mathbb{N}$. Let us define the spread of I in J as the function $s_I : \{1, \dots, k\} \to \mathbb{N}\cup\{+\infty\}$ by backwards induction, setting

$$s_{I,J}(k) = \inf\left(J \cap [i_k + 1, \infty)\right)$$

and, for all $l \leqslant k - 1$,

$$s_{I,J}(l) = \inf\left(J \cap [i_l + 1, \infty) \cap \{s_{I,J}(l + 1), \dots, s_{I,J}(k)\}^c\right),$$

where the c in exponent denotes the complement in $\mathbb{N}$. The picture is the same as in Section 6, except that now only the spaces whose labels belong to J are open. It may happen that a bike does not find any space, in which case the function $s_{I,J}$ takes the value $+\infty$.

We say that the couple (I, J) is a *restricted parking function* of size (n, k) if I belongs to $\{1, \dots, n-1\}^k$, J is a subset of size k of $\{2, \dots, n\}$ and s_I is a bijection from $\{1, \dots, k\}$ onto J.

Proposition 17. *Let k be a positive integer. Let $I = (i_1, \dots, i_k)$ be a sequence of positive integers. Let $J = \{j_1 < \dots < j_k\}$ be a subset of $\mathbb{N}$. The couple (I, J) is a restricted parking function if and only if the non-decreasing reordering $(a_1 \leqslant \dots \leqslant a_k)$ of I satisfies, for all $l \in \{1, \dots, k\}$, the inequality $a_l < j_l$.*

With $J = \{2, \dots, k + 1\}$, one recovers the classical situation and the usual condition $a_l \leqslant l$.

Proof. Let us assume that (I, J) is a restricted parking function. The mapping $s_{I,J}$ is injective by construction and satisfies, for all $l \in \{1, \dots, k\}$, the inequality $i_l < s_{I,J}(l)$.

Thus, for all l, there at least $k-l+1$ elements of J in the interval $(a_l, +\infty)$. The elements $j_k, j_{k-1}, \ldots, j_l$, being the $k-l+1$ greatest elements of J, thus belong to this interval, and in particular we have $a_l < j_l$.

Let us prove the converse by induction on k. For $k = 1$, the assertion is true. Consider $k > 1$ and assume that it has been proved when I and J have size $k-1$. Consider I and J of size k such that for all $l \in \{1, \ldots, k\}$, the inequality $a_l < j_l$ holds. Let $l \in \{1, \ldots, k\}$ be the smallest integer such that $i_k < j_l$. Observe that there is such an integer, because $i_k \leqslant a_k < j_k$. We have $s_{I,J}(k) = j_l$. Then set $I' = (i_1, \ldots, i_{k-1})$ and $J' = J \setminus \{j_l\}$. By construction of $s_{I,J}$, its restriction to $\{1, \ldots, k-1\}$ coincides with $s_{I',J'}$. Let $p \in \{1, \ldots, k\}$ be such that $i_k = a_p$. Since $a_p = i_k \geqslant j_{l-1}$, we have $p \geqslant l$. Let $(a'_1 \leqslant \ldots \leqslant a'_{k-1})$ be the reordering of I'. We have on one hand

$$a'_1 = a_1 < j_1 = j'_1, \ldots, a'_{l-1} = a_{l-1} < j_{l-1} = j'_{l-1}$$

and on the other hand

$$a'_l \in \{a_l, a_{l+1}\} < j_{l+1} = j'_l, \ldots, a'_{k-1} \in \{a_{k-1}, a_k\} < j_k = j'_{k-1},$$

so that we can apply the induction hypothesis to (I', J') to find that $s_{I',J'}$ is a bijection from $\{1, \ldots, k-1\}$ onto J'. Thus, (I, J) is a restricted parking function. $\square$

Proposition 18. *The number of restricted parking functions of size* (n, k) *is* $\binom{n}{k+1} n^{k-1}$.

Proof. We use a classical conjugation argument. Let us consider $I \in \{1, \ldots, n\}^k$ and J a subset of $\{1, \ldots, n\}$ with $k+1$ elements. Consider the n-cycle $\sigma = (1 \ldots n)$. Let us define the circular spread of I in J by setting

$$c_{I,J}(k) = \sigma^r(k), \text{ where } r = \min\{s \in \mathbb{N}^* : \sigma^s(k) \in J\}$$

and, for all $l \leqslant k-1$,

$$c_{I,J}(l) = \sigma^r(l), \text{ where } r = \min\{s \in \mathbb{N}^* : \sigma^s(l) \in J \setminus \{c_{I,J}(l+1), \ldots, c_{I,J}(k)\}\}.$$

The bike shed is now circular and has one more available space than there are bikes to park. The map $c_{I,J}$ is an injection of $\{1, \ldots, k\}$ into J and we denote by $e(I, J) = J \setminus c_{I,J}(\{1, \ldots, k\})$ the space which is left free at the end of the parking process. We say that (I, J) is a restricted circular parking function of size (n, k) if $e(I, J) = 1$.

On one hand, if $(I, J) \in \{1, \ldots, n-1\}^k \times \binom{\{2,\ldots,n\}}{k}$ is a restricted parking function, then $(I, J \cup \{1\})$ is a restricted circular parking function. On the other hand, if (I, J) is a circular restricted parking function, then the fact that the space 1 is left free in the process indicates that n does not appear in I and no bike has ever to go past the space labelled n before it finds an available space, so that $(I, J \setminus \{1\})$ is a restricted parking function.

This argument allows us to conclude that there are exactly as many restricted parking functions of a given size as there are restricted circular parking functions of the same size. This allows us to conclude easily. Indeed, the group $\mathbb{Z}/n\mathbb{Z}$ acts freely on $\{1, \ldots, n\}^k \times \binom{\{1,\ldots,n\}}{k+1}$ and on $\{1, \ldots, n\}$, the action of the generator 1 being that of the n-cycle $(1 \ldots n)$. The mapping $e : (I, J) \mapsto e(I, J)$ is equivariant under this action, so that every orbit contains exactly one restricted circular parking function. Thus, there are $\frac{1}{n}\binom{n}{k+1} n^k$ such functions. $\square$

8 Prefixes of minimal factorisations

We are now able to count prefixes of minimal factorisations. For this, we establish a bijection between the set $\mathscr{F}_{n,k}$ of k-prefixes of minimal factorisations of $(1 \ldots n)$ on one hand and the set, which we denote by $\mathscr{P}_{n,k}$, of couples $(I,J) \in \{1,\ldots,n-1\}^k \times \binom{\{2,\ldots,n\}}{k}$ which are restricted parking functions of size (n,k) on the other hand. Let us describe this bijection.

Let $\gamma = ((i_1\, j_1),\ldots,(i_k\, j_k))$ be an element of $\mathscr{F}_{n,k}$. Let $\sigma \in \mathfrak{S}_k$ be a permutation such that $\sigma \cdot \gamma = ((a_1\, b_1),\ldots,(a_k\, b_k))$ belongs to $\mathscr{F}^{\uparrow}_{n,k}$. Set $\mathcal{I}(\gamma) = (i_1,\ldots,i_k)$ and $\mathcal{J}(\gamma) = \{b_1,\ldots,b_k\}$.

Lemma 19. *The couple* $(\mathcal{I}(\gamma),\mathcal{J}(\gamma))$ *is an element of* $\mathscr{P}_{n,k}$.

Proof. For each $l \in \{1,\ldots,k\}$, we have $1 \leqslant i_l < j_l \leqslant n$, so that $\mathcal{I}(\gamma)$ belongs to $\{1,\ldots,n-1\}^k$. We also have $1 \leqslant a_l < b_l \leqslant n$ for all l, so that $b_l \geqslant 2$, and by the third assertion of Lemma 7, $\mathcal{J}(\gamma)$ belongs to $\binom{\{2,\ldots,n\}}{k}$. Since for all $l \in \{1,\ldots,k\}$ we have $a_l < b_l$, it follows from Proposition 17 that $(\mathcal{I}(\gamma),\mathcal{J}(\gamma))$ is a restricted parking function. $\square$

Let us denote by $P : \mathscr{F}_{n,k} \to \mathscr{P}_{n,k}$ the map which we just defined. Let us now describe the reciprocal mapping. Let (I,J) be a restricted parking function. Let $\sigma \in \mathfrak{S}_k$ be such that $\sigma \cdot I$ is non-decreasing.

Lemma 20. *The sequence* $\sigma^{-1} \cdot ((i_{\sigma^{-1}(1)}\, s_{\sigma\cdot I,J}(1)),\ldots,(i_{\sigma^{-1}(n-1)}\, s_{\sigma\cdot I,J}(n-1)))$ *belongs to* $\mathscr{F}_{n,k}$.

Proof. It suffices to prove that $((i_{\sigma^{-1}(1)}\, s_{\sigma\cdot I,J}(1)),\ldots,(i_{\sigma^{-1}(n-1)}\, s_{\sigma\cdot I,J}(n-1)))$ belongs to $\mathscr{F}_{n,k}$, indeed to $\mathscr{F}^{\uparrow}_{n,k}$. For all $m \in \{1,\ldots,k\}$, the fact that $s_{\sigma\cdot I,J}(m)$ is the smallest element of $J \setminus \{s_{\sigma\cdot I,J}(m+1),\ldots,s_{\sigma\cdot I,J}(k)\}$ which is strictly larger than $i_{\sigma^{-1}(m)}$ implies that for all $l \in \{1,\ldots,m-1\}$, one either has $s_{\sigma\cdot I,J}(l) \leqslant i_{\sigma^{-1}(m)}$ or $s_{\sigma\cdot I,J}(l) > s_{\sigma\cdot I,J}(l)$. By Proposition 10, this suffices to imply that the factorisation which we are considering belongs to $\mathscr{F}_{n,k}$. $\square$

Let us denote by $F : \mathscr{P}_{n,k} \to \mathscr{F}_{n,k}$ this second mapping. It is straightforward to check the following fact.

Proposition 21. *The mappings F and P are inverse of each other.*

The main result, Theorem 1, follows immediately from this result and Proposition 18.

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