Triangular Fully Packed Loop Configurations of Excess 2

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Abstract

Triangular fully packed loop configurations (TFPLs) came up in the study of fully packed loop configurations on a square (FPLs) corresponding to link patterns with a large number of nested arches. To a TFPL is assigned a triple $(u, v; w)$ of 01-words encoding its boundary conditions that must necessarily satisfy $\mathrm{d}(u) + \mathrm{d}(v) \leqslant \mathrm{d}(w)$, where $\mathrm{d}(u)$ denotes the number of inversions in u. Wieland gyration, on the other hand, was invented to show the rotational invariance of the numbers of FPLs having given link patterns. Later, Wieland drift – a map on TFPLs that is based on Wieland gyration – was defined. The main contribution of this article will be a linear expression for the number of TFPLs with boundary $(u, v; w)$ where $\mathrm{d}(w) - \mathrm{d}(u) - \mathrm{d}(v) = 2$ in terms of numbers of stable TFPLs, that is, TFPLs invariant under Wieland drift. This linear expression generalises already existing enumeration results for TFPLs with boundary $(u, v; w)$ that satisfies $\mathrm{d}(w) - \mathrm{d}(u) - \mathrm{d}(v) \in \{0, 1\}$.

1 Introduction

The objects considered in this article are *fully packed loops* on the square grid. They have their origin in *six vertex* (or *square ice*) configurations of statistical mechanics. A fully packed loop configuration (FPL) of size n is a subgraph F of the $n \times n$ square grid together with $2n$ external edges that satisfies the following two conditions: each of the n^2 vertices of the square grid is of degree 2 in F and every other external edge is occupied by F starting with the topmost horizontal external edge on the left side. See Figure 1 for an example. FPLs are well-established in combinatorics. This is partly due to the

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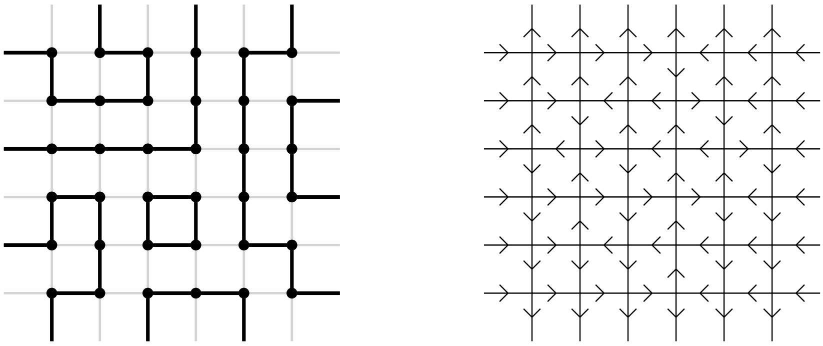
fact that they are in one-to-one correspondence to *alternating sign matrices* (ASMs). The number of FPLs of size n thus is given by the famous formula for the number of ASMs of size n proved in [11] and [6]. On the other hand, FPLs allow a study in dependency on the connectivity of the occupied external edges (these connections are encoded as a *link pattern*).



$$\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

Figure 1: An FPL, a six vertex configuration and an ASM.

The study of FPLs having a link pattern with *nested arches* is an example for one such study. It was conjectured in [12] and proved in [4] that the number of FPLs having a fixed link pattern $\pi \cup m$ consisting of a link pattern π of size n and m nested arches is a polynomial in m. In the course of the proof of this conjecture *triangular fully packed loop configurations* (TFPLs) came up. To be more precise, the following expression for the number $A_\pi(m)$ of FPLs having link pattern $\pi \cup m$ in which the numbers $t_{u,v}^{w}$ of TFPLs with *boundary* $(u, v; w)$ – u, v and w are 01-words – is proved:

$$A_\pi(m) = \sum_{u,v \in \mathcal{D}_n} P_{\lambda(u)}(m - 2n + 1)\, t_{u',v'}^{\mathbf{w}(\pi)}\, P_{\lambda(v)}(n), \tag{1}$$

where $\mathcal{D}_n$ denotes the set of Dyck words of length $2n$, u' denotes the 01-word obtained from a Dyck word u by deleting the first 0 and the last 1, $\mathbf{w}(\pi)$ denotes the Dyck word corresponding to the link pattern π, $\lambda(u)$ denotes the Young diagram associated with a 01-word u and

$$P_\lambda(n) = \prod_{x \in \lambda} \frac{n + c(x)}{h(x)}$$

is the hook content formula.

Apparently, Equation (1) motivates the study of TFPLs. Another motivation comes from the many nice properties of TFPLs that have been discovered since their emergence, see for instance [9], [7] and [5]. An example of one such property is that the boundary $(u, v; w)$ of a TFPL has to satisfy that $\mathrm{d}(u) + \mathrm{d}(v) \leqslant \mathrm{d}(w)$, where $\mathrm{d}(\omega)$ denotes the number of inversions of ω. The integer

$$\mathrm{exc}(u, v; w) = \mathrm{d}(w) - \mathrm{d}(u) - \mathrm{d}(v)$$

is said to be the *excess* of u, v, w. To study TFPLs with respect to the excess of their boundary turned out to be fruitful: in [5] enumeration results for TFPLs with boundary

$(u, v; w)$ where $\mathrm{exc}(u, v; w) \in \{0, 1\}$ were proved. Apart from those, only for a class of *oriented* TFPLs enumeration results are known up to now.

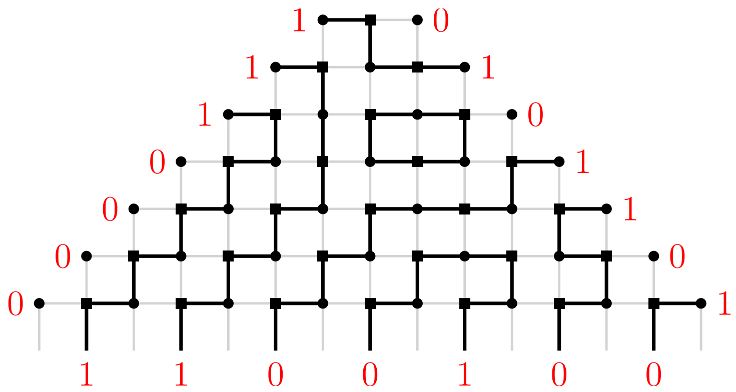


Figure 2: A TFPL with boundary $(0000111, 1011010; 1100100)$.

Wieland gyration, on the other hand, is an operation on FPLs that was invented in [10] to prove the rotational invariance of the numbers of FPLs having fixed link patterns. Later it was heavily used by Cantini and Sportiello [3] to prove the Razumov–Stroganov conjecture. In connection with TFPLs, Wieland gyration first appeared in [9]. It was later explicitly defined for TFPLs – under the name *Wieland drift* – and studied in [2]. For instance, it is shown in [2] that Wieland drift is eventually periodic with period 1. Thus, Wieland drift offers a tool to assign to a TFPL a *stable* TFPL, that is, a TFPL invariant under the application of Wieland drift.

This article focuses on TFPLs with boundary $(u, v; w)$ that satisfies $\mathrm{exc}(u, v; w) = 2$. The main contribution is a linear expression for $t^w_{u,v}$ (where $\mathrm{exc}(u, v; w) = 2$) in terms of numbers of stable TFPLs. This expression generalises already existing enumeration results for TFPLs whose boundary has excess 0 or 1.

In the following, let $|u|_i$ be the number of occurrences of i in a 01-word u for $i = 0, 1$ and $s^w_{u^+,v^+}$ the number of stable TFPLs with boundary $(u^+, v^+; w)$.

Definition 1. Let u and u^+ be two 01-words of the same length that satisfy $|u|_1 = |u^+|_1$ and $\lambda(u) \subseteq \lambda(u^+)$. Then let G_{u,u^+} be the set of semi-standard Young tableaux of skew shape $\lambda(u^+)/\lambda(u)$ with entries in the i-th column – when counted from the left – restricted to $1, 2, \ldots, |u|_1 - i + 1$ and g_{u,u^+} its cardinality.

The main result of this article now is the following:

Theorem 2. *Let u, v, w be words of the same length that satisfy* $\mathrm{exc}(u, v; w) = 2$. *Then*

$$t^w_{u,v} = \sum_{\substack{u^+,v^+ \text{ words of length } N: \\ \lambda(u)\subseteq\lambda(u^+),\, \lambda(v)\subseteq\lambda(v^+)}} g_{u,u^+}\ g_{v,v^+}\ s^w_{u^+,v^+}. \tag{2}$$

It is shown in [2] that it follows from results in [5] that for 01-words u, v and w of length N with $\mathrm{exc}(u, v; w) = 0$ it holds that

$$t^w_{u,v} = s^w_{u,v}. \tag{3}$$

For words u, v and w that satisfy $\mathrm{exc}(u,v;w)=1$ in [5, Theorem 6.16(5)] an expression for $t^w_{u,v}-s^w_{u,v}$ in terms of TFPLs of excess 0 is proved. This expression is equivalent to the following:

$$t^w_{u,v}=s^w_{u,v}+\sum_{u^+:\ \lambda(u)\subset\lambda(u^+)} g_{u,u^+}\ s^w_{u^+,v}+\sum_{v^+:\ \lambda(v)\subset\lambda(v^+)} g_{v,v^+}\ s^w_{u,v^+}. \tag{4}$$

Thus, the linear expression stated in Theorem 2 indeed generalises already existing enumeration results for TFPLs whose boundary has excess 0 or 1. This suggests a study of TFPLs with boundary $(u,v;w)$ where $\mathrm{exc}(u,v;w)\geqslant 3$ that uses Wieland drift in order to obtain expressions for the numbers $t^w_{u,v}$ in terms of stable TFPLs. A poster about this work was presented at FPSAC 2015, see [1].

2 Preliminaries

2.1 Words and Young diagrams

A *word* ω of length N is a finite sequence $\omega=\omega_1\omega_2\cdots\omega_N$ where $\omega_i\in\{0,1\}$ for all $1\leqslant i\leqslant N$. Given a word ω the number of occurrences of 0 (*resp.* 1) in ω is denoted by $|\omega|_0$ (*resp.* $|\omega|_1$). Furthermore, two words ω and σ of length N with $|\omega|_1=|\sigma|_1$ are said to satisfy $\omega\leqslant\sigma$ if $|\omega_1\cdots\omega_n|_1\leqslant|\sigma_1\cdots\sigma_n|_1$ holds for all $1\leqslant n\leqslant N$. Finally, the number of inversions of ω, that is, pairs $1\leqslant i<j\leqslant N$ satisfying $\omega_i=1$ and $\omega_j=0$ is denoted by $\mathrm{d}(\omega)$.

Throughout this article, a Young diagram $\lambda(\omega)$ is associated with a word ω as follows: to begin with, a path on the square lattice is constructed by drawing a $(0,1)$-step if $\omega_i=0$ and a $(1,0)$-step if $\omega_i=1$ for i from 1 to n. Thereafter, a vertical line through the paths start point and a horizontal line through its end point are drawn. Then the region enclosed by the lattice path and the two lines is a Young diagram, which shall be the image of ω under λ. In Figure 3, an example of a word and its corresponding Young diagram is given. For two words ω and σ of length N it then holds $\omega\leqslant\sigma$ if and only if $\lambda(\omega)$ is contained in $\lambda(\sigma)$. Furthermore, the number of cells of $\lambda(\omega)$ equals $\mathrm{d}(\omega)$.

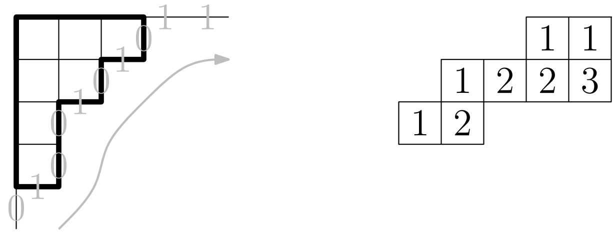


Figure 3: The Young diagram $\lambda(0100101011)$ and a semi-standard Young tableau of skew shape $\lambda(011011100)/\lambda(001011011)$.

A Young diagram of skew shape is a *horizontal strip* if each of its columns contains at most one cell. Now, given two words ω and σ of length N that satisfy $|\omega|_1=|\sigma|_1$ and $\omega\leqslant\sigma$

the skew shape $\lambda(\sigma)/\lambda(\omega)$ is a horizontal strip if and only if for each $j \in \{1, \ldots, |\omega|_1\}$ the following holds: *If ω_i is the j-th one in ω then σ_{i-1} or σ_i is the j-th one in σ.* Throughout this article, if the skew shape $\lambda(\sigma)/\lambda(\omega)$ is a horizontal strip it is written $\omega \xrightarrow{\mathrm{h}} \sigma$.

Finally, recall that semi-standard Young tableaux of skew shape λ^+/λ with entries $1, 2, \ldots, m$ are in one-to-one correspondence with sequences of Young diagrams

$$\lambda = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_{m-1} \subseteq \lambda_m = \lambda^+,$$

where λ_i/λ_{i-1} is a horizontal strip for every $1 \leqslant i \leqslant m$. (The horizontal strip λ_i/λ_{i-1} corresponds to the cells of the semi-standard Young tableau whose entry is i.)

Thus, for non-negative integers N_1 and N_0 and Young diagrams λ and λ^+ that satisfy $\lambda \subseteq \lambda^+$ and that λ and λ^+ have at most N_1 many columns and N_0 many rows the set of semi-standard Young tableaux of skew shape λ^+/λ with entries $1, 2, \ldots, m$ is in bijection with the set of sequences of 01-words

$$\tau^0 \xrightarrow{\mathrm{h}} \tau^1 \xrightarrow{\mathrm{h}} \cdots \xrightarrow{\mathrm{h}} \tau^{m-1} \xrightarrow{\mathrm{h}} \tau^m$$

that satisfy $|\tau^i|_0 = N_0$ and $|\tau^i|_1 = N_1$ for every $1 \leqslant i \leqslant m$, $\lambda(\tau^0) = \lambda$ and $\lambda(\tau^m) = \lambda^+$. For instance, for $N_1 = 5$, $N_0 = 4$ and $m = 3$ the semi-standard Young tableau of skew shape $\lambda(011011100)/\lambda(001011011)$ in Figure 3 corresponds to the sequence

$$001011011 \xrightarrow{\mathrm{h}} 010101110 \xrightarrow{\mathrm{h}} 011011010 \xrightarrow{\mathrm{h}} 011011100.$$

2.2 Triangular fully packed loop configurations

To give the definition of triangular fully packed loop configurations the following graph is needed:

Definition 3. Let N be a positive integer. The graph G^N is defined as the induced subgraph of the square grid made up of N consecutive centred rows of $3, 5, \ldots, 2N+1$ vertices from top to bottom together with $2N+1$ vertical *external* edges incident to the $2N+1$ bottom vertices.

In Figure 4, the graph G^7 is depicted. From now on, the vertices of G^N are partitioned into *odd* and *even* vertices in a chessboard manner where by convention the leftmost vertex of the top row of G^N is odd. In the figures, odd vertices are represented by circles and even vertices by squares.

There are vertices of G^N that play a special role: let $\mathcal{L}^N = \{L_1, L_2, \ldots, L_N\}$ (*resp.* $\mathcal{R}^N = \{R_1, R_2, \ldots, R_N\}$) be the set made up of the vertices that are leftmost (*resp.* rightmost) in each of the N rows of G^N and let $\mathcal{B}^N = \{B_1, B_2, \ldots, B_N\}$ be the set made up of the even vertices of the bottom row of G^N. The vertices in $\mathcal{L}^N$ and $\mathcal{R}^N$ are numbered from bottom to top and the vertices in $\mathcal{B}^N$ from left to right. Furthermore, the $N(N+1)$ unit squares of G^N including external unit squares that have three surrounding edges only are said to be the *cells* of G^N. They are partitioned into *odd* and *even* cells in a chessboard manner where by convention the cell that L_1 is a vertex of is odd.

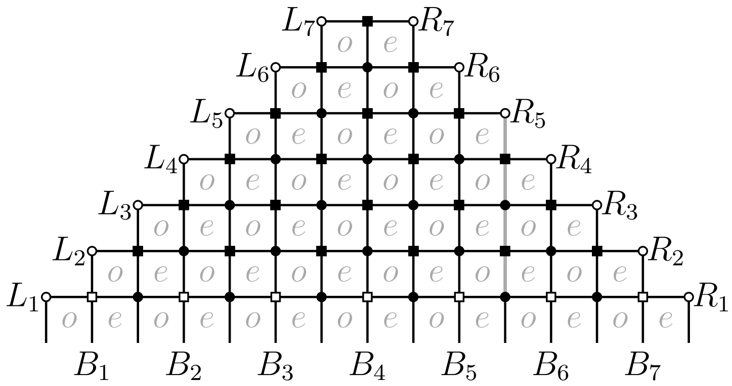


Figure 4: The graph G^7.

Definition 4. Let N be a positive integer. A *triangular fully packed loop configuration* (TFPL) of size N is a subgraph f of G^N such that:

1. Precisely those external edges that are incident to a vertex in $\mathcal{B}^N$ belong to f.
2. The $2N$ vertices in $\mathcal{L}^N \cup \mathcal{R}^N$ have degree 0 or 1.
3. All other vertices of G^N have degree 2.
4. A path in f neither connects two vertices of $\mathcal{L}^N$ nor two vertices of $\mathcal{R}^N$.

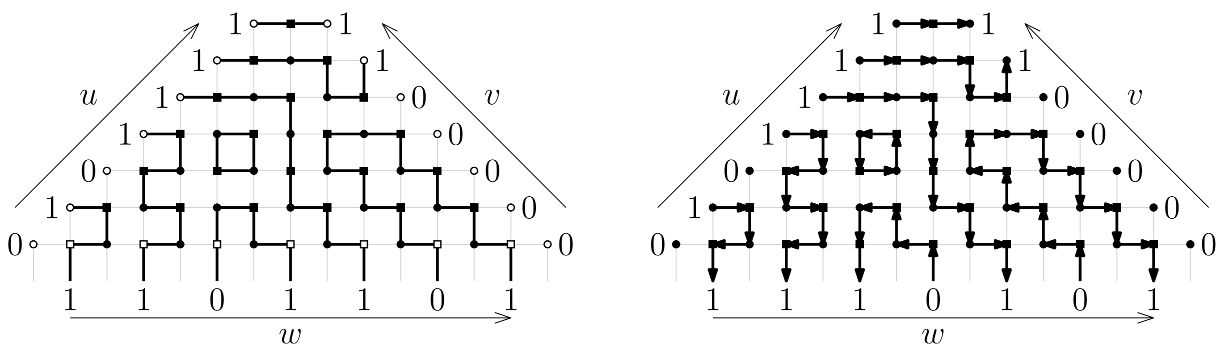


Figure 5: A TFPL of size 7 and an oriented TFPL of size 7.

Figure 5 displays a TFPL. A *cell of* f is a cell of G^N together with those of its surrounding edges that are occupied by f.

Definition 5. Let f be a TFPL of size N. A triple $(u, v; w)$ of words of length N is assigned to f as follows:

1. For $i = 1, \ldots, N$ set $u_i = 1$ if the vertex $L_i \in \mathcal{L}^N$ has degree 1 and $u_i = 0$ otherwise.
2. For $i = 1, \ldots, N$ set $v_i = 1$ if the vertex $R_i \in \mathcal{R}^N$ has degree 1 and $v_i = 0$ otherwise.

3. For $i = 1, \ldots, N$ set $w_i = 1$ if in f the vertex $B_i \in \mathcal{B}^N$ is connected with a vertex in $\mathcal{L}^N$ or with a vertex B_h for an $h < i$ and $w_i = 0$ otherwise.

The triple $(u, v; w)$ is said to be the *boundary* of f. Furthermore, the set of TFPLs with boundary $(u, v; w)$ is denoted by $T^w_{u,v}$ and its cardinality by $t^w_{u,v}$.

For example, the triple $(0101111, 0000011; 1101101)$ is the boundary of the TFPL depicted in Figure 5. In the definition of a TFPL condition (4) is global. It can be omitted when adding an orientation to each edge of a TFPL.

Definition 6. An *oriented TFPL* of size N is a TFPL of size N together with an orientation of its edges such that the edges attached to $\mathcal{L}^N$ are outgoing, the edges attached to $\mathcal{R}^N$ are incoming and all other vertices of G^N are incident to an incoming and an outgoing edge.

In Figure 5, an example of an oriented TFPL of size 7 is given. From the constraints that edges that are incident to vertices in $\mathcal{L}^N$ must be outgoing and that vertices of degree two must be have in-degree 1 and out-degree 1 it follows that no path in an oriented TFPL joins two vertices in $\mathcal{L}^N$. The same is true for vertices in $\mathcal{R}^N$. Thus, in the underlying TFPL of an oriented TFPL condition (4) in Definition 4 is satisfied and therefore may be disregarded.

Definition 7. To an oriented TFPL f a triple $(u, v; w)$ of words is assigned as follows:

1. For $i = 1, \ldots, N$ set $u_i = 1$ if the vertex $L_i \in \mathcal{L}^N$ has degree 1 and $u_i = 0$ otherwise.
2. For $i = 1, \ldots, N$ set $v_i = 1$ if the vertex $R_i \in \mathcal{R}^N$ has degree 1 and $v_i = 0$ otherwise.
3. For $i = 1, \ldots, N$ set $w_i = 1$ if the external edge attached to the vertex $B_i \in \mathcal{B}^N$ is outgoing and $w_i = 0$ otherwise.

The triple $(u, v; w)$ is said to be the *boundary* of f.

While u and v coincide with the respective boundary word in the underlying ordinary TFPL this may not be the case for w. Instead of the connectivity of the paths for an oriented TFPL w encodes the orientation of the external edges. Only in the case when in an oriented TFPL all paths between two vertices B_i and B_j of $\mathcal{B}^N$ are oriented from B_i to B_j if $i < j$ the boundary word w coincides with the respective boundary word of the underlying TFPL.

Definition 8. The *canonical orientation* of a TFPL of size N is defined as the orientation of the edges of the TFPL that in addition to the conditions in Definition 6 satisfies that each path between two vertices $B_i, B_j \in \mathcal{B}^N$ is oriented from B_i to B_j if $i < j$ and that all closed paths are oriented clockwise.

It is shown in [9, 5] that the boundary $(u, v; w)$ of both ordinary and oriented TFPLs must satisfy

$$\mathrm{d}(w) - \mathrm{d}(u) - \mathrm{d}(v) \geqslant 0.$$

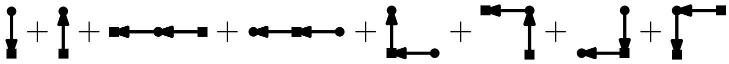
Definition 9 ([5])**.** Let u, v, w be words of length N. Then the *excess* of u, v and w is defined as

$$\mathrm{exc}(u, v; w) = \mathrm{d}(w) - \mathrm{d}(u) - \mathrm{d}(v).$$

If $\mathrm{exc}(u, v; w) = k$ then both ordinary and oriented TFPLs with boundary $(u, v; w)$ are said to be of excess k.

In [5] it was furthermore proved that the excess has the following interpretation in terms of numbers of occurrences of certain local configurations in an oriented TFPL:

Proposition 10 ([5, Theorem 4.3])**.** *Let f be an oriented TFPL with boundary $(u, v; w)$. Then*

$\mathrm{exc}(u, v; w) =$  (5)

where by , , *etc. the numbers of occurrences of the local configurations* , , *etc. in f are denoted.*

2.3 Blue-red path tangles

In this subsection an alternative representation of oriented TFPLs is introduced, namely *blue-red path tangles*. They were introduced in [5] and play a crucial rule in the proof of Theorem 2. In the following, let u, v and w be words of length N that satisfy $|u|_0 = |v|_1 = |w|_0$ and therefore $|u|_1 = |v|_0 = |w|_1$.

To begin with, let i_k be the index of the k-th 0 in w and j_k the index of the k-th 0 in u for each $k = 1, \dots, |w|_0$. Then set $D_k = (2i_k - 2, 0)$ and $E_k = (j_k - 1, j_k - 1)$ and define $\mathcal{P}(D_k, E_k)$ as the set of lattice paths from D_k to E_k with steps in $S_{\mathrm{blue}} = \{(-1, 1), (-1, -1), (-2, 0)\}$ that do not go below the x-axis for $k = 1, \dots, |w|_0$. Finally, let $\mathcal{P}(u, w)$ be the set of $|w|_0$-tuples of non-intersecting lattice paths $(P_1, \dots, P_{|w|_0})$ where $P_k \in \mathcal{P}(D_k, E_k)$. Figure 6 displays a pair of non-intersecting lattice paths in $\mathcal{P}(01101, 10110)$.

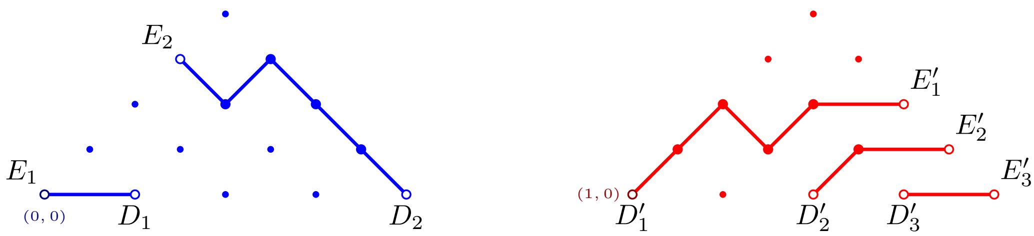


Figure 6: An element of $\mathcal{P}(01101, 10110)$ and an element of $\mathcal{P}'(00011, 10110)$.

On the other hand, let i'_ℓ be the index of the ℓ-th 1 in w and j'_ℓ the index of the $(|v|_0 - \ell + 1)$-st 0 in v. Then set $D'_\ell = (2i'_\ell - 1, 0)$ and $E'_\ell = (N - 1 + j'_\ell, j'_\ell - 1)$ and define $\mathcal{P}'(D'_\ell, E'_\ell)$ as the set of paths from D'_ℓ to E'_ℓ with steps in $S_{\mathrm{red}} = \{(1, 1), (1, -1), (2, 0)\}$ that do not go below the x-axis for $\ell = 1, \dots, |w|_1$. Finally, let $\mathcal{P}'(v, w)$ be the set of $|w|_1$-tuples of non-intersecting lattice paths $(P'_1, \dots, P'_{|w|_1})$ where $P'_\ell \in \mathcal{P}'(D'_\ell, E'_\ell)$ for every $1 \leqslant \ell \leqslant |w|_1$. Figure 6 shows a triple of non-intersecting lattice paths in $\mathcal{P}'(00011, 10110)$.

Definition 11. A *blue-red path tangle with boundary* $(u, v; w)$ is a pair (B, R) in $\mathcal{P}(u, w) \times \mathcal{P}'(v, w)$ that satisfies the following conditions:

1. No diagonal step of R crosses a diagonal step of B.

2. Each middle point of a horizontal step in B (*resp.* R) is used by a step in R (*resp.* B).

The set of blue-red path tangles with boundary $(u, v; w)$ is denoted by $\mathrm{BlueRed}(u, v; w)$.

Proposition 12 ([5, Theorem 4.1])**.** *The set of oriented TFPLs with boundary* $(u, v; w)$ *is in bijection with the set of blue-red path tangles with boundary* $(u, v; w)$.

An example of an oriented TFPL and its corresponding blue-red path tangle is given in Figure 7.

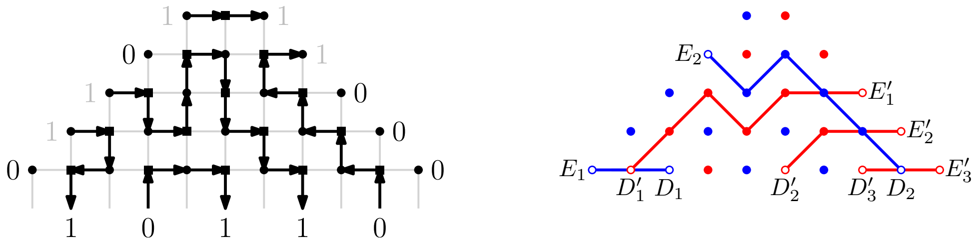


Figure 7: An oriented TFPL with boundary $(01101, 00011; 10110)$ and its corresponding blue-red path tangle with boundary $(01101, 00011; 10110)$.

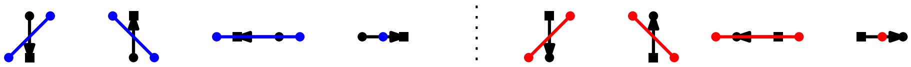


Figure 8: From oriented TFPLs to blue-red path tangles.

Proof. Here the bijection given in [5] is repeated: let f be an oriented TFPL of size N and with boundary $(u, v; w)$. As a start blue vertices are inserted in the middle of each horizontal edge of G^N that has an odd vertex to its left and red vertices are inserted in the middle of each horizontal edge of G^N that has an even vertex to its left. Next, blue edges are inserted as indicated in the left part of Figure 8 and red edges are inserted as indicated in the right part of Figure 8.

Then the blue vertices together with the blue edges give rise to an N_0-tuple of non-intersecting paths $B = (P_1, P_2, \dots, P_{N_0})$ in $\mathcal{P}(u, w)$ and the red vertices together with the red edges give rise to an N_1–tuple of non-intersecting paths $R = (P'_1, P'_2, \dots, P'_{N_1})$ in $\mathcal{P}'(v, w)$. The condition that no diagonal step of R crosses a diagonal step of B is equivalent to that there is a unique orientation of each vertical edge in f. On the other hand, the condition that each middle point of a horizontal step in B (*resp.* R) is used by a step in R (*resp.* B) is equivalent to that each even vertex in f must be incident to an outgoing (*resp.* incoming) edge. Thus, $(B, R) \in \mathrm{BlueRed}(u, v; w)$. $\square$

2.4 Wieland drift

Wieland drift is composed of the same local operations as Wieland gyration for fully packed loop configurations. These are the following operations on either the odd or even cells of an FPL: let c be an odd *resp.* even cell of an FPL. If c contains precisely two edges on opposite sides then Wieland gyration W leaves c invariant. Otherwise, the effect of W on c is that edges and non-edges are exchanged. In Figure 9, the action of W is illustrated.

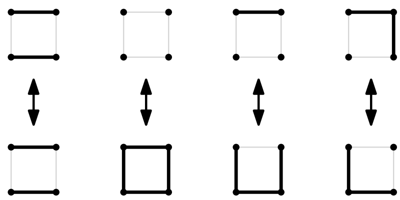


Figure 9: Up to rotation, the action of W on the internal cells of an FPL.

Definition 13. Let f be a TFPL with left boundary u and let u^- be a word that satisfies $|u^-|_1 = |u|_1$, $|u^-|_0 = |u|_0$ and $u^- \xrightarrow{\mathrm{h}} u$. The *image of f under left-Wieland drift with respect to u^-* is determined as follows:

1. Insert a vertex L'_i to the left of L_i for $1 \leqslant i \leqslant N$. Then run through the occurrences of ones in u^-: Let $\{i_1 < i_2 < \ldots < i_{|u|_1}\} = \{i | u^-_i = 1\}$.
   (a) If u_{i_j} is the j-th one in u, add a horizontal edge between L'_{i_j} and L_{i_j}.
   (b) If u_{i_j-1} is the j-th one in u, add a vertical edge between L'_{i_j} and L_{i_j-1}.
2. Apply W to each odd cell of f.
3. Delete all vertices in $\mathcal{R}^N$ and their incident edges.

After shifting the whole construction one unit to the right, one obtains the desired image $\mathrm{WL}_{u^-}(f)$. In the case $u^- = u$, simply write $\mathrm{WL}(f)$ and say the *image of f under left-Wieland drift*.

Figure 10 displays a TFPL and its image under left-Wieland drift. It is shown in [2, Proposition 2.2] that the image of a TFPL with boundary $(u, v; w)$ under left-Wieland drift with respect to u^- is again a TFPL and has boundary $(u^-, v^+; w)$ for a word v^+ that satisfies $|v^+|_1 = |v|_1$, $|v^+|_0 = |v|_0$ and $v \xrightarrow{\mathrm{h}} v^+$. This is why the excess of the image of a TFPL with boundary $(u, v; w)$ under left-Wieland drift (with respect to u) must be smaller than or equal to $\mathrm{exc}(u, v; w)$.

Right-Wieland drift – denoted WR_{v^-} respectively WR – is defined as the vertically symmetric version of left-Wieland drift and it shall simply be illustrated by the example in Figure 11. The image of a TFPL with boundary $(u, v; w)$ under right-Wieland drift with respect to v^- is a TFPL with boundary $(u^+, v^-; w)$ for a word u^+ that satisfies

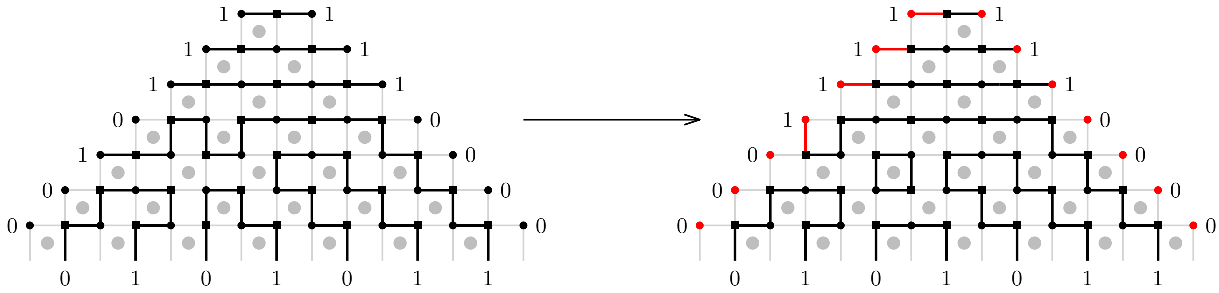


Figure 10: A TFPL and its image under left-Wieland drift with respect to 0001111. The grey points indicate the cells to which W is applied.

$|u^+|_1 = |u|_1$, $|u^+|_0 = |u|_0$ and $u \xrightarrow{\text{h}} u^+$. As for left-Wieland drift, the excess of the image of a TFPL with boundary $(u, v; w)$ under right-Wieland drift (with respect to v) must be smaller than or equal to $\text{exc}(u, v; w)$.

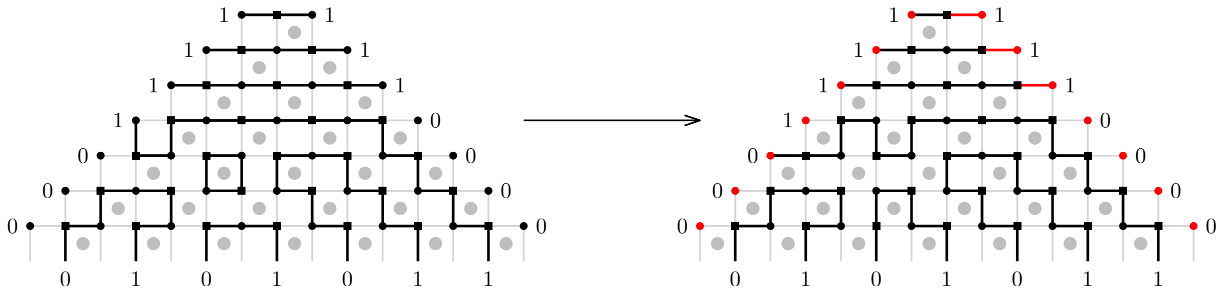


Figure 11: A TFPL and its image under right-Wieland drift.

Given a TFPL with right boundary v the effect of left-Wieland drift along the right boundary of the TFPL is inverted by right-Wieland drift with respect to v. On the other hand, given a TFPL with left boundary u the effect of right-Wieland drift along the left boundary is inverted by left-Wieland drift with respect to u. Since Wieland gyration W is an involution it follows:

Proposition 14 ([2, Theorem 2]).

1. *Let f be a TFPL with boundary $(u^+, v; w)$ and u be a word such that $|u|_1 = |u^+|_1$, $|u|_0 = |u^+|_0$ and $u \xrightarrow{\text{h}} u^+$. Then*
$$\text{WR}_v(\text{WL}_u(f)) = f.$$
2. *Let f be a TFPL with boundary $(u, v^+; w)$ and v be a word such that $|v|_1 = |v^+|_1$, $|v|_0 = |v^+|_0$ and $v \xrightarrow{\text{h}} v^+$. Then*
$$\text{WL}_u(\text{WR}_v(f)) = f.$$

Note that by Proposition 14 a TFPL is invariant under left-Wieland drift if and only if it is invariant under right-Wieland drift.

Definition 15. A TFPL is said to be *stable* if it is invariant under left-Wieland drift. Otherwise, it is said to be *instable*. The set of stable TFPLs with boundary $(u, v; w)$ is denoted by $S_{u,v}^{w}$ and its cardinality by $s_{u,v}^{w}$.

Now, stable TFPLs have the following nice characterisation:

Proposition 16 ([2, Theorem 4]). *A TFPL is stable if and only if it contains no edge of the form* .

Definition 17. The edge is said to be a *drifter*.

Given a TFPL f the sequence $(\mathrm{WL}^m(f))_{m \geqslant 0}$ is eventually periodic since there are only finitely many TFPLs of a fixed size. The length of its period is in fact always 1.

Proposition 18 ([2, Theorem 3]). *Let f be a TFPL of size N. Then $\mathrm{WL}^{2N-1}(f)$ is stable, so that the following holds for all $m \geqslant 2N-1$:*

$$\mathrm{WL}^m(f) = \mathrm{WL}^{2N-1}(f).$$

The same holds for right-Wieland drift.

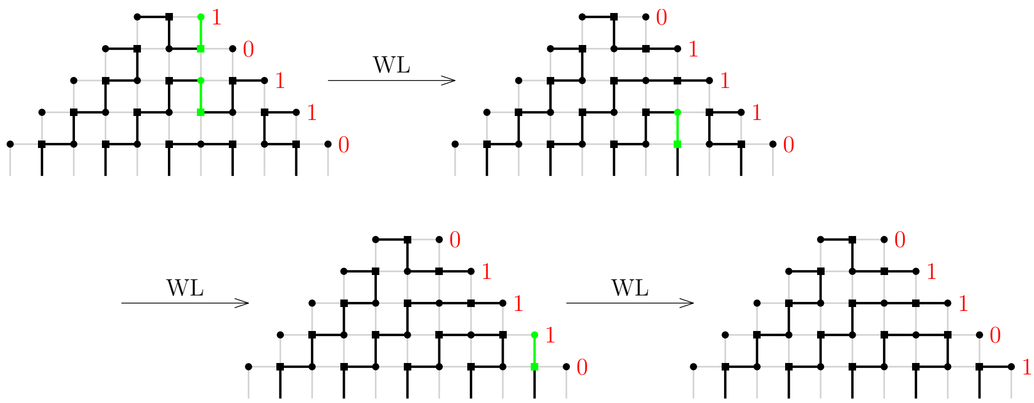


Figure 12: A TFPL and its images under left-Wieland drift.

In Figure 12, a TFPL and its images under left-Wieland drift are depicted. From now on, for an instable TFPL f denote by $L = L(f)$ the positive integer L such that $\mathrm{WL}^{\ell}(f)$ is instable for each $0 \leqslant \ell \leqslant L$ and $\mathrm{WL}^{L+1}(f)$ is stable and by $R = R(f)$ the positive integer such that $\mathrm{WR}^{r}(f)$ is instable for each $0 \leqslant r \leqslant R$ and $\mathrm{WR}^{R+1}(f)$ is stable.

Definition 19. Let f be an instable TFPL. The *path* of f is defined as the sequence of all TFPLs that can be reached by an iterated application of left- respectively right-Wieland drift to f, that is,

$$\mathrm{Path}(f) = \left(\mathrm{WR}^{R+1}(f), \ldots, \mathrm{WR}(f), f, \mathrm{WL}(f), \ldots, \mathrm{WL}^{L+1}(f)\right).$$

Furthermore, the stable TFPL $\mathrm{WR}^{R+1}(f)$ is denoted by $\mathrm{Right}(f)$ and the stable TFPL $\mathrm{WL}^{L+1}(f)$ by $\mathrm{Left}(f)$.

In conclusion, when v^ℓ denotes the right boundary of $\mathrm{WL}^\ell(f)$ for each $0 \leqslant \ell \leqslant L+1$ and $v^+ = v^{L+1}$ then the sequence

$$v = v^0 \xrightarrow{\mathrm{h}} v^1 \xrightarrow{\mathrm{h}} \cdots \xrightarrow{\mathrm{h}} v^L \xrightarrow{\mathrm{h}} v^{L+1} = v^+$$

corresponds to a semi-standard Young tableau $T(f)$ of skew shape $\lambda(v^+)/\lambda(v)$ with entries $1, 2, \ldots, L+1$. On the other hand, when u^r denotes the left boundary of $\mathrm{WR}^r(f)$ for each $0 \leqslant r \leqslant R+1$ and $u^+ = u^{R+1}$ then the sequence

$$u = u^0 \xrightarrow{\mathrm{h}} u^1 \xrightarrow{\mathrm{h}} \cdots \xrightarrow{\mathrm{h}} u^R \xrightarrow{\mathrm{h}} u^{R+1} = u^+$$

corresponds to a semi-standard Young tableau $S(f)$ of skew shape $\lambda(u^+)/\lambda(u)$.

It is shown below that for an instable TFPL f with boundary $(u, v; w)$ of excess 2 it is impossible that $S(f) \in G_{u,u^+}$ and $T(f) \in G_{v,v^+}$ at the same time. Thus, f may be associated with $S(f)$ and the stable TFPL $\mathrm{Right}(f)$ in the case when $S(f) \in G_{u,u^+}$ and with the stable TFPL $\mathrm{Left}(f)$ and $T(f)$ in the case when $T(f) \in G_{v,v^+}$. Finally, when neither $S(f) \in G_{u,u^+}$ nor $T(f) \in G_{v,v^+}$ to the TFPL f a stable TFPL with boundary $(u^+, v^+; w)$, where $u^+ > u$ and $v^+ > v$, and non-empty semi-standard Young tableaux in $G_{u,u+}$ and G_{v,v^+} are assigned. The stable TFPL will be obtained from f by a number of local transformations of f. These transformations derive from Wieland drift and, in fact, are part of a set of transformations that completely determine the effect of left- and right-Wieland drift on a TFPL of excess 2.

3 An alternative description of the effect of Wieland drift

The main contribution of this section is a description of the effect of Wieland drift on TFPLs of excess at most 2 as a composition of moves. In Figure 13, the moves that form the basis for that description are depicted.

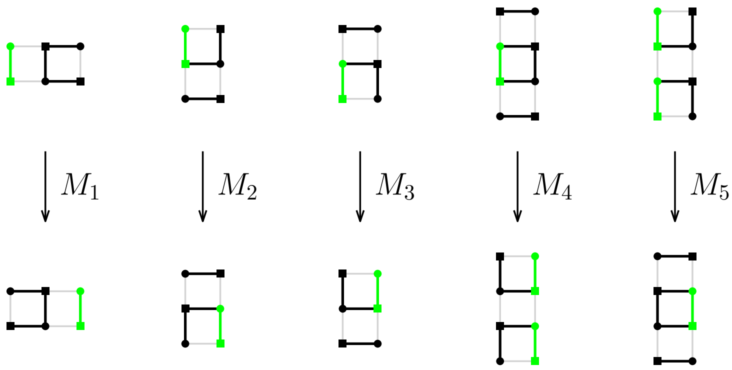


Figure 13: The moves that describe the effect of left-Wieland drift on an instable TFPL of excess at most 2.

Proposition 20. *Let u, v and w be words of length N that satisfy $\mathrm{exc}(u, v; w) \leqslant 2$. If f is an instable TFPL with boundary $(u, v; w)$ then the image of f under left-Wieland drift is determined as follows:*

1. *For $i = 2, \dots, N$ if $R_i \in \mathcal{R}^N$ is incident to a drifter then delete that drifter and add a horizontal edge incident to R_{i-1}. Denote the so-obtained TFPL by f'.*
2. *Let $\mathcal{I} = \{2 \leqslant i \leqslant 2N : f'$ has a drifter in the i-th column of $G^N\}$, where the columns of G^N are counted from left to right.*
   (a) *If $\mathcal{I} = \{i < j\}$ apply a move in $\{M_1, M_2, M_3\}$ to the drifter incident to vertices of the j-th column and thereafter apply a move in $\{M_1, M_2, M_3\}$ to the drifter incident to vertices of the i-th column;*
   (b) *If $\mathcal{I} = \{i\}$ perform a move in $\{M_4, M_5\}$ or if this is not possible apply a move in $\{M_1, M_2, M_3\}$ to each of the drifters in f' in the following order (if there are two drifters in f'): if the odd cell that contains the upper drifter is not of the form $\mathfrak{o}_9$ (see Figure 15 below) move the upper drifter first. Otherwise, move the lower drifter first.*

The effect of right-Wieland drift on TFPLs of excess at most 2 can be described in the vertically symmetric way as the composition of the moves M_1^{-1}, M_2^{-1}, M_3^{-1}, M_4^{-1} and M_5^{-1}.

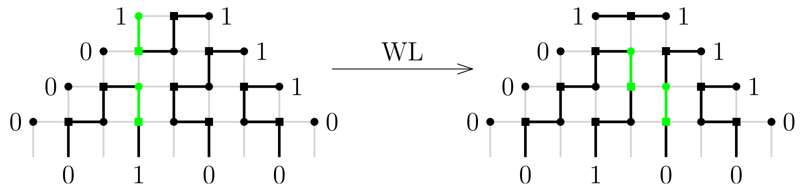


Figure 14: A TFPL of excess 2 with two drifters and its image under left-Wieland drift.

Figure 14 shows a TFPL of excess 2 with two drifters and its image under left-Wieland drift. Note that if the move M_4 (*resp.* M_5) is applied by left-Wieland drift to a TFPL of excess 2 then no other move is performed. Now, in the proof of Proposition 20 the effect of left-Wieland drift is checked cell by cell. From the set of cells that can occur in a TFPL of excess at most 2 the following cells can be excluded:

Lemma 21. *In a TFPL of excess at most 2, none of the following cells can occur:*

Since the proofs in this section work by studying the cells of a TFPL it is convenient to fix notations for all the odd and even cells that can occur in a TFPL. In total, there are 16 different odd and 16 different even internal cells that can occur in a TFPL. By Lemma 21 below fourteen of those odd and fourteen of those even internal cells may occur in a TFPL of excess at most 2. The odd respectively even cells that can occur in a TFPL of excess at most 2 will be numbered by 1 up to 14 and are listed in Figure 15, whereas the two excluded odd respectively even internal cells will be numbered by 15 and 16 as indicated in Lemma 21.

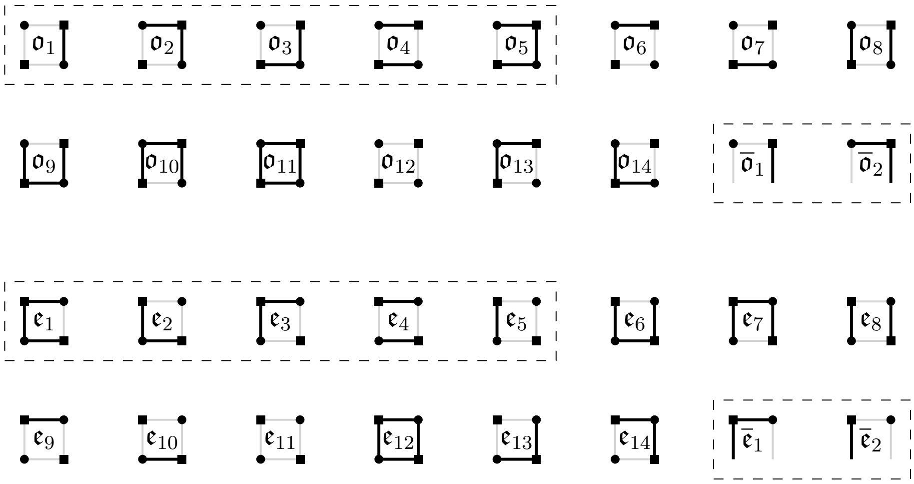


Figure 15: A list of the cells of a TFPL of excess at most 2 in which $\mathfrak{O} = \{\mathfrak{o}_1, \mathfrak{o}_2, \mathfrak{o}_3, \mathfrak{o}_4, \mathfrak{o}_5, \overline{\mathfrak{o}}_1, \overline{\mathfrak{o}}_2\}$ and $\mathfrak{E} = \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4, \mathfrak{e}_5, \overline{\mathfrak{e}}_1, \overline{\mathfrak{e}}_2\}$ are indicated.

Proof. First, let f be a TFPL that contains a cell c of the form $\mathfrak{o}_{15}$ and $\overrightarrow{f}$ its canonical orientation. Then the directed edges of c give rise to two configurations in $\overrightarrow{f}$ that contribute to the excess, see Proposition 10. Additionally, the right vertex of the horizontal edge in c that is oriented from right to left either is adjacent to the vertex to its right or is incident to a drifter. Thus, $\overrightarrow{f}$ contains in summary three configurations that contribute to the excess, which is impossible. For the same reasons, a TFPL of excess at most 2 cannot contain a cell of the form $\mathfrak{e}_{16}$.

Now, let f be a TFPL that contains a cell c of type $\mathfrak{o}_{16}$. Then both the top and the bottom rightmost vertex of c must be incident to a drifter. Therefore, f contains at least three drifters, which is impossible by Proposition 10. By the same argument, a TFPL of excess at most 2 cannot contain a cell of the form $\mathfrak{e}_{15}$. $\square$

In the following, to distinguish between the cells of a TFPL and the cells of its image under left-Wieland drift given a cell c of G^N it is written c when it is referred to the cell c of the TFPL and c' when it is referred to the cell c of the image of the TFPL under left-Wieland drift. When the cells of a TFPL and of its image under left-Wieland drift are compared it has to be kept in mind that in the last step of left-Wieland drift the whole configuration is shifted one unit to the right. For that reason, for each odd cell o of a TFPL and the even cell e to the right of o the following holds when disregarding the distinction between odd and even vertices:

$$\boxed{e' = \mathrm{W}(o)}$$

The cells $\mathfrak{O} = \{\mathfrak{o}_1, \mathfrak{o}_2, \mathfrak{o}_3, \mathfrak{o}_4, \mathfrak{o}_5, \overline{\mathfrak{o}}_1, \overline{\mathfrak{o}}_2\}$ and $\mathfrak{E} = \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{e}_4, \mathfrak{e}_5, \overline{\mathfrak{e}}_1, \overline{\mathfrak{e}}_2\}$ play a special role in connection with Wieland drift:

Lemma 22 ([2]). *Let f be a TFPL, o an odd cell of f and e the even cell to the right of o. If no vertex of o and e is incident to a drifter, then*

$$(o, e) \in \{(\mathfrak{o}_1, \mathfrak{e}_1), (\mathfrak{o}_2, \mathfrak{e}_2), (\mathfrak{o}_3, \mathfrak{e}_3), (\mathfrak{o}_4, \mathfrak{e}_4), (\mathfrak{o}_5, \mathfrak{e}_5), (\overline{\mathfrak{o}}_1, \overline{\mathfrak{e}}_1), (\overline{\mathfrak{o}}_2, \overline{\mathfrak{e}}_2)\}.$$

In particular, $e' = e$.

To study the effect of left-Wieland drift on a TFPL it suffices to study its effect on the even cells of the TFPL. That is because edges of a TFPL that are not edges of an even cell have to be incident to a vertex in $\mathcal{L}^N$ and in the image of a TFPL under left-Wieland drift all edges incident to a vertex in $\mathcal{L}^N$ must be horizontal. In addition, Lemma 22 says that to determine the effect of left-Wieland drift on a TFPL it suffices to determine its effect, on the one hand, on all even cells of the TFPL whereof a vertex is incident to a drifter and, on the other hand, on all even cells where the odd cells to their left contain drifters.

Now, given a drifter $\mathfrak{d}$ in an instable TFPL there are at most three even cells whereof a vertex is incident to $\mathfrak{d}$ and there is at most one even cell such that the odd cell to its left contains $\mathfrak{d}$. In Figure 16, these four even cells together with the odd cells to their left are depicted. Note that all four such even cells exist if and only if $\mathfrak{d}$ is not incident to a vertex in $\mathcal{L}^N \cup \mathcal{R}^N$. From now on, these even cells and the odd cells to their left are denoted as indicated in Figure 16.

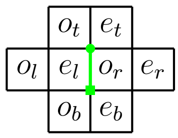


Figure 16: The even cells surrounding a drifter.

Given a drifter in a TFPL of excess at most 2 the effect of left-Wieland drift on the cell e_t, e_l respectively e_b can be uniformly described as long as o_t, o_l respectively o_b are in $\mathfrak{O}$:

Lemma 23. *Let f be an instable TFPL of excess at most 2, o an odd cell of f and e the even cell to the right of o. If o is in $\mathfrak{O}$ and a vertex of e is incident to a drifter then e' and e coincide with the following sole exceptions:*

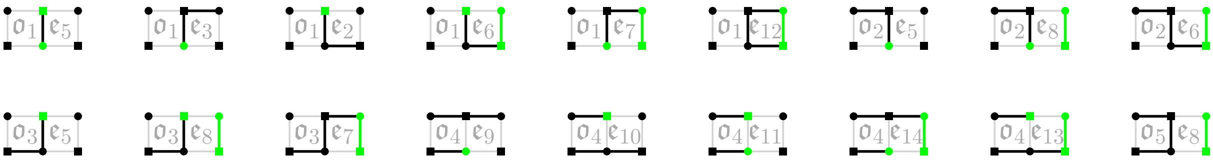
1. *If in e there is a drifter, then in e' there is none,*

2. *if the top left vertex of e is incident to a drifter, then there is no horizontal edge between the two top vertices of e but there is one between the two top vertices of e',*

3. *if the bottom left vertex of e is incident to a drifter, then there is no horizontal edge between the two bottom vertices of e but there is one between the two bottom vertices of e'.*

Proof. If o and e are external cells, then $o = \bar{\mathfrak{o}}_1$, $e = \bar{\mathfrak{e}}_2$ and $e' = \bar{\mathfrak{e}}_1$. Thus, e' and e coincide with the sole exception that there is no horizontal edge between the two top vertices of e whereas there is one between the two top vertices of e'.

Suppose that o and e are internal cells. Then (o, e) can only occur as part of one of the following pairs:



Now, $e' = \mathfrak{e}_1$ if $o = \mathfrak{o}_1$, $e' = \mathfrak{e}_2$ if $o = \mathfrak{o}_2$, $e' = \mathfrak{e}_3$ if $o = \mathfrak{o}_3$, $e' = \mathfrak{e}_4$ if $o = \mathfrak{o}_4$ and $e' = \mathfrak{e}_5$ if $o = \mathfrak{o}_5$. It can easily be checked that in any case e' and e satisfy the assertions. $\square$

To begin with, the assertion of Proposition 20 is proved for TFPLs that exhibit precisely one drifter.

Lemma 24. *Let f be an instable TFPL of excess at most 2 that exhibits precisely one drifter. If that drifter is incident to a vertex R_i in $\mathcal{R}^N$ then* $\mathrm{WL}(f)$ *and f coincide with the sole exception that in* $\mathrm{WL}(f)$ *no drifter is incident to R_i but a horizontal edge is incident to R_{i-1}. On the other hand, if the drifter in f is not incident to a vertex in $\mathcal{R}^N$ then* $\mathrm{WL}(f)$ *and f coincide with the sole exception that to the drifter in f one of the moves M_1, M_2, M_3 or M_4 has been applied by* WL.

Proof. Let f be an instable TFPL of excess at most 2 that contains precisely one drifter $\mathfrak{d}$. First, the case when $\mathfrak{d}$ is incident to a vertex R_i in $\mathcal{R}^N$ is considered. In that case, the cells o_l, e_l, o_b and e_b exist. Additionally, both o_l and o_b are in $\mathfrak{O}$ since in f there is only one drifter. Thus, by Lemma 23, on the one hand, e_l' and e_l coincide with the sole exception that in e_l' there is no drifter while in e_l there is one and, on the other hand, e_b' and e_b coincide with the sole exception that in e_b' the two top vertices are adjacent while in e_b they are not. In conclusion, by Lemma 22 the effect of left-Wieland drift on f is that the drifter incident to R_i is replaced by a horizontal edge incident to R_{i-1} while the rest of f is preserved.

It remains to consider the case when $\mathfrak{d}$ is not incident to a vertex in $\mathcal{R}^N$. In that case, the cells o_r, e_r, o_b and e_b of f exist and o_r is part of $\{\mathfrak{o}_8, \mathfrak{o}_9, \mathfrak{o}_{10}, \mathfrak{o}_{11}\}$ by Lemma 21. It will be proceeded by treating each of the four possible forms of o_r separately.

First, the case when $o_r = \mathfrak{o}_8$ is regarded. In that case, $e_r = \mathfrak{e}_1$ because f contains precisely one drifter. Thus, $e_r' = \mathfrak{e}_8$. Additionally, e_b' and e_b coincide with the sole exception that the two top vertices in e_b' are adjacent while in e_b they are not by Lemma 23. If the cells o_t, e_t, o_l and e_l exist, then both o_t and o_l are in $\mathfrak{O}$ and for that reason e_t' and e_t coincide with the sole exception that the two bottom vertices in e_t' are adjacent whereas in e_t they are not and e_l' and e_l coincide with the sole exception that in e_l' there is no drifter whereas in e_l there is one by Lemma 23. Thus, by Lemma 22 the effect of left-Wieland drift on f is that the move M_1 is applied to $\mathfrak{d}$ while the rest of f remains unchanged.

Next, the case when $o_r = \mathfrak{o}_9$ is considered. In that case, $e_r = \mathfrak{e}_3$, $o_b = \mathfrak{o}_7$ and $e_b = \mathfrak{e}_4$. This is why $e_r' = \mathfrak{e}_9$ and $e_b' = \mathfrak{e}_7$. If, in addition, the cells o_l, e_l, o_t and e_t exist, then o_t and

o_l are in $\mathfrak{O}$. To conclude, by Lemma 23 and Lemma 22 the effect of left-Wieland drift on f is that the move M_2 is applied to $\mathfrak{d}$ while the rest of f is preserved.

Next, the case when $o_r = \mathfrak{o}_{10}$ is regarded. In that case, o_t, e_t, o_l, e_l, o_b and e_b exist and $e_r = \mathfrak{e}_2$, $o_t = \mathfrak{o}_6$ and $e_t = \mathfrak{e}_4$. Therefore, $e'_r = \mathfrak{e}_{10}$ and $e'_t = \mathfrak{e}_6$. Since o_b and o_l are in $\mathfrak{O}$ by Lemma 23 and Lemma 22 the effect of left-Wieland drift on f is that the move M_3 is applied to $\mathfrak{d}$ while the rest of f is preserved.

Finally, the case when $o_r = \mathfrak{o}_{11}$ is checked. In that case o_t, e_t, o_l, e_l, o_b and e_b exist and $e_r = \mathfrak{e}_5$, $o_b = \mathfrak{o}_7$, $e_b = \mathfrak{e}_4$, $o_l = \mathfrak{o}_5$, $e_l = \mathfrak{e}_8$, $o_t = \mathfrak{o}_6$ and $e_t = \mathfrak{e}_4$. Therefore, $e'_r = \mathfrak{e}_{11}$, $e'_b = \mathfrak{e}_7$, $e'_l = \mathfrak{e}_5$ and $e'_t = \mathfrak{e}_6$. In summary, by Lemma 22 the effect of left-Wieland drift on f is that the move M_4 is applied to $\mathfrak{d}$ while the rest of f is preserved. $\square$

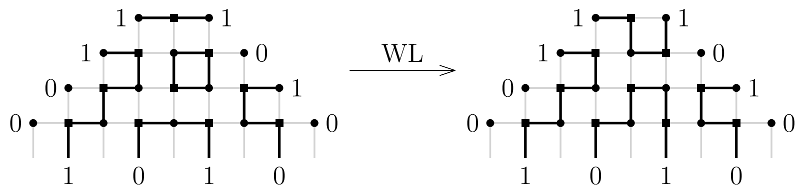


Figure 17: A TFPL with boundary $(0011, 0101; 1010)$ and its image under left-Wieland drift.

Now, Proposition 20 is proved for TFPLs that exhibit two drifters. In doing so, first the case when at least one of the two drifters is incident to a vertex in $\mathcal{R}^N$ is treated:

Lemma 25. *Let f be a TFPL of excess 2 that exhibits two drifters. If at least one of the drifters in f is incident to a vertex in $\mathcal{R}^N$ then the effect of left-Wieland drift on f is determined as follows:*

1. *For $i = 2, \dots, N$ if $R_i \in \mathcal{R}^N$ is incident to a drifter then delete that drifter and add a horizontal edge incident to R_{i-1}. Denote the so-obtained TFPL by f'.*

2. *If f' exhibits a drifter apply M_1, M_2 or M_3 to it.*

Proof. To begin with, suppose that both drifters in f are incident to vertices in $\mathcal{R}^N$. In the following, denote the two drifter in f by $\mathfrak{d}$ and $\mathfrak{d}^*$ and let R_i and R_{i^*} respectively be the vertices in $\mathcal{R}^N$ to which $\mathfrak{d}$ and $\mathfrak{d}^*$ are incident. Then the cells o_l, e_l, o_b and e_b and the cells o_l^*, e_l^*, o_b^* and e_b^* exist. Additionally, the cells o_l, o_b, o_l^* and o_b^* are all in $\mathfrak{O}$, which is why by Lemma 23 and Lemma 22 the effect of left-Wieland drift on f is that both drifters are replaced by horizontal edges incident to R_{i-1} and R_{i^*-1} while the rest of f remains unchanged.

Now, suppose that $\mathfrak{d}$ is incident to a vertex R_i in $\mathcal{R}^N$, while $\mathfrak{d}^*$ is not. To begin with, note that f neither contains a cell of type $\mathfrak{o}_{11}$ nor of type $\mathfrak{e}_{12}$. That is because if it was then in the canonical orientation of f such a cell would give rise to two local configurations that contribute to the excess, which would imply that f is of excess greater than 2. This is impossible. As a start, suppose that no vertex of o_l and o_b is incident to $\mathfrak{d}^*$. In that case e'_l and e_l coincide with the sole exception that in e'_l there is no drifter and e'_b and e_b

coincide with the sole exception that in e_b' the two top vertices are adjacent whereas in e_b they are not by Lemma 23. On the other hand, since $\mathfrak{d}^*$ is not incident to a vertex in $\mathcal{R}^N$ the cells o_r^*, e_r^*, o_b^* and e_b^* exist. Furthermore, no vertex of o_r^*, e_r^*, o_b^* and e_b^* is incident to $\mathfrak{d}$. If the cells o_l^*, e_l^*, o_t^* and e_t^* exist then also no vertex of these cells is incident to $\mathfrak{d}$. For those reasons, by analogous arguments as in the proof of Proposition 20(1) the effect of left-Wieland drift on f is that $\mathfrak{d}$ is replaced by a horizontal edge incident to R_{i-1} before a unique move of $\{M_1, M_2, M_3\}$ is applied to $\mathfrak{d}^*$. The rest of f is preserved by left-Wieland drift.

Now, if the bottom right vertex of o_b is incident to $\mathfrak{d}^*$, then $o_r^* \in \{\mathfrak{o}_8, \mathfrak{o}_9, \mathfrak{o}_{10}\}$. If $o_r^* = \mathfrak{o}_8$, then $e_r^* = \mathfrak{e}_1$. Furthermore, o_l, o_b, o_l^* and o_b^* do not contain a drifter and are not in $\{\mathfrak{o}_6, \mathfrak{o}_7, \mathfrak{o}_{12}\}$. Thus, the effect of left-Wieland drift is that $\mathfrak{d}$ is replaced by a horizontal edge incident to R_{i-1} before the move M_1 is applied to $\mathfrak{d}^*$ while the rest of f is preserved. If $o_r^* = \mathfrak{o}_9$, then $e_r^* = \mathfrak{e}_3$, $o_b^* = \mathfrak{o}_7$ and $e_b^* = \mathfrak{e}_4$. Additionally, o_l, o_b and o_l do not contain a drifter and are not in $\{\mathfrak{o}_6, \mathfrak{o}_7, \mathfrak{o}_{12}\}$. Therefore, the effect of left-Wieland drift is that $\mathfrak{d}$ is replaced by a horizontal edge incident to R_{i-1} before the move M_2 is applied to $\mathfrak{d}^*$ while the rest of f is preserved. Finally, if $o_r^* = \mathfrak{o}_{10}$, then $e_r^* = \mathfrak{e}_2$, $o_b = \mathfrak{o}_6$ and $e_b = \mathfrak{e}_{10}$. Furthermore, o_l, o_l^* and o_b^* do not contain a drifter and are not in $\{\mathfrak{o}_6, \mathfrak{o}_7, \mathfrak{o}_{12}\}$. For those reasons, the effect of left-Wieland drift is that $\mathfrak{d}$ is replaced by a horizontal edge incident to R_{i-1} before the move M_3 is applied to $\mathfrak{d}^*$ while the rest of f is preserved.

Next, if o_l contains $\mathfrak{d}^*$, then $o_l = \mathfrak{o}_{10}$, $e_l = \mathfrak{e}_6$, $o_t^* = \mathfrak{o}_6$, $e_t^* = \mathfrak{e}_4$, $o_b = \mathfrak{o}_4$, $e_b = \mathfrak{e}_{10}$ and $o_b^* \notin \{\mathfrak{o}_6, \mathfrak{o}_7, \mathfrak{o}_{12}\}$. Therefore, $e_l' = \mathfrak{e}_{10}$, $e_t' = \mathfrak{e}_6$, $e_b' = \mathfrak{e}_4$ and by Lemma 23 the cells $e_b^{*\prime}$ and e_b^* coincide with the sole exception that in $e_b^{*\prime}$ there is an edge between the two top vertices whereas in e_b^* there is none. For those reasons, the effect of left-Wieland drift on f is that $\mathfrak{d}$ is replaced by a horizontal edge incident to R_{i-1} before the move M_3 is applied to $\mathfrak{d}^*$ while the rest of f is preserved.

Finally, if o_b contains $\mathfrak{d}^*$, then $o_b \in \{\mathfrak{o}_8, \mathfrak{o}_9\}$. If $o_b = \mathfrak{o}_8$, then $e_b = \mathfrak{e}_2$. Furthermore, none of the cells o_l, o_l^* and o_b^* is in $\{\mathfrak{o}_6, \mathfrak{o}_7, \mathfrak{o}_{12}\}$. Thus, the effect of left-Wieland drift on f is that $\mathfrak{d}$ is replaced by a horizontal edge incident to R_{i-1} before the move M_1 is applied to $\mathfrak{d}^*$ while the rest of f is preserved. On the other hand, if $o_b = \mathfrak{o}_9$, then $e_b = \mathfrak{e}_5$, $o_b^* = \mathfrak{o}_7$ and $e_b^* = \mathfrak{e}_4$. Additionally, o_l and o_l^* do not contain a drifter and are not in $\{\mathfrak{o}_6, \mathfrak{o}_7, \mathfrak{o}_{12}\}$. Therefore, the effect of left-Wieland drift on f is that $\mathfrak{d}$ is replaced by a horizontal edge incident to R_{i-1} before the move M_2 is applied to $\mathfrak{d}^*$ while the rest of f is preserved. $\square$

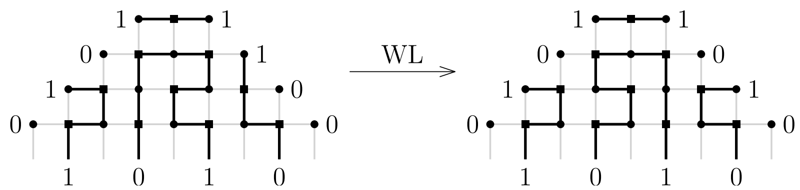


Figure 18: A TFPL with boundary $(0101, 0011; 1010)$ and its image under left-Wieland drift.

To conclude the proof of Proposition 20 TFPLs of excess 2 that exhibit two drifters that are both not incident to a vertex in $\mathcal{R}^N$ need to be considered.

Proof of Proposition 20. Let f be a TFPL of excess 2 that contains two drifters whereof none is incident to a vertex in $\mathcal{R}^N$. In that case the cells o_r, e_r, o_b, e_b and the cells o_r^*, e_r^*, o_b^*, e_b^* exist. In addition, both o_r and o_r^* must be in $\{\mathfrak{o}_8, \mathfrak{o}_9, \mathfrak{o}_{10}, \mathfrak{o}_{13}, \mathfrak{o}_{14}\}$. It is started with the case when no vertex of the cells o_r, e_r, o_b, e_b is incident to $\mathfrak{d}^*$ and no vertex of the cells o_r^*, e_r^*, o_b^*, e_b^* is incident to $\mathfrak{d}$. This implies that if the cells o_l, e_l, o_t and e_t exist then none of their vertices is incident to $\mathfrak{d}^*$ and if the cells o_l^*, e_l^*, o_t^* and e_t^* exist then none of their vertices is incident to $\mathfrak{d}$. Therefore, by the same arguments as in the proof of Lemma 24 the effect of left-Wieland drift on f is that to each of the two drifters $\mathfrak{d}$ and $\mathfrak{d}^*$ a unique move in $\{M_1, M_2, M_3\}$ is applied while the rest of f is conserved. Since the moves can be performed simultaneously they can be performed in the order stated in Proposition 20.

It remains to study the case when a vertex of o_r, e_r, o_b or e_b is incident to $\mathfrak{d}^*$ or a vertex of o_r^*, e_r^*, o_b^* or e_b^* is incident to $\mathfrak{d}$. Hence, without loss of generality assume that a vertex of the cells o_r, e_r, o_b, e_b that is not the top right vertex of o_r is incident to the drifter $\mathfrak{d}^*$. Then o_r does not equal $\mathfrak{o}_{14}$ and o_r^* does not equal $\mathfrak{o}_{13}$.

First, the case when the bottom right vertex of o_b is incident to $\mathfrak{d}^*$ is considered. In that case, $\mathfrak{d}$ and $\mathfrak{d}^*$ have the same x-coordinate and $\mathfrak{d}$ has the larger y-coordinates than $\mathfrak{d}^*$. If the cells o_t, e_t, o_l and e_l exist then o_t neither equals $\mathfrak{o}_7$ nor $\mathfrak{o}_{12}$. Furthermore, if $o_t = \mathfrak{o}_6$ then $e_t = \mathfrak{e}_4$, $o_r = \mathfrak{o}_{10}$ and $e_r = \mathfrak{e}_2$. Thus, $e_t' = \mathfrak{e}_6$ and $e_r' = \mathfrak{e}_{10}$. On the other hand, if o_t does not equal $\mathfrak{o}_6$ then e_t' and e_t coincide with the sole exception that in e_t' there is a horizontal edge between its two bottom vertices while in e_t there is none by Lemma 23. Since o_l and o_l^* are in $\mathfrak{O}$ the cells e_l' and e_l (*resp.* $e_l^{*\prime}$ and e_l^*) coincide with the sole exception that in e_l' (*resp.* $e_l^{*\prime}$) there is no drifter by Lemma 23. Finally, o_b^* does neither equal $\mathfrak{o}_6$ nor $\mathfrak{o}_{12}$ and if it equals $\mathfrak{o}_7$ then $e_b^* = \mathfrak{e}_4$, $o_r^* = \mathfrak{o}_9$, $e_r^* = \mathfrak{e}_3$, $e_b^{*\prime} = \mathfrak{e}_7$ and $e_r^{*\prime} = \mathfrak{e}_2$.

By Lemma 22, it remains to study the cells o_r, e_r, o_b, e_b, o_r^*, e_r^*, e_r', e_b' and $e_r^{*\prime}$. A list of all possible configurations in the cells o_r, e_r, o_b, e_b, o_r^*, e_r^*, e_r', e_b' and $e_r^{*\prime}$ in f is given in Table 1.

o_r	$\mathfrak{o}_8$	$\mathfrak{o}_8$	$\mathfrak{o}_8$	$\mathfrak{o}_8$	$\mathfrak{o}_8$	$\mathfrak{o}_9$	$\mathfrak{o}_9$	$\mathfrak{o}_9$	$\mathfrak{o}_{10}$	$\mathfrak{o}_{10}$	$\mathfrak{o}_{10}$	$\mathfrak{o}_{10}$
e_r	$\mathfrak{e}_1$	$\mathfrak{e}_1$	$\mathfrak{e}_1$	$\mathfrak{e}_1$	$\mathfrak{e}_1$	$\mathfrak{e}_3$	$\mathfrak{e}_3$	$\mathfrak{e}_3$	$\mathfrak{e}_2$	$\mathfrak{e}_2$	$\mathfrak{e}_2$	$\mathfrak{e}_2$
o_r^*	$\mathfrak{o}_8$	$\mathfrak{o}_8$	$\mathfrak{o}_9$	$\mathfrak{o}_9$	$\mathfrak{o}_{10}$	$\mathfrak{o}_8$	$\mathfrak{o}_9$	$\mathfrak{o}_{10}$	$\mathfrak{o}_8$	$\mathfrak{o}_8$	$\mathfrak{o}_9$	$\mathfrak{o}_{10}$
e_r^*	$\mathfrak{e}_1$	$\mathfrak{e}_1$	$\mathfrak{e}_3$	$\mathfrak{e}_3$	$\mathfrak{e}_2$	$\mathfrak{e}_1$	$\mathfrak{e}_3$	$\mathfrak{e}_2$	$\mathfrak{e}_1$	$\mathfrak{e}_1$	$\mathfrak{e}_3$	$\mathfrak{e}_2$
o_b	$\mathfrak{o}_1$	$\mathfrak{o}_4$	$\mathfrak{o}_1$	$\mathfrak{o}_4$	$\mathfrak{o}_6$	$\mathfrak{o}_7$	$\mathfrak{o}_7$	$\mathfrak{o}_{12}$	$\mathfrak{o}_1$	$\mathfrak{o}_4$	$\mathfrak{o}_4$	$\mathfrak{o}_6$
e_b	$\mathfrak{e}_5$	$\mathfrak{e}_{11}$	$\mathfrak{e}_5$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{10}$	$\mathfrak{e}_9$	$\mathfrak{e}_9$	$\mathfrak{e}_4$	$\mathfrak{e}_5$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{11}$	$\mathfrak{e}_{10}$
e_r'	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_9$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{10}$
$e_r^{*\prime}$	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$
e_b'	$\mathfrak{e}_{16}$	$\mathfrak{e}_4$	$\mathfrak{e}_{16}$	$\mathfrak{e}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_7$	$\mathfrak{e}_{12}$	$\mathfrak{e}_{16}$	$\mathfrak{e}_4$	$\mathfrak{e}_4$	$\mathfrak{e}_6$

Table 1: The cells $o_r, e_r, o_r^*, e_r^*, o_b$ and e_b of f and the cells $e_r', e_r^{*\prime}$ and e_b' of $\mathrm{WL}(f)$ in the case when $\mathfrak{d}^*$ is incident to the bottom right vertex of o_b in f.

In summary, left-Wieland drift has the following effect:

- The move M_5 is applied if $o_r = \mathfrak{o}_9$ and $o_r^* = \mathfrak{o}_{10}$.

- The move M_1/M_2 is applied to $\mathfrak{d}^*$ before the move M_2 is applied to $\mathfrak{d}$ if $o_r = \mathfrak{o}_9$ and $o_r^* = \mathfrak{o}_8/\mathfrak{o}_9$.

- The move M_1 is applied to $\mathfrak{d}$ before the move $M_1/M_2/M_3$ is applied to $\mathfrak{d}^*$ if $o_r = \mathfrak{o}_8$ and $o_r^* = \mathfrak{o}_8/\mathfrak{o}_9/\mathfrak{o}_{10}$.

- The move M_3 is applied to $\mathfrak{d}$ before the move $M_1/M_2/M_3$ is applied to $\mathfrak{d}^*$ if $o_r = \mathfrak{o}_{10}$ and $o_r^* = \mathfrak{o}_8/\mathfrak{o}_9/\mathfrak{o}_{10}$.

In all cases the rest of f is preserved by left-Wieland drift.

Next, the case when the drifter $\mathfrak{d}^*$ is contained in e_r is studied. In that case the x-coordinate of $\mathfrak{d}^*$ is larger than the one of $\mathfrak{d}$. Note that $(o_r, e_r) \in \{(\mathfrak{o}_9, \mathfrak{e}_7), (\mathfrak{o}_{10}, \mathfrak{e}_6)\}$ since f contains neither of the cells $\mathfrak{o}_{11}$ and $\mathfrak{e}_{12}$. Now, if $o_r = \mathfrak{o}_9$ then $o_b = \mathfrak{o}_7$, $e_b = \mathfrak{e}_4$, $o_t^* = \mathfrak{o}_4$, $e_t^* = \mathfrak{e}_9$ and $(o_r^*, e_r^*) \in \{(\mathfrak{o}_8, \mathfrak{e}_1), (\mathfrak{o}_9, \mathfrak{e}_3)\}$. Furthermore, if $o_r^* = \mathfrak{o}_9$ then $o_b^* = \mathfrak{o}_7$ and $e_b^* = \mathfrak{e}_4$. Thus, $e_r' = \mathfrak{e}_9$, $e_b' = \mathfrak{e}_7$, $e_t^{*\prime} = \mathfrak{e}_4$ and if $o_r^* = \mathfrak{o}_9$ then $e_r^{*\prime} = \mathfrak{e}_9$ and $e_b^{*\prime} = \mathfrak{e}_7$. On the other hand, if $o_r = \mathfrak{o}_{10}$ then $o_t = \mathfrak{o}_6$, $e_t = \mathfrak{e}_4$, $o_b^* = \mathfrak{o}_4$, $e_b^* = \mathfrak{e}_{10}$ and $(o_r^*, e_r^*) \in \{(\mathfrak{o}_8, \mathfrak{e}_1), (\mathfrak{o}_{10}, \mathfrak{e}_2)\}$. Furthermore, if $o_r^* = \mathfrak{o}_{10}$ then $o_t^* = \mathfrak{o}_6$ and $e_t^* = \mathfrak{e}_4$. Thus, $e_r' = \mathfrak{e}_{10}$, $e_t' = \mathfrak{e}_6$ and $e_b^{*\prime} = \mathfrak{e}_4$ and if $o_r^* = \mathfrak{o}_{10}$ then $e_r^{*\prime} = \mathfrak{e}_{10}$ and $e_t^{*\prime} = \mathfrak{e}_6$. By Lemma 22 and Lemma 23 the effect of left-Wieland drift is the following:

- The move M_1/M_2 is applied to $\mathfrak{d}^*$ before the move M_2 is applied to $\mathfrak{d}$ if $o_r = \mathfrak{o}_9$ and $o_r^* = \mathfrak{o}_8/\mathfrak{o}_9$.

- The move M_1/M_3 is applied to $\mathfrak{d}^*$ before the move M_3 is applied to $\mathfrak{d}$ if $o_r = \mathfrak{o}_{10}$ and $o_r^* = \mathfrak{o}_8/\mathfrak{o}_{10}$.

In both cases the rest of f is preserved by left-Wieland drift.

Next, the case when $\mathfrak{d}^*$ is contained in e_b is regarded. In that case the x-coordinate of $\mathfrak{d}^*$ is larger than the one of $\mathfrak{d}$. The cells o_l and o_b are both not contained in $\{\mathfrak{o}_6, \mathfrak{o}_7, \mathfrak{o}_{12}\}$. For instance, it is not possible that o_b equals $\mathfrak{o}_7$ because then e_b would have to equal e_{15} or the bottom right vertex of o_b would be incident to a drifter. As a start, if o_t exists then it cannot be in $\{\mathfrak{o}_7, \mathfrak{o}_{12}\}$. Furthermore, if $o_t = \mathfrak{o}_6$ then $e_t = \mathfrak{e}_4$, $o_r = \mathfrak{o}_{10}$, $e_r = \mathfrak{e}_2$, $e_t' = \mathfrak{e}_6$ and $e_r' = \mathfrak{e}_{10}$. On the other hand, o_b^* cannot be in $\{\mathfrak{o}_6, \mathfrak{o}_{12}\}$. Furthermore, if $o_b^* = \mathfrak{o}_7$ then $e_b^* = \mathfrak{e}_4$, $o_r^* = \mathfrak{o}_9$, $e_r^* = \mathfrak{e}_3$, $e_b^{*\prime} = \mathfrak{e}_7$ and $e_r^{*\prime} = \mathfrak{e}_9$. To determine the effect of left-Wieland drift on f it remains to study the cells o_r^*, e_r^*, o_r, e_r, $e_r^{*\prime}$ and e_r'. In Table 2 all possible configurations in these cells are listed.

In summary, the effect of left-Wieland drift on f is the following:

- The move M_1 is applied to $\mathfrak{d}^*$ before the move M_1/M_3 is applied to $\mathfrak{d}$ if $o_r^* = \mathfrak{o}_8$ and $o_r = \mathfrak{o}_8/\mathfrak{o}_{10}$.

- The move M_2 is applied to $\mathfrak{d}^*$ before the move M_1/M_3 is applied to $\mathfrak{d}$ if $o_r^* = \mathfrak{o}_9$ and $o_r = \mathfrak{o}_8/\mathfrak{o}_{10}$.

- The move M_3 is applied to $\mathfrak{d}^*$ before it also is applied to $\mathfrak{d}$ if $o_r^* = \mathfrak{o}_{10}$ and $o_r = \mathfrak{o}_{13}$.

o_r^*	$\mathfrak{o}_8$	$\mathfrak{o}_8$	$\mathfrak{o}_9$	$\mathfrak{o}_9$	$\mathfrak{o}_{10}$
e_r^*	$\mathfrak{e}_1$	$\mathfrak{e}_1$	$\mathfrak{e}_3$	$\mathfrak{e}_3$	$\mathfrak{e}_2$
o_r	$\mathfrak{o}_8$	$\mathfrak{o}_{10}$	$\mathfrak{o}_8$	$\mathfrak{o}_{10}$	$\mathfrak{o}_{13}$
e_r	$\mathfrak{e}_3$	$\mathfrak{e}_5$	$\mathfrak{e}_3$	$\mathfrak{e}_5$	$\mathfrak{e}_4$
$e_r^{*\prime}$	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$
e_r'	$\mathfrak{e}_8$	$\mathfrak{e}_{10}$	$\mathfrak{e}_8$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{13}$

Table 2: The cells o_r^*, e_r^*, o_r and e_r of f and the cells $e_r^{*\prime}$ and e_r' of $\mathrm{WL}(f)$ in the case when $\mathfrak{d}^*$ is contained in e_b.

In all cases the rest of f is preserved by left-Wieland drift.

The last case that is to be considered is the case when $\mathfrak{d}^*$ is contained in o_b. In that case the x-coordinate of $\mathfrak{d}$ is larger than the one of $\mathfrak{d}^*$. Furthermore, the cells o_l and o_l^* are not contained in $\{\mathfrak{o}_6, \mathfrak{o}_7, \mathfrak{o}_12\}$, if o_t exists then it cannot be in $\{\mathfrak{o}_7, \mathfrak{o}_{12}\}$ and o_b^* cannot be in $\{\mathfrak{o}_6, \mathfrak{o}_{12}\}$. On the other hand, if $o_t = \mathfrak{o}_6$ then $e_t = \mathfrak{e}_4$, $o_r = \mathfrak{o}_{10}$, $e_r = \mathfrak{e}_2$, $e_t' = \mathfrak{e}_6$ and $e_r' = \mathfrak{e}_{10}$ and if $o_b^* = \mathfrak{o}_7$ then $e_b^* = \mathfrak{e}_4$, $o_r^* = \mathfrak{o}_9$, $e_r^* = \mathfrak{e}_3$, $e_b^{*\prime} = \mathfrak{e}_7$ and $e_r^{*\prime} = \mathfrak{e}_9$. To determine the effect of left-Wieland drift on f it remains to study the cells o_r^*, e_r^*, o_r, e_r, $e_r^{*\prime}$ and e_r'. In Table 3 all possible configurations in these cells are listed.

o_r	$\mathfrak{o}_8$	$\mathfrak{o}_8$	$\mathfrak{o}_9$	$\mathfrak{o}_{10}$	$\mathfrak{o}_{10}$
e_r	$\mathfrak{e}_1$	$\mathfrak{e}_1$	$\mathfrak{e}_3$	$\mathfrak{e}_2$	$\mathfrak{e}_2$
o_b	$\mathfrak{o}_8$	$\mathfrak{o}_9$	$\mathfrak{o}_{14}$	$\mathfrak{o}_8$	$\mathfrak{o}_9$
e_b	$\mathfrak{e}_2$	$\mathfrak{e}_5$	$\mathfrak{e}_4$	$\mathfrak{e}_2$	$\mathfrak{e}_5$
e_r'	$\mathfrak{e}_8$	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_{10}$	$\mathfrak{e}_{10}$
e_b'	$\mathfrak{e}_8$	$\mathfrak{e}_9$	$\mathfrak{e}_2$	$\mathfrak{e}_8$	$\mathfrak{e}_9$

Table 3: The cells o_r, e_r, o_b and e_b of f and the cells e_r' and e_b' of $\mathrm{WL}(f)$ in the case when $\mathfrak{d}^*$ is contained in o_b.

By Lemma 22 and Lemma 23 the effect of left-Wieland drift on f is the following:

- The move M_1 is applied to $\mathfrak{d}$ before the move M_1/M_2 is applied to $\mathfrak{d}^*$ if $o_r = \mathfrak{o}_8$ and $o_b = \mathfrak{o}_8/\mathfrak{o}_9$.
- The move M_2 is first applied to $\mathfrak{d}$ and then to $\mathfrak{d}^*$ if $o_r = \mathfrak{o}_9$ and $o_b = \mathfrak{o}_{14}$.
- The move M_3 is applied to $\mathfrak{d}$ before the move M_1/M_2 is applied to $\mathfrak{d}^*$ if $o_r = \mathfrak{o}_{10}$ and $o_b = \mathfrak{o}_8/\mathfrak{o}_9$. □

4 The path of a drifter under Wieland drift

The focus of this section is on studying how many iterations of left- (*resp.* right-) Wieland drift are needed until a drifter in an instable TFPL of excess 2 is incident to a vertex in

$\mathcal{R}^N$ (*resp.* $\mathcal{L}^N$). For this purpose, it is necessary to specify which drifter in the image of a TFPL under left- (*resp.* right-) Wieland drift is assigned to which drifter in the initial TFPL. To begin with, in both the preimage and the image of each of the moves M_1, M_2 and M_3 (*resp.* M_1^{-1}, M_2^{-1} and M_3^{-1}) there is precisely one drifter. Thus, from now on, the drifter in the image is assigned to the drifter in the preimage for each of these moves.

In contrast to the moves M_1, M_2 and M_3 (*resp.* M_1^{-1}, M_2^{-1} and M_3^{-1}) the preimage and the image of both M_4 and M_5 (*resp.* M_4^{-1} and M_5^{-1}) do not exhibit the same number of drifters. Hence, fix a drifter in the image of M_4 (*resp.* M_5^{-1}) that is from now on assigned to the drifter in the preimage of M_4 (*resp.* M_5^{-1}). Reversely, if to a drifter the move M_5 (*resp.* M_4^{-1}) is applied then the drifter in the image of the move M_5 (*resp.* M_4^{-1}) is assigned to this drifter.

Now, let f be an instable TFPL of excess 2 and $\mathfrak{d}$ a drifter in f. Then denote by $\mathfrak{d}(\ell)$ (*resp.* $\mathfrak{d}(-r)$) the drifter in $\mathrm{WL}^\ell(f)$ (*resp.* $\mathrm{WR}^r(f)$) that is assigned to $\mathfrak{d}$ and with $L_f(\mathfrak{d})$ (*resp.* $R_f(\mathfrak{d})$) the unique non-negative integer such that $\mathfrak{d}(L_f(\mathfrak{d}))$ (*resp.* $\mathfrak{d}(R_f(\mathfrak{d}))$) is incident to a vertex in $\mathcal{R}^N$ (*resp.* $\mathcal{L}^N$). Thus, $\mathfrak{d}$ can be understood as a map on $[-R_f(\mathfrak{d}), L_f(\mathfrak{d})]$ that assigns to $\ell \in [0; L_f(\mathfrak{d})]$ a drifter in $\mathrm{WL}^\ell(f)$ and to $-r \in [-R_f(\mathfrak{d}); -1]$ a drifter in $\mathrm{WR}^r(f)$. (Note that $\mathfrak{d}(0) = \mathfrak{d}$.)

Definition 26. Let f be an instable TFPL of excess at most 2 and $\mathfrak{d}$ be a drifter in f. The *path* of $\mathfrak{d}$ is defined as the sequence of all instable TFPLs that contain $\mathfrak{d}$ and can be reached by an iterated application of left- or right-Wieland drift to f, that is,

$$\mathrm{Path}(f;\mathfrak{d}) = \left(\mathrm{WR}^{R_f(\mathfrak{d})}(f), \ldots, \mathrm{WR}(f), f, \mathrm{WL}(f), \ldots, \mathrm{WL}^{L_f(\mathfrak{d})}(f)\right).$$

In addition, $\mathrm{Right}(f;\mathfrak{d})$ (*resp.* $\mathrm{Left}(f;\mathfrak{d})$) is defined as the TFPL $\mathrm{WR}^{R_f(\mathfrak{d})}(f)$ (*resp.* $\mathrm{WL}^{L_f(\mathfrak{d})}(f)$) and $\mathrm{HeigthR}(\mathfrak{d})$ (*resp.* $\mathrm{HeightL}(f;\mathfrak{d})$) as the positive integer h such that $\mathfrak{d}$ is incident to L_{h+1} (*resp.* R_{h+1}) in $\mathrm{Right}(f;\mathfrak{d})$ (*resp.* $\mathrm{Left}(f;\mathfrak{d})$).

By definition it holds $|\mathrm{Path}(f;\mathfrak{d})| = L_f(\mathfrak{d}) + R_f(\mathfrak{d}) + 1$. For $i = 1,2,3,4,5$ set

$$\begin{aligned}\#M_i(\mathfrak{d}) = |\{0 \leqslant r \leqslant R_f(\mathfrak{d}) - 1 :\ &\text{by WR the move } M_i^{-1} \text{ is applied to } \mathfrak{d}(r) \text{ in } \mathrm{WR}^r(f)\}| \\ + |\{0 \leqslant \ell \leqslant L_f(\mathfrak{d}) - 1 :\ &\text{by WL the move } M_i \text{ is applied to } \mathfrak{d}(\ell) \text{ in } \mathrm{WL}^\ell(f)\}|.\end{aligned}$$

Thus, $|\mathrm{Path}(f;\mathfrak{d})| = \#M_1(\mathfrak{d}) + \#M_2(\mathfrak{d}) + \#M_3(\mathfrak{d}) + \#M_4(\mathfrak{d}) + \#M_5(\mathfrak{d}) + 1$ and in summary

$$L_f(\mathfrak{d}) + R_f(\mathfrak{d}) = \#M_1(\mathfrak{d}) + \#M_2(\mathfrak{d}) + \#M_3(\mathfrak{d}) + \#M_4(\mathfrak{d}) + \#M_5(\mathfrak{d}).$$

Definition 27. Let f be an instable TFPL of excess at most 2, $\mathfrak{d}$ a drifter in f, $u^{R_f(\mathfrak{d})}$ the left boundary of $\mathrm{WR}^{R_f(\mathfrak{d})}(f)$ and $v^{L_f(\mathfrak{d})}$ the right boundary of $\mathrm{WL}^{L_f(\mathfrak{d})}(f)$. Then define $R_i(u^{R_f(\mathfrak{d})})$ (*resp.* $R_i(v^{L_f(\mathfrak{d})})$) as the number of occurrences of i among the last $N - 1 - \mathrm{HeightR}(f;\mathfrak{d})$ (*resp.* $N - 1 - \mathrm{HeightL}(f;\mathfrak{d})$) letters of $u^{R_f(\mathfrak{d})}$ (*resp.* $v^{L_f(\mathfrak{d})}$) for $i = 0, 1$.

Proposition 28. *Let f, $\mathfrak{d}$, $u^{R_f(\mathfrak{d})}$ and $v^{L_f(\mathfrak{d})}$ as in Definition 27. Then*

$$\#M_1(\mathfrak{d}) + \#M_2(\mathfrak{d}) + \#M_3(\mathfrak{d}) + \#M_4(\mathfrak{d}) + \#M_5(\mathfrak{d}) = R_1(u^{R_f(\mathfrak{d})}) + R_1(v^{L_f(\mathfrak{d})}) + 1.$$

The proof of Proposition 28 will be the content of the rest of this section. The crucial idea is to regard TFPLs together with their canonical orientation. Before starting with the proof a crucial corollary of Proposition 28 is stated.

Corollary 29. *Let f, $\mathfrak{d}$, $u^{R_f(\mathfrak{d})}$ and $v^{L_f(\mathfrak{d})}$ as in Definition 27. Then*

$$L_f(\mathfrak{d}) + R_f(\mathfrak{d}) = R_1(u^{R_f(\mathfrak{d})}) + R_1(v^{L_f(\mathfrak{d})}) + 1. \tag{6}$$

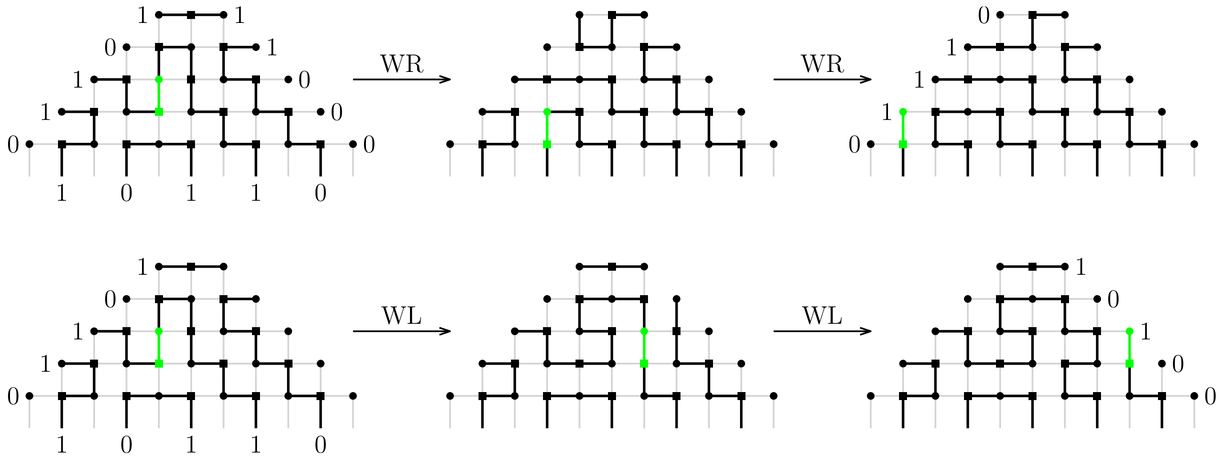


Figure 19: A TFPL with boundary $(01101, 00011; 10110)$ and the path of the drifter that is indicated in green.

The following lemmas are immediate consequences of Proposition 10 and Proposition 20 and describe the effect of Wieland drift on canonically oriented TFPLs of excess at most 2. The moves that form the basis for this description derive from the moves in Figure 13 and are depicted in Figure 20.

Remark 30. The moves $\overrightarrow{M}_{1,1}$, $\overrightarrow{M}_{1,2}$, $\overrightarrow{M}_{1,3}$, $\overrightarrow{M}_{1,4}$, $\overrightarrow{M}_{2,1}$ and $\overrightarrow{M}_{3,1}$ in Figure 20 coincide with the moves BB, BR, RR, RB, B and R respectively in [5].

Lemma 31. *Let u, v and w be words of length N such that $\mathrm{exc}(u,v;w) \leqslant 2$ and f an instable TFPL with boundary $(u,v;w)$ in which not all drifters are incident to a vertex in $\mathcal{R}^N$. When $\overrightarrow{f}$ denotes f together with the canonical orientation of its edges, then the effect of left-Wieland drift on f translates into the following effect on $\overrightarrow{f}$:*

1. *If in $\overrightarrow{f}$ there is precisely one drifter then by left-Wieland drift one of the moves $\overrightarrow{M}_{1,1}$, $\overrightarrow{M}_{1,2}$, $\overrightarrow{M}_{1,3}$, $\overrightarrow{M}_{1,4}$, $\overrightarrow{M}_{2,1}$, $\overrightarrow{M}_{2,2}$, $\overrightarrow{M}_{2,3}$, $\overrightarrow{M}_{3,1}$, $\overrightarrow{M}_{3,2}$, $\overrightarrow{M}_{3,3}$ or $\overrightarrow{M}_{4}$ is performed while the rest of $\overrightarrow{f}$ remains unchanged.*

2. *If in $\overrightarrow{f}$ there are two drifters and none of those drifters is incident to a vertex in $\mathcal{R}^N$ then by left-Wieland drift either $\overrightarrow{M}_5$ is performed or to each drifter one of the moves $\overrightarrow{M}_{1,1}$, $\overrightarrow{M}_{1,2}$, $\overrightarrow{M}_{1,3}$, $\overrightarrow{M}_{1,4}$, $\overrightarrow{M}_{2,1}$ or $\overrightarrow{M}_{3,1}$ is applied in the same order as in Proposition 20. The rest of $\overrightarrow{f}$ remains unchanged.*

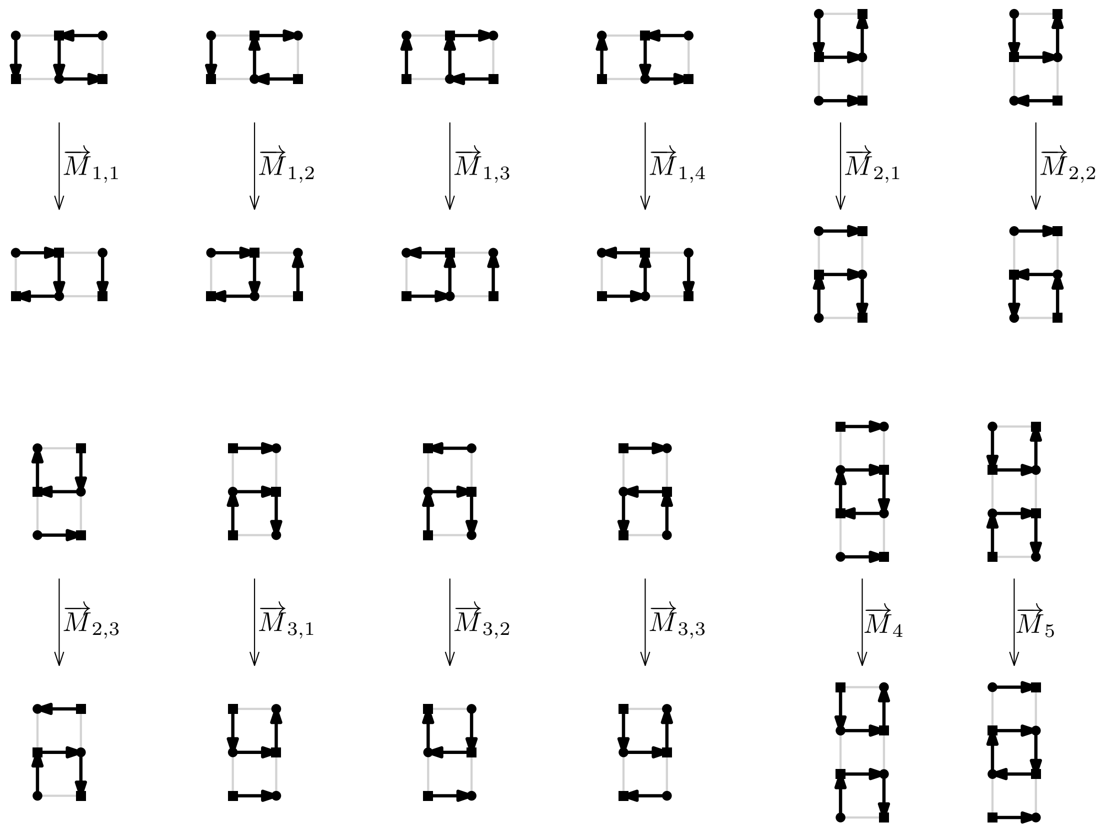


Figure 20: The moves describing the effect of Wieland drift on TFPLs of excess at most 2 that are equipped with the canonical orientation.

3. *Finally, if in $\overrightarrow{f}$ there are two drifters whereof one is incident to a vertex in $R_i \in \mathcal{R}^N$ then by left-Wieland drift the drifter incident to R_i is replaced by a horizontal edge incident to R_{i-1} before to the remaining drifter one of the moves $\overrightarrow{M}_{1,1}$, $\overrightarrow{M}_{1,2}$, $\overrightarrow{M}_{1,3}$, $\overrightarrow{M}_{1,4}$, $\overrightarrow{M}_{2,1}$ or $\overrightarrow{M}_{3,1}\}$ is applied. The rest of $\overrightarrow{f}$ remains unchanged.*

The effect of right-Wieland drift on a canonically oriented TFPL of excess at most 2 can be described in the vertically symmetric way as the composition of the moves $\overrightarrow{M}_{1,1}^{-1}$, $\overrightarrow{M}_{1,2}^{-1}$, $\overrightarrow{M}_{1,3}^{-1}$, $\overrightarrow{M}_{1,4}^{-1}$, $\overrightarrow{M}_{2,1}^{-1}$, $\overrightarrow{M}_{2,2}^{-1}$, $\overrightarrow{M}_{2,3}^{-1}$, $\overrightarrow{M}_{3,1}^{-1}$, $\overrightarrow{M}_{3,2}^{-1}$, $\overrightarrow{M}_{3,3}^{-1}$, $\overrightarrow{M}_{4}^{-1}$ and $\overrightarrow{M}_{5}^{-1}$.

Now, for $(i,j) = (1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$ set

$$\begin{aligned}\#\overrightarrow{M}_{i,j}(\mathfrak{d}) = &|\{0 \leqslant r < R_f(\mathfrak{d}) : \overrightarrow{M}_{i,j}^{-1} \textit{ is applied to } \mathfrak{d}(r) \textit{ in } \mathrm{WR}^r(\overrightarrow{f}) \text{ by } \mathrm{WR}\}| \\ &+ |\{0 \leqslant \ell < L_f(\mathfrak{d}) : \overrightarrow{M}_{i,j} \textit{ is applied to } \mathfrak{d}(\ell) \textit{ in } \mathrm{WL}^\ell(\overrightarrow{f}) \textit{ by } \mathrm{WL}\}|.\end{aligned}$$

For the moves $\overrightarrow{M}_{4}^{-1}$ and $\overrightarrow{M}_5$ it has to be distinguished which of the two drifters in the respective image is identified with the one in the preimage. Hence, for $i = 4, 5$ set

$$\begin{aligned}\#\overrightarrow{M}_i = |\{0 \leqslant r < R_f(\mathfrak{d}) : \overrightarrow{M}_i^{-1} \textit{ is applied to } \mathfrak{d}(r) \textit{ in } \mathrm{WR}^r(\overrightarrow{f}) \textit{ by } \mathrm{WR}\}| \\ + |\{0 \leqslant \ell < L_f(\mathfrak{d}) : \overrightarrow{M}_i \textit{ is applied to } \mathfrak{d}(\ell) \textit{ in } \mathrm{WL}^\ell(\overrightarrow{f}) \textit{ by } \mathrm{WL}\}|.\end{aligned}$$

and indicate by a t or b whether in the respective image the top or the bottom drifter is identified with the drifter in the preimage. In that way, one obtains the notations $\#\overrightarrow{M}_i^b$ respectively $\#\overrightarrow{M}_i^t$ for $i = 4, 5$.

Lemma 32. *Let f be an instable TFPL with boundary $(u, v; w)$ where* $\mathrm{exc}(u, v; w) \leqslant 2$ *and $\mathfrak{d}$ a drifter in f. When $\overrightarrow{f}$ denotes f together with the canonical orientation of its edges, then the following hold:*

1. $\sum_{i=1}^{4} \#\overrightarrow{M}_{1,i}(\mathfrak{d}) + \sum_{j=1}^{3} \#\overrightarrow{M}_{3,j}(\mathfrak{d}) + \#\overrightarrow{M}_4^t(\mathfrak{d}) + \#\overrightarrow{M}_5^b(\mathfrak{d}) = N - \mathrm{HeightR}(f; \mathfrak{d})$,
2. $\sum_{i=1}^{4} \#\overrightarrow{M}_{1,i}(\mathfrak{d}) + \sum_{j=1}^{3} \#\overrightarrow{M}_{2,j}(\mathfrak{d}) + \#\overrightarrow{M}_4^b(\mathfrak{d}) + \#\overrightarrow{M}_5^t(\mathfrak{d}) = N - \mathrm{HeightL}(f; \mathfrak{d})$,
3. $\#\overrightarrow{M}_{1,1}(\mathfrak{d}) + \#\overrightarrow{M}_{1,4}(\mathfrak{d}) + \#\overrightarrow{M}_{2,3}(\mathfrak{d}) - \#\overrightarrow{M}_{3,3}(\mathfrak{d}) + \#\overrightarrow{M}_4^b(\mathfrak{d}) = R_0(v^{L_f(\mathfrak{d})})$,
4. $\#\overrightarrow{M}_{1,2}(\mathfrak{d}) + \#\overrightarrow{M}_{1,3}(\mathfrak{d}) - \#\overrightarrow{M}_{2,3}(\mathfrak{d}) + \#\overrightarrow{M}_{3,3}(\mathfrak{d}) - \#\overrightarrow{M}_4^b(\mathfrak{d}) = R_0(u^{R_f(\mathfrak{d})}) + 1$.

Remark 33. The identities in Lemma 32 generalise identities proved in Proposition 6.11 and Proposition 6.12 in [5] for $\#BB$, $\#BR$, $\#RR$, $\#RB$, $\#B$ and $\#R$.

The proof of Lemma 32 is given in terms of blue-red path tangles and generalises the proofs of Proposition 6.11 and Proposition 6.12 in [5].

Proof. As to the first identity, observe that the set of odd vertices in G^N is decomposed of $N+1$ sets such that in each such set the odd vertices are aligned with slope -1. These sets are in the following denoted the \-diagonals of G^N. For instance, $\mathcal{R}^N$ is a \-diagonal of G^N. It can be seen from Figure 13 that if a move M_1 or M_3 is applicable to $\mathfrak{d}(\ell)$ in $\mathrm{WL}^\ell(f)$, then the odd vertex incident to $\mathfrak{d}(\ell+1)$ in $\mathrm{WL}^{\ell+1}(f)$ lies on a \-diagonal to the right of the one on which the odd vertex incident to $\mathfrak{d}(\ell)$ lies. The same is true if M_4 or M_5 is applicable to $\mathfrak{d}(\ell)$ and $\mathfrak{d}(\ell+1)$ is chosen to be the top drifter in the image of M_4 respectively $\mathfrak{d}(\ell)$ is the bottom drifter in the preimage of the move M_5. In all the other cases the odd vertices incident to $\mathfrak{d}(\ell)$ and $\mathfrak{d}(\ell+1)$ lie on the same \-diagonal of G^N. The first identity now follows since the odd vertex incident to $\mathfrak{d}(R_f(\mathfrak{d}))$ in $\mathrm{Right}_f(\mathfrak{d})$ (*resp.* $\mathfrak{d}(L_f(\mathfrak{d}))$ in $\mathrm{Left}_f(\mathfrak{d})$) lies on the $\mathrm{HeightR}_f(\mathfrak{d})$-th (*resp.* $N+1$-st) \-diagonal of G^N when counted from the left.

The second identity can be shown using analogous arguments as for the first identity by decomposing the odd vertices of G^N into /-diagonals, that is, sets in which the odd vertices are aligned with slope 1.

For the third identity, observe that in the blue-red path tangle corresponding to $\mathrm{WR}^{R_f(\mathfrak{d})}(\overrightarrow{f})$ the drifter $\mathfrak{d}(R_f(\mathfrak{d}))$ corresponds to a blue $(-1,-1)$-step, while $\mathfrak{d}(L_f(\mathfrak{d}))$

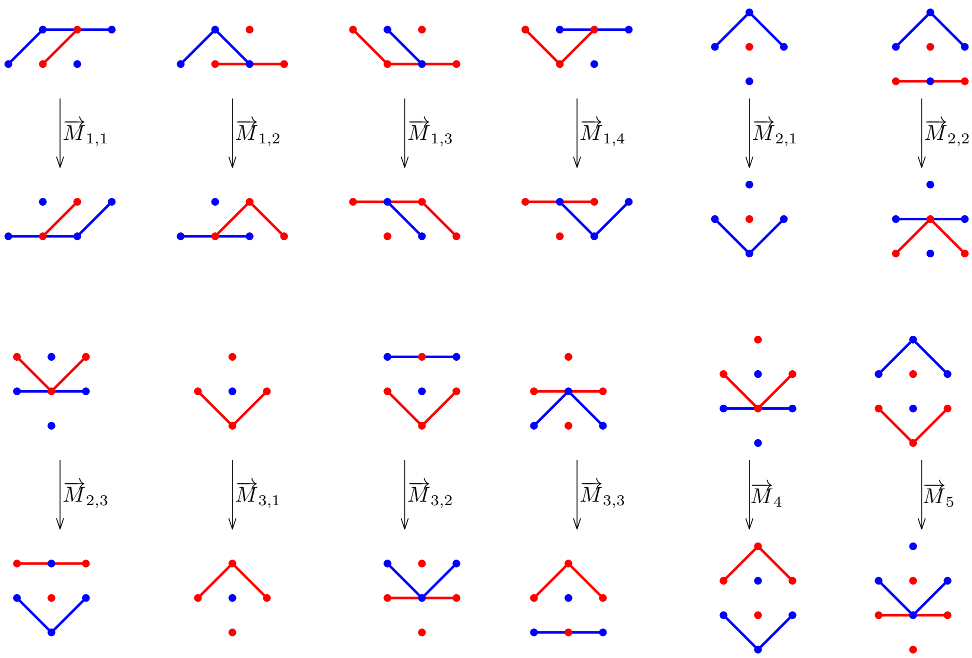


Figure 21: The moves of Figure 20 in terms of blue-red path tangles.

is a red $(1,-1)$-step in the blue-red path tangle corresponding to $\mathrm{WL}^{L_f(\mathfrak{d})}(\overrightarrow{f})$. Furthermore, in the blue-red path tangle corresponding to $\mathrm{WL}^{L_f(\mathfrak{d})}$ there are $L_1(v^{L_f(\mathfrak{d})})$ many red paths that intersect the right boundary above the red path on which the edge corresponding to $\mathfrak{d}(L_f(\mathfrak{d}))$ lies. Thus, when running through $r \in \{-R_f(\mathfrak{d}),\dots,-1,0\}$ and $\ell \in \{1,\dots,L_f(\mathfrak{d})\}$ in increasing order, $\mathfrak{d}$ must overcome $L_1(v^{L_f(\mathfrak{d})})$ red paths. On the other hand, a red path is overcome precisely by the moves $\overrightarrow{M}_{1,1}$, $\overrightarrow{M}_{1,4}$, $\overrightarrow{M}_{2,3}$ or $\overrightarrow{M}_4^b$ as it can be seen in Figure 21. That is because by the moves $\overrightarrow{M}_{2,3}$ or $\overrightarrow{M}_4^b$ the drifter is transformed from a red into a blue down step that lies in the area below the red path. On the other hand, the move $\overrightarrow{M}_{3,3}$ is the only move by which the drifter is transformed from a blue down step is into a red down step that lies in the area above the blue path. For that reason, $\#\overrightarrow{M}_{3,3}(\mathfrak{d})$ must be subtracted. Finally, by no move is a blue down step moved from the area below a red path into the area above the very same red path.

The last identity follows by analogous arguments as the third. Instead of the red paths blue paths need to be considered. $\square$

Proof of Proposition 28. By subtracting (4) from (1) in Lemma 32 one obtains

$$R_1(u^{R_f(\mathfrak{d})}) = \#\overrightarrow{M}_{1,1}(\mathfrak{d}) + \#\overrightarrow{M}_{1,4}(\mathfrak{d}) + \#\overrightarrow{M}_{2,3}(\mathfrak{d}) + \#\overrightarrow{M}_{3,1}(\mathfrak{d}) + \#\overrightarrow{M}_{3,2}(\mathfrak{d}) + \#\overrightarrow{M}_4^b(\mathfrak{d}) + \#\overrightarrow{M}_4^t(\mathfrak{d}) + \#\overrightarrow{M}_5^b(\mathfrak{d}).$$

On the other hand, by subtracting (3) from (2) in Lemma 32 one obtains

$$\begin{aligned} R_1(v^{L_f(\mathfrak{d})})+1 = {} & \#\overrightarrow{M}_{1,2}(\mathfrak{d}) + \#\overrightarrow{M}_{1,3}(\mathfrak{d}) + \#\overrightarrow{M}_{2,1}(\mathfrak{d}) + \#\overrightarrow{M}_{2,2}(\mathfrak{d}) + \#\overrightarrow{M}_{3,3}(\mathfrak{d}) \\ & + \#\overrightarrow{M}^t_5(\mathfrak{d}). \end{aligned}$$

Summing these two identities gives the assertion. □

5 Proof of Theorem 2

5.1 The map Φ

To begin with, recall that by Proposition 28 precisely one of the inequalities

$$\boxed{L_f(\mathfrak{d}) \leqslant R_1(v^{L_f(\mathfrak{d})}) \text{ and } R_f(\mathfrak{d}) \leqslant R_1(u^{R_f(\mathfrak{d})})}$$

is satisfied for each drifter $\mathfrak{d}$ in an instable TFPL f of excess 2. Which of the two inequalities $\mathfrak{d}$ satisfies is decisive for the direction in which $\mathfrak{d}$ is moved: let f be a TFPL with boundary $(u,v;w)$ that satisfies $\mathrm{exc}(u,v;w)=2$. Then a triple $(S(f),g(f),T(f))$ in $\bigcup_{u^+,\ v^+:\ u^+\geqslant u,\ v^+\geqslant v} G_{u,u^+} \times S^w_{u^+,v^+} \times G_{v,v^+}$ is assigned to f as follows:

1. If f is stable, then set $g(f)=f$, $S(f)$ the empty semi-standard Young tableau in $G_{u,u}$ and $T(f)$ the empty semi-standard Young tableau in $G_{v,v}$.

2. If in f for each drifter $\mathfrak{d}$ it holds $R_f(\mathfrak{d}) \leqslant R_1(u^{R_f(\mathfrak{d})})$, then set $g(f)=\mathrm{Right}(f)$, $S(f)$ the semi-standard Young tableau of skew shape $\lambda(u^+)/\lambda(u)$ corresponding to the sequence

$$u=u^0 \xrightarrow{\mathrm{h}} u^1 \xrightarrow{\mathrm{h}} \cdots \xrightarrow{\mathrm{h}} u^{R(f)} \xrightarrow{\mathrm{h}} u^{R(f)+1}=u^+$$

in which u^r denotes the left boundary of $\mathrm{WR}^r(f)$ for each $0 \leqslant r \leqslant R(f)+1$ and $T(f)$ the empty semi-standard Young tableau in $G_{v,v}$.

3. If in f for each drifter $\mathfrak{d}$ it holds $L_f(\mathfrak{d}) \leqslant R_1(v^{L_f(\mathfrak{d})})$, then set $g(f)=\mathrm{Left}(f)$, $S(f)$ is the empty semi-standard Young tableau in $G_{u,u}$ and $T(f)$ is the semi-standard Young tableau of skew shape $\lambda(v^+)/\lambda(v)$ corresponding to the sequence

$$v=v^0 \xrightarrow{\mathrm{h}} v^1 \xrightarrow{\mathrm{h}} \cdots \xrightarrow{\mathrm{h}} v^{L(f)} \xrightarrow{\mathrm{h}} v^{L(f)+1}=v^+$$

in which v^ℓ denotes the right boundary of $\mathrm{WL}^\ell(f)$ for each $0 \leqslant \ell \leqslant L(f)+1$.

4. If in f there are two drifters $\mathfrak{d}_\mathfrak{r}$ and $\mathfrak{d}_\mathfrak{l}$ such that $R_f(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})})$ and $L_f(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})})$, then set $g(f)$ the stable TFPL with boundary $(u^+,v^+;w)$ for words $u^+>u$ and $v^+>v$ that is obtained from f as follows: the drifter $\mathfrak{d}_\mathfrak{l}$ is moved to the right boundary using the moves M_1, M_2, M_3 and there replaced by a horizontal edge; thereafter, the drifter $\mathfrak{d}_\mathfrak{r}$ is moved to the left boundary using the moves M_1^{-1}, M_2^{-1}, M_3^{-1} and there replaced by a horizontal edge. Finally, set $S(f)$ the semi-standard Young tableau of skew shape $\lambda(u^+)/\lambda(u)$ with entry $R_f(\mathfrak{d}_\mathfrak{r})+1$ and $T(f)$ the semi-standard Young tableau of skew shape $\lambda(v^+)/\lambda(v)$ with entry $L_f(\mathfrak{d}_\mathfrak{l})+1$.

In Figure 22, the TFPL of excess 2 displayed in Figure 12 and the triple (S, g, T) associated with it are depicted.

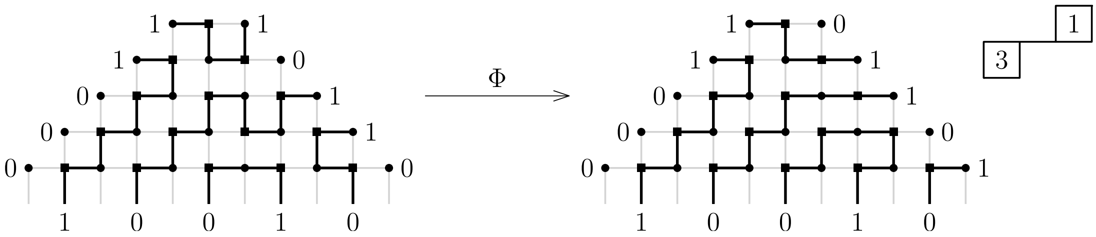


Figure 22: An instable TFPL with boundary $(00011, 01101; 10010)$ and its image under Φ.

Theorem 34. *Let u, v and w be words of length N such that $|u|_1 = |v|_0 = |w|_1$ and $\operatorname{exc}(u, v; w) = 2$. Then the map*

$$\begin{aligned} \Phi : T_{u,v}^{w} &\longrightarrow \bigcup_{u^+,v^+:\ u^+\geqslant u,\, v^+\geqslant v} G_{u,u^+} \times S_{u^+,v^+}^{w} \times G_{v,v^+} \\ f &\longmapsto (S(f), g(f), T(f)) \end{aligned}$$

is a bijection.

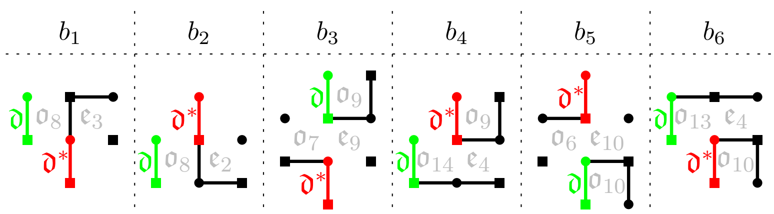
Remark 35. Theorem 2 immediately follows from Theorem 34.

Proposition 36. *Let f be an instable TFPL of excess 2 that contains two drifters $\mathfrak{d}_{\mathfrak{r}}$ and $\mathfrak{d}_{\mathfrak{l}}$ that satisfy $R_f(\mathfrak{d}_{\mathfrak{r}}) \leqslant R_1(u^{R_f(\mathfrak{d}_{\mathfrak{r}})})$ and $L_f(\mathfrak{d}_{\mathfrak{l}}) \leqslant R_1(v^{L_f(\mathfrak{d}_{\mathfrak{l}})})$. Then $\mathfrak{d}_{\mathfrak{l}}$ can be moved to the right boundary by the moves M_1, M_2 and M_3 and $\mathfrak{d}_{\mathfrak{r}}$ can thereafter be moved to the left boundary by the moves M_1^{-1}, M_2^{-1} and M_3^{-1}.*

Note that the drifters $\mathfrak{d}$ and $\mathfrak{d}^*$ in a TFPL of excess 2 that exhibits the preimage of the move M_5 (*resp.* M_4^{-1}) must satisfy $L_f(\mathfrak{d}) = L_f(\mathfrak{d}^*)$ (*resp.* $R_f(\mathfrak{d}) = R_f(\mathfrak{d}^*)$). Therefore, such a TFPL does not meet the preconditions of Proposition 36 by Corollary 29.

The proof of Proposition 36 now is based on the following two lemmas.

Lemma 37. *Let f be an instable TFPL of excess 2 that contains two drifters $\mathfrak{d}$ and $\mathfrak{d}^*$, whereof $\mathfrak{d}$ is not incident to a vertex in $\mathcal{R}^N$. If none of the moves M_1, M_2 or M_3 can be applied to $\mathfrak{d}$, then f exhibits one of the following blockades:*



Proof. Since $\mathfrak{d}$ is not incident to a vertex in $\mathcal{R}^N$ it is contained in an odd cell o of f. Now, $o \in \{\mathfrak{o}_8, \mathfrak{o}_9, \mathfrak{o}_{10}, \mathfrak{o}_{13}, \mathfrak{o}_{14}\}$ by Proposition 10. Here, only the case when $o = \mathfrak{o}_8$ is considered. In that case, the even cell e to the right of o satisfies $e \in \{\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3\}$. The case $e = \mathfrak{e}_1$ is impossible since by assumption the move M_1 cannot be applied to $\mathfrak{d}$. Thus, $e = \mathfrak{e}_2$ or $e = \mathfrak{e}_3$ yielding the blockades b_2 and b_1 respectively. □

Lemma 38. *Let f be a TFPL of excess 2 with two drifters $\mathfrak{d}$ and $\mathfrak{d}^*$ of which $\mathfrak{d}$ is not incident to a vertex in $\mathcal{R}^N$. If $\mathfrak{d}$ cannot be moved by M_1, M_2 or M_3 until it is incident to a vertex in $\mathcal{R}^N$ then it must hold that*

$$L_f(\mathfrak{d}) - L_f(\mathfrak{d}^*) > R_1(v^{L_f(\mathfrak{d})}) - R_1(v^{L_f(\mathfrak{d}^*)}).$$

If already in f no move in $\{M_1, M_2, M_3\}$ can be applied to $\mathfrak{d}$ then it holds that $L_f(\mathfrak{d}) - L_f(\mathfrak{d}^*) = R_1(v^{L_f(\mathfrak{d})}) - R_1(v^{L_f(\mathfrak{d}^*)}) + 1$.

A crucial idea in the proof of Lemma 38 is to consider TFPLs of excess 2 together with their canonical orientation and then represent them in terms of blue-red path tangles. When doing so the blockades in Lemma 37 translate into the blockades depicted in Figure 23 and Figure 24 respectively.

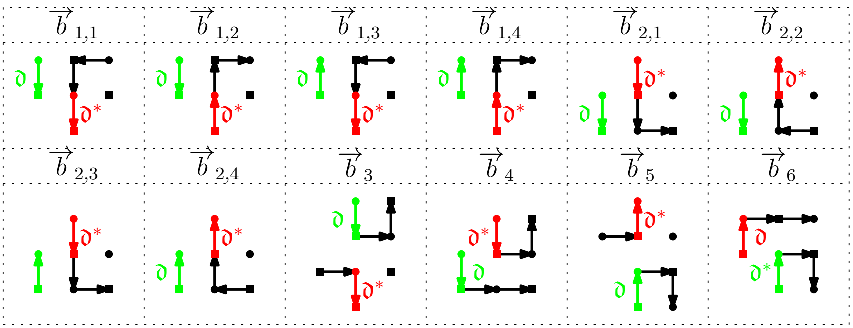


Figure 23: The blockades of Lemma 37 when equipping TFPLs of excess 2 with the canonical orientation.

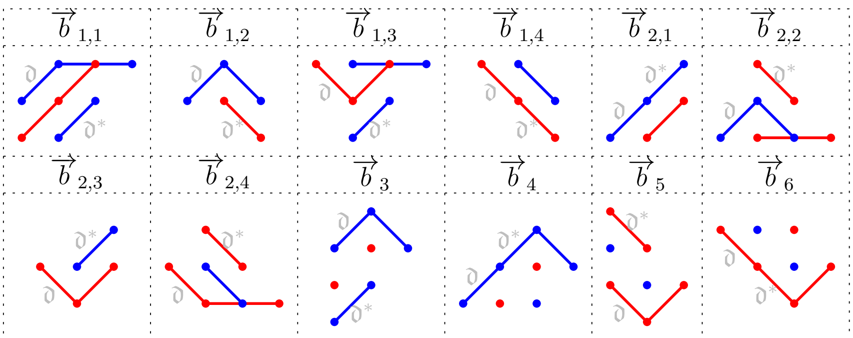


Figure 24: The blockades in Figure 23 in terms of blue-red path tangles.

Proof. To begin with, observe that if $\mathfrak{d}$ cannot be moved by M_1, M_2 or M_3 until it is incident to a vertex in $\mathcal{R}^N$ a non-negative integer k exists such that $\mathfrak{d}$ is blocked by $\mathfrak{d}^*$ by one of the barricades in Lemma 37 after it has been moved k many times by M_1, M_2 or M_3. In the following, denote by f' the TFPL in which $\mathfrak{d}$ is barred by $\mathfrak{d}^*$. Additionally, let $\overrightarrow{f}$ be the canonical orientation of f and

$$\#\overrightarrow{ML}_{i,j}(\mathfrak{d}) = |\{k \leqslant \ell < L_f(\mathfrak{d}) : \overrightarrow{M}_{i,j} \textit{ is applied to } \mathfrak{d}(\ell) \textit{ in } \mathrm{WL}^{\ell}(\overrightarrow{f}) \textit{ by } \mathrm{WL}\}|.$$

and

$$\#\overrightarrow{ML}_{i,j}(\mathfrak{d}^*) = |\{0 \leqslant \ell < L_f(\mathfrak{d}^*) : \overrightarrow{M}_{i,j} \textit{ is applied to } \mathfrak{d}^*(\ell) \textit{ in } \mathrm{WL}^{\ell}(\overrightarrow{f}) \textit{ by } \mathrm{WL}\}|.$$

for $(i,j) \in \{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (2,1),\ (2,2),\ (2,3),\ (3,1),\ (3,2),\ (3,3)\}$. The numbers $\#\overrightarrow{ML}^b_4(\mathfrak{d})$, $\#\overrightarrow{ML}^t_4(\mathfrak{d})$, $\#\overrightarrow{ML}^b_5(\mathfrak{d})$, $\#\overrightarrow{ML}^t_5(\mathfrak{d})$, $\#\overrightarrow{ML}^b_4(\mathfrak{d}^*)$, $\#\overrightarrow{ML}^t_4(\mathfrak{d}^*)$, $\#\overrightarrow{ML}^b_5(\mathfrak{d}^*)$ and $\#\overrightarrow{ML}^t_5(\mathfrak{d}^*)$ are defined analogously. Finally, fix the following notation:

$$\begin{aligned}
\mathrm{D}_{NW}(\mathfrak{d},\mathfrak{d}^*) &= \sum_{i=1}^{4} \#\overrightarrow{ML}_{1,i}(\mathfrak{d}) + \sum_{j=1}^{3} \#\overrightarrow{ML}_{3,j}(\mathfrak{d}) + \#\overrightarrow{ML}^t_4(\mathfrak{d}) + \#\overrightarrow{ML}^b_5(\mathfrak{d}) \\
&\quad - \sum_{i=1}^{4} \#\overrightarrow{ML}_{1,i}(\mathfrak{d}^*) - \sum_{j=1}^{3} \#\overrightarrow{ML}_{3,j}(\mathfrak{d}^*) - \#\overrightarrow{ML}^t_4(\mathfrak{d}^*) - \#\overrightarrow{ML}^b_5(\mathfrak{d}^*), \\
\mathrm{D}_{NE}(\mathfrak{d},\mathfrak{d}^*) &= N - \mathrm{HeightL}(f;\mathfrak{d}) - \sum_{i=1}^{4} \#\overrightarrow{ML}_{1,i}(\mathfrak{d}) - \sum_{j=1}^{3} \#\overrightarrow{ML}_{2,j}(\mathfrak{d}) - \#\overrightarrow{ML}^b_4(\mathfrak{d}) \\
&\quad - \#\overrightarrow{ML}^t_5(\mathfrak{d}) \\
&\quad - N + \mathrm{HeightL}(f;\mathfrak{d}^*) + \sum_{i=1}^{4} \#\overrightarrow{ML}_{1,i}(\mathfrak{d}^*) + \sum_{j=1}^{3} \#\overrightarrow{ML}_{2,j}(\mathfrak{d}^*) + \#\overrightarrow{ML}^b_4(\mathfrak{d}^*) \\
&\quad + \#\overrightarrow{ML}^t_5(\mathfrak{d}^*), \\
\mathrm{Red}(\mathfrak{d},\mathfrak{d}^*) &= R_0(v^{L_f(\mathfrak{d})}) - \#\overrightarrow{ML}_{1,1}(\mathfrak{d}) - \#\overrightarrow{ML}_{1,4}(\mathfrak{d}) - \#\overrightarrow{ML}_{2,3}(\mathfrak{d}) + \#\overrightarrow{ML}_{3,3}(\mathfrak{d}) \\
&\quad - \#\overrightarrow{ML}^b_4(\mathfrak{d}) \\
&\quad - R_0(v^{L_f(\mathfrak{d}^*)}) + \#\overrightarrow{ML}_{1,1}(\mathfrak{d}^*) + \#\overrightarrow{ML}_{1,4}(\mathfrak{d}^*) + \#\overrightarrow{ML}_{2,3}(\mathfrak{d}^*) - \#\overrightarrow{ML}_{3,3}(\mathfrak{d}^*) \\
&\quad + \#\overrightarrow{ML}^b_4(\mathfrak{d}^*), \\
\mathrm{Blue}(\mathfrak{d},\mathfrak{d}^*) &= \#\overrightarrow{ML}_{1,2}(\mathfrak{d}) + \#\overrightarrow{ML}_{1,3}(\mathfrak{d}) - \#\overrightarrow{ML}_{2,3}(\mathfrak{d}) + \#\overrightarrow{ML}_{3,3}(\mathfrak{d}) - \#\overrightarrow{ML}^b_4(\mathfrak{d}) \\
&\quad - \#\overrightarrow{ML}_{1,2}(\mathfrak{d}^*) - \#\overrightarrow{ML}_{1,3}(\mathfrak{d}^*) + \#\overrightarrow{ML}_{2,3}(\mathfrak{d}^*) - \#\overrightarrow{ML}_{3,3}(\mathfrak{d}^*) + \#\overrightarrow{ML}^b_4(\mathfrak{d}^*).
\end{aligned}$$

An easy computation shows that

$$\begin{aligned}
&\mathrm{D}_{NW}(\mathfrak{d},\mathfrak{d}^*) - \mathrm{D}_{NE}(\mathfrak{d},\mathfrak{d}^*) + \mathrm{Red}(\mathfrak{d},\mathfrak{d}^*) - \mathrm{Blue}(\mathfrak{d},\mathfrak{d}^*) \\
&\qquad = L_f(\mathfrak{d}) - k - L_f(\mathfrak{d}^*) - R_1(v^{L_f(\mathfrak{d})}) + R_1(v^{L_f(\mathfrak{d}^*)}).
\end{aligned} \tag{7}$$

It is in the following shown that the left-hand side in (7) equals 1. For this purpose, decompose the odd vertices of G^N into \-diagonals as in the proof of the first identity in Lemma 32. A \-diagonal d_r is said to lie to the right of a \-diagonal d_l if the odd vertex with the smallest y-coordinate in d_r has a larger x-coordinate than the odd vertex with the smallest y-coordinate in d_l. Now, number the \-diagonals of G_N from left to right. If $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) is incident to a vertex of the $i(\mathfrak{d})$-th (*resp.* $i(\mathfrak{d}^*)$-th) \-diagonal in $\mathrm{WL}^k(f)$ (*resp.* f) then it holds that $\mathrm{D}_{NW}(\mathfrak{d},\mathfrak{d}^*) = i(\mathfrak{d}^*) - i(\mathfrak{d})$. This follows from analogous arguments as Lemma 32(1).

Next, decompose the odd vertices of G^N into /-diagonals as in the proof of the second identity in Lemma 32. Similar as for \-diagonals, a /-diagonal d_r is said to lie to the right of a /-diagonal d_l if the odd vertex with the smallest y-coordinate in d_r has a larger x-coordinate than the odd vertex with the smallest y-coordinate in d_l. Now, number the /-diagonals of G^N from left to right. If $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) is part of the $j(\mathfrak{d})$-th (*resp.* $j(\mathfrak{d}^*)$-th) /-diagonal in $\mathrm{WL}^k(f)$ (*resp.* f) then it holds that $\mathrm{D}_{NE}(\mathfrak{d},\mathfrak{d}^*) = j(\mathfrak{d}) - j(\mathfrak{d}^*)$. This follows from analogous arguments as Lemma 32(2).

From now on, let (B_k, R_k) (*resp.* (B, R)) be the blue-red path tangle associated with $\mathrm{WL}^k(\overrightarrow{f})$ (*resp.* $\overrightarrow{f}$) and $R_k = (P'_{k,1}, \ldots, P'_{k,|w|_1})$ (*resp.* $R = (P'_1, \ldots, P'_{|w|_1})$). Now, if the edge corresponding to $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) is red in (B_k, R_k) (*resp.* (B, R)) then let $r(\mathfrak{d})+1$ (*resp.* $r(\mathfrak{d}^*)+1$) be the index of the red path the edge corresponding to $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) is part of. (That is, the edge corresponding to $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) is an edge of $P'_{k,r(\mathfrak{d})+1}$ (*resp.* $P'_{r(\mathfrak{d}^*)+1}$).) On the other hand, if the edge corresponding to $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) is blue then it lies to the left of $P'_{k,1}$ (*resp.* P'_1), between $P'_{k,r}$ and $P'_{k,r+1}$ (*resp.* P'_{r^*} and P'_{r^*+1}) or to the right of $P'_{k,|w|_1}$ (*resp.* $P'_{|w|_1}$) in (B_k, R_k) (*resp.* (B, R)). In the first case set $r(\mathfrak{d}) = 0$ (*resp.* $r(\mathfrak{d}^*) = 0$), in the second case set $r(\mathfrak{d}) = r$ (*resp.* $r(\mathfrak{d}^*) = r^*$) and in the last case set $r(\mathfrak{d}) = |w|_1$ (*resp.* $r(\mathfrak{d}^*) = |w|_1$). Then it must hold that $\mathrm{Red}(\mathfrak{d},\mathfrak{d}^*) = r(\mathfrak{d}) - r(\mathfrak{d}^*)$. This follows from analogous arguments as Lemma 32(3).

Finally, let $B_k = (P_{k,1}, \ldots, P_{k,|w|_0})$ (*resp.* $B = (P_1, \ldots, P_{|w|_0})$). Now, if the edge corresponding to $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) is blue in (B_k, R_k) (*resp.* (B, R)) then let $s(\mathfrak{d})+1$ (*resp.* $s(\mathfrak{d}^*)+1$) be the index of the blue path the edge corresponding to $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) is part of. On the other hand, if the edge corresponding to $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) is blue then it lies to the left of $P_{k,1}$ (*resp.* P_1), between $P_{k,s}$ and $P_{k,s+1}$ (*resp.* P_{s^*} and P_{s^*+1}) or to the right of $P_{k,|w|_0}$ (*resp.* $P_{|w|_0}$) in (B_k, R_k) (*resp.* (B, R)). In the first case set $s(\mathfrak{d}) = 0$ (*resp.* $s(\mathfrak{d}^*) = 0$), in the second case set $s(\mathfrak{d}) = s$ (*resp.* $s(\mathfrak{d}^*) = s^*$) and in the last case set $s(\mathfrak{d}) = |w|_0$ (*resp.* $s(\mathfrak{d}^*) = |w|_0$). Then it must hold that $\mathrm{Blue}(\mathfrak{d},\mathfrak{d}^*) = |w|_0 - s(\mathfrak{d}) - |w|_0 + s(\mathfrak{d}^*) = s(\mathfrak{d}^*) - s(\mathfrak{d})$. This follows from analogous arguments as Lemma 32(4).

Now, $\mathrm{D}_{NW}(\mathfrak{d},\mathfrak{d}^*)$, $\mathrm{D}_{NE}(\mathfrak{d},\mathfrak{d}^*)$, $\mathrm{Red}(\mathfrak{d},\mathfrak{d}^*)$ and $\mathrm{Blue}(\mathfrak{d},\mathfrak{d}^*)$ can be computed separately for each blockade in f' by looking at Figure 23 and Figure 24, see Table 4.

In summary, it holds that

$$\mathrm{D}_{NW}(\mathfrak{d},\mathfrak{d}^*) - \mathrm{D}_{NE}(\mathfrak{d},\mathfrak{d}^*) + \mathrm{Red}(\mathfrak{d},\mathfrak{d}^*) - \mathrm{Blue}(\mathfrak{d},\mathfrak{d}^*) = 1.$$

Therefore, $L_f(\mathfrak{d}) - L_f(\mathfrak{d}^*) - R_1(v^{L_f(\mathfrak{d})}) + R_1(v^{L_f(\mathfrak{d}^*)}) = k + 1 > 0$. □

Proposition 36 now immediately follows from Lemma 38.

	$\overrightarrow{b_{1,1}}$	$\overrightarrow{b_{1,2}}$	$\overrightarrow{b_{1,3}}$	$\overrightarrow{b_{1,4}}$	$\overrightarrow{b_{2,1}}$	$\overrightarrow{b_{2,2}}$	$\overrightarrow{b_{2,3}}$	$\overrightarrow{b_{2,4}}$	$\overrightarrow{b_3}$	$\overrightarrow{b_4}$	$\overrightarrow{b_5}$	$\overrightarrow{b_6}$
$\mathrm{D}_{NW}(\mathfrak{d},\mathfrak{d}^*)$	0	0	0	0	1	1	1	1	-1	1	1	0
$\mathrm{D}_{NE}(\mathfrak{d},\mathfrak{d}^*)$	-1	-1	-1	-1	0	0	0	0	-1	0	1	-1
$\mathrm{Red}(\mathfrak{d},\mathfrak{d}^*)$	-1	0	-1	0	0	1	0	1	0	0	1	0
$\mathrm{Blue}(\mathfrak{d},\mathfrak{d}^*)$	-1	0	-1	0	0	1	0	1	-1	0	0	0

Table 4: The numbers $\mathrm{D}_{NW}(\mathfrak{d},\mathfrak{d}^*)$, $\mathrm{D}_{NE}(\mathfrak{d},\mathfrak{d}^*)$, $\mathrm{Red}(\mathfrak{d},\mathfrak{d}^*)$ and $\mathrm{Blue}(\mathfrak{d},\mathfrak{d}^*)$ computed separately for each type of blockade.

Proof of Proposition 36. Let f be an instable TFPL of excess 2 that contains two drifters $\mathfrak{d}_\mathfrak{l}$ and $\mathfrak{d}_\mathfrak{r}$ that satisfy $R_f(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})})$ and $L_f(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})})$. Suppose that $\mathfrak{d}_\mathfrak{l}$ cannot be moved to the right boundary using the moves M_1, M_2 and M_3. Then, by Lemma 38

$$L_f(\mathfrak{d}_\mathfrak{r}) < L_f(\mathfrak{d}_\mathfrak{l}) - R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})}) + R_1(v^{L_f(\mathfrak{d}_\mathfrak{r})}).$$

Thus, $L_f(\mathfrak{d}_\mathfrak{r}) < R_1(v^{L_f(\mathfrak{d}_\mathfrak{r})})$ and equivalently $R_f(\mathfrak{d}_\mathfrak{r}) > R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})}) + 1$, which is a contradiction. Therefore, $\mathfrak{d}_\mathfrak{l}$ can be moved to the right boundary using the moves M_1, M_2 and M_3. Since there are only two drifters in f the drifter $\mathfrak{d}_\mathfrak{r}$ can be moved to the left boundary after $\mathfrak{d}_\mathfrak{l}$ has been deleted. $\square$

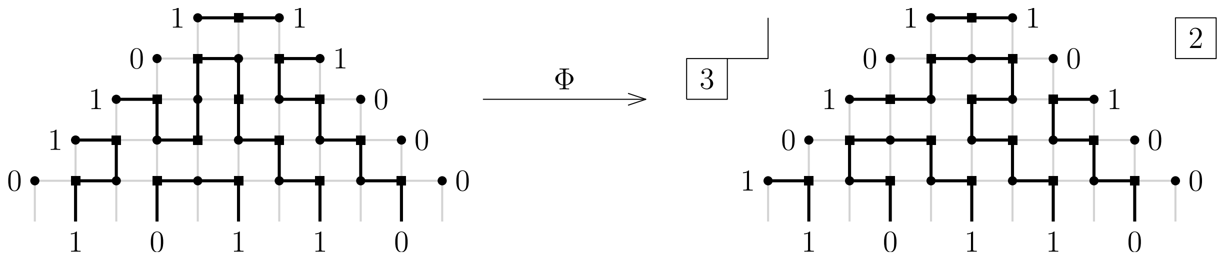


Figure 25: The instable TFPL of Figure 4 and its image under Φ.

Proposition 39. *Let f be an instable TFPL of excess 2 that contains two drifters $\mathfrak{d}_\mathfrak{r}$ and $\mathfrak{d}_\mathfrak{l}$ that satisfy $R_f(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})})$ and $L_f(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})})$. Then for $\ell = 1, \dots, L_f(\mathfrak{d}_\mathfrak{l})$ the move that is applied to $\mathfrak{d}_\mathfrak{l}(\ell-1)$ in $\mathrm{WL}^{\ell-1}(f)$ by WL coincides with the ℓ-th move in $\{M_1, M_2, M_3\}$ that is applied to $\mathfrak{d}_\mathfrak{l}$ starting from f. In particular, starting from f the drifter $\mathfrak{d}_\mathfrak{l}$ is moved $L_f(\mathfrak{d}_\mathfrak{l})$ many times by the moves M_1, M_2 and M_3 until it is incident to the vertex $R_{\mathrm{HeightL}(f;\mathfrak{d}_\mathfrak{l})+1}$ in $\mathcal{R}^N$.*

Proof. Let f be an instable TFPL of excess 2 that contains two drifters $\mathfrak{d}_\mathfrak{r}$ and $\mathfrak{d}_\mathfrak{l}$ that satisfy $R_f(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})})$ and $L_f(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})})$. Then the move M_4 is not performed in the course of the application of WL to $\mathrm{WL}^{\ell-1}(f)$ for any $1 \leqslant \ell \leqslant L(f)$. That is because if it was then $L_f(\mathfrak{d}_\mathfrak{r}) = L_f(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})}) = R_1(v^{L_f(\mathfrak{d}_\mathfrak{r})})$, which is a contradiction.

Now, assume that for an $1 \leqslant \ell_0 \leqslant L_f(\mathfrak{d}_\mathfrak{l})$ the move that is applied to $\mathfrak{d}_\mathfrak{l}(\ell_0 - 1)$ in $\mathrm{WL}^{\ell_0 - 1}(f)$ by WL does not coincide with the ℓ_0-th move that is applied to $\mathfrak{d}_\mathfrak{l}$ in the course of the generation of $g(f)$. In the following, choose ℓ_0 minimal and let o be the odd cell of G^N that $\mathfrak{d}_\mathfrak{l}(\ell_0 - 1)$ is an edge of in $\mathrm{WL}^{\ell_0 - 1}(f)$.

Since ℓ_0 is chosen minimal $\mathfrak{d}_\mathfrak{l}$ is part of o after it has been moved $\ell_0 - 1$ many times starting from f. What is more, at that point M_1, M_2 or M_3 can be applied to $\mathfrak{d}_\mathfrak{l}$ by Proposition 36. This is why there must exist an $0 \leqslant \ell' < \ell_0 - 1$ such that in $\mathrm{WL}^{\ell'}(f)$ the drifter $\mathfrak{d}_\mathfrak{r}(\ell')$ is part of one of the following two configurations:

Now, if beginning with $\mathrm{WL}^{\ell'+1}(f)$ the drifter $\mathfrak{d}_\mathfrak{l}(\ell' + 1)$ is moved $\ell_0 - \ell'$ many times by M_1, M_2 and M_3 then it must be part of o, where it is barred by $\mathfrak{d}_\mathfrak{r}(\ell' + 1)$ by way of the blockade b_1 or the blockade b_2 in Lemma 37. Thus, by Lemma 38 it must hold that $L_f(\mathfrak{d}_\mathfrak{l}) - L_f(\mathfrak{d}_\mathfrak{r}) > R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})}) - R_1(v^{L_f(\mathfrak{d}_\mathfrak{r})})$. Since $L_f(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})})$ it therefore must hold that $L_f(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(v^{L_f(\mathfrak{d}_\mathfrak{r})})$, which is a contradiction. Therefore, for $\ell = 1, \ldots, L_f(\mathfrak{d}_\mathfrak{l})$ the move that is applied to $\mathfrak{d}_\mathfrak{l}(\ell - 1)$ in $\mathrm{WL}^{\ell-1}(f)$ by WL coincides with the ℓ-th move in $\{M_1, M_2, M_3\}$ that is applied to $\mathfrak{d}_\mathfrak{l}$ starting from f. $\square$

Proposition 40. *Let f be an instable TFPL of excess 2 that contains two drifters $\mathfrak{d}_\mathfrak{r}$ and $\mathfrak{d}_\mathfrak{l}$ that satisfy $R_f(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})})$ and $L_f(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})})$. In addition, let f' be the TFPL obtained from f by moving $\mathfrak{d}_\mathfrak{l}$ by M_1, M_2 and M_3 until it is incident to $R_{\mathrm{HeightL}(f;\mathfrak{d}_\mathfrak{l})+1}$ and thereafter replacing $\mathfrak{d}_\mathfrak{l}$ by a horizontal edge incident to $R_{\mathrm{HeightL}(f;\mathfrak{d}_\mathfrak{l})}$. Then for $r = 1, \ldots, R_f(\mathfrak{d}_\mathfrak{r})$ the move that is applied to $\mathfrak{d}_\mathfrak{r}(r - 1)$ in $\mathrm{WR}^{r-1}(f)$ by WR coincides with the r-th move in $\{M_1^{-1}, M_2^{-1}, M_3^{-1}\}$ that is applied to $\mathfrak{d}_\mathfrak{r}$ beginning with f'. In particular, starting from f' the drifter $\mathfrak{d}_\mathfrak{r}$ is moved $R_f(\mathfrak{d}_\mathfrak{r})$ many times by the moves M_1^{-1}, M_2^{-1} and M_3^{-1} until it is incident to the vertex $L_{\mathrm{HeightR}(f;\mathfrak{d}_\mathfrak{r})+1}$ in $\mathcal{L}^N$.*

To prove Proposition 40 the following more general version of Lemma 38 is necessary.

Lemma 41. *Let f, $\mathfrak{d}$ and $\mathfrak{d}^*$ be as in Lemma 38. If f can be transformed into a TFPL in which $\mathfrak{d}$ is barred by $\mathfrak{d}^*$ by way of one of the blockades in Lemma 37 by moving the drifter $\mathfrak{d}$ by M_1, M_2 and M_3 and the drifter $\mathfrak{d}^*$ by M_1^{-1}, M_2^{-1} and M_3^{-1} then it must hold that*

$$L_f(\mathfrak{d}) - L_f(\mathfrak{d}^*) > R_1(v^{L_f(\mathfrak{d})}) - R_1(v^{L_f(\mathfrak{d}^*)}).$$

Proof. Let k (*resp.* s) be the number of moves that are applied to $\mathfrak{d}$ (*resp.* $\mathfrak{d}^*$) in the course of transforming f into a TFPL in which $\mathfrak{d}$ is barred by $\mathfrak{d}^*$ by way of one of the blockades in Lemma 37. Then starting from f for $1 \leqslant \ell \leqslant k$ the ℓ-th move in $\{M_1, M_2, M_3\}$ that is applied to $\mathfrak{d}$ coincides with the move that is applied to $\mathfrak{d}(\ell - 1)$ in $\mathrm{WL}^{\ell-1}(f)$ by WL. The analogous is true for $\mathfrak{d}^*$.

In the following, the definitions of the proof of Lemma 38 are retained except the definitions of $\#\overrightarrow{ML}_{i,j}(\mathfrak{d}^*)$, $\#\overrightarrow{ML}{}^b_4(\mathfrak{d}^*)$, $\#\overrightarrow{ML}{}^t_4(\mathfrak{d}^*)$, $\#\overrightarrow{ML}{}^b_5(\mathfrak{d}^*)$ and $\#\overrightarrow{ML}{}^t_5(\mathfrak{d}^*)$. They are replaced by the following more general definitions. To begin with, set

$$\begin{aligned}\#\overrightarrow{ML}_{i,j}(\mathfrak{d}^*) = &|\{0 \leqslant \ell < L_f(\mathfrak{d}^*) : \overrightarrow{M}_{i,j} \textit{ is applied to } \mathfrak{d}^* \textit{ in } \mathrm{WL}^\ell(\overrightarrow{f}) \textit{ by } \mathrm{WL}\}| \\ &+ |\{0 \leqslant r < s : \overrightarrow{M}{}^{-1}_{i,j} \textit{ is applied to } \mathfrak{d}^*(r) \textit{ in } \mathrm{WR}^r(\overrightarrow{f}) \textit{ by } \mathrm{WR}\}|\end{aligned}$$

for $(i,j) \in \{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (2,1),\ (2,2),\ (2,3),\ (3,1),\ (3,2),\ (3,3)\}$. The numbers $\#\overrightarrow{ML}{}^b_4(\mathfrak{d}^*)$, $\#\overrightarrow{ML}{}^t_4(\mathfrak{d}^*)$, $\#\overrightarrow{ML}{}^b_5(\mathfrak{d}^*)$ and $\#\overrightarrow{ML}{}^t_5(\mathfrak{d}^*)$ are defined analogously. Then $\mathfrak{d}$ and $\mathfrak{d}^*$ must satisfy the following:

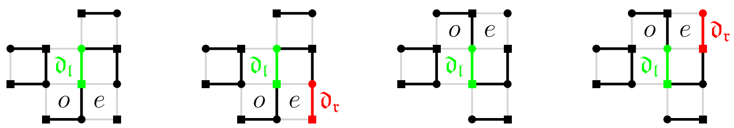
$$\begin{aligned}&\mathrm{D}_{NW}(\mathfrak{d},\mathfrak{d}^*) - \mathrm{D}_{NE}(\mathfrak{d},\mathfrak{d}^*) + \mathrm{Red}(\mathfrak{d},\mathfrak{d}^*) - \mathrm{Blue}(\mathfrak{d},\mathfrak{d}^*) \\ &\qquad = L_f(\mathfrak{d}) - k - L_f(\mathfrak{d}^*) - s - R_1(v^{L_f(\mathfrak{d})}) + R_1(v^{L_f(\mathfrak{d}^*)}). \qquad (8)\end{aligned}$$

By similar arguments as in the proof of Lemma 38 ($\mathrm{WR}^s(f)$ is considered instead of f in connection with $\mathfrak{d}^*$) it now follows that the left-hand side in (8) equals 1, which concludes the proof. □

Proof of Proposition 40. Let f be an instable TFPL of excess 2 that contains two drifters $\mathfrak{d}_\mathfrak{r}$ and $\mathfrak{d}_\mathfrak{l}$ that satisfy $R_f(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})})$ and $L_f(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(v^{L_f(\mathfrak{d}_\mathfrak{l})})$. Then the move M_5^{-1} is not performed in the course of the application of WR to $\mathrm{WR}^{r-1}(f)$ for every $1 \leqslant r \leqslant R(f)$. That is because if it was then it would hold that $R_f(\mathfrak{d}_\mathfrak{l}) = R_f(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})}) = R_1(u^{R_f(\mathfrak{d}_\mathfrak{l})})$, which is a contradiction. Therefore, starting from f both $\mathfrak{d}_\mathfrak{l}$ and $\mathfrak{d}_\mathfrak{r}$ are solely moved by M_1^{-1}, M_2^{-1} and M_3^{-1} by right-Wieland drift.

Now, assume that for an $1 \leqslant r_0 \leqslant R_f(\mathfrak{d}_\mathfrak{r})$ the move that is applied to $\mathfrak{d}_\mathfrak{r}(r_0-1)$ in $\mathrm{WR}^{r_0-1}(f)$ by WR does not coincide with the r_0-th move that is applied to $\mathfrak{d}_\mathfrak{r}$ in the course of the generation of $g(f)$. In the following, choose r_0 minimal and let e be the even cell of G^N that $\mathfrak{d}_\mathfrak{r}(r_0-1)$ is an edge of in $\mathrm{WR}^{r_0-1}(f)$.

To begin with, observe that the edges of e that belong to f may differ from the edges of e that belong to f'. If they do differ then there exists a non-negative integer k such that starting from f at the point at which $\mathfrak{d}_\mathfrak{l}$ has been moved k many times by a move in $\{M_1,\ M_2,\ M_3\}$ it is part of one of the following configurations:



(9)

That is because r_0 is chosen minimal and therefore must be an edge of e after it is moved $\ell_0 - 1$ many times by M_1^{-1}, M_2^{-1} and M_3^{-1} beginning with f'. For the same reason, in the TFPL that is obtained from f by applying k moves in $\{M_1,\ M_2,\ M_3\}$ to $\mathfrak{d}_\mathfrak{l}$ and $\ell_0 - 1$ moves in $\{M_1^{-1},\ M_2^{-1},\ M_3^{-1}\}$ to $\mathfrak{d}_\mathfrak{r}$ the drifter $\mathfrak{d}_\mathfrak{r}$ is prevented by the drifter $\mathfrak{d}_\mathfrak{l}$ from being moved by M_1^{-1}, M_2^{-1} or M_3^{-1}. Thus, by vertical symmetry and by Lemma 41 it must hold that $R_f(\mathfrak{d}_\mathfrak{r}) - R_f(\mathfrak{d}_\mathfrak{l}) > R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})}) - R_1(u^{R_f(\mathfrak{d}_\mathfrak{l})})$. Since $R_f(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_f(\mathfrak{d}_\mathfrak{r})})$ this implies that $R_f(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(u^{R_f(\mathfrak{d}_\mathfrak{l})})$, which is a contradiction.

This is why the edges in e that are occupied by f and those that are occupied by f' must be the same. In that case, $\mathfrak{d}_{\mathfrak{r}}$ must satisfy $R_f(\mathfrak{d}_{\mathfrak{l}}) \leqslant R_1(u^{R_f(\mathfrak{d}_{\mathfrak{l}})})$ by analogous arguments as in the proof of Proposition 39. This is again a contradiction, which is why for $r = 1, \ldots, R_f(\mathfrak{d}_{\mathfrak{r}})$ the move that is applied to $\mathfrak{d}_{\mathfrak{r}}(r-1)$ in $\mathrm{WR}^{r-1}(f)$ by WR coincides with the r-th move in $\{M_1^{-1}, M_2^{-1}, M_3^{-1}\}$ that is applied to $\mathfrak{d}_{\mathfrak{r}}$ beginning with f'. $\square$

It finally can be shown that $\Phi(f)$ indeed is an element of $G_{u,u^+} \times S^w_{u^+,v^+} \times G_{v,v^+}$ for a $u^+ \geqslant u$ and a $v^+ \geqslant v$.

Proposition 42. *Let u, v and w be words of length N that satisfy $\mathrm{exc}(u,v;w) \leqslant 2$. Then for every instable TFPL f with boundary $(u,v;w)$ it holds that*

$$\Phi(f) \in \bigcup_{u^+,\, v^+ \colon u^+ \geqslant u,\, v^+ \geqslant v} G_{u,u^+} \times S^w_{u^+,v^+} \times G_{v,v^+}.$$

Proof. Let $f \in T^w_{u,v}$ be instable and $\Phi(f) = (S(f), g(f), T(f))$. To begin with, $S(f) \in G_{u,u^+}$ for the following reason: let c be a cell in $\lambda(u^+)/\lambda(u)$. Then the entry of c in $S(f)$ must be $R_f(\mathfrak{d}) + 1$ for a drifter $\mathfrak{d}$ in f. Now, if c is part of the i-th column in $\lambda(u^+)/\lambda(u)$ when counted from the left then $R_1(u^{R_f(\mathfrak{d})}) = |u|_1 - i$. Finally, since $\mathfrak{d}$ must satisfy $R_f(\mathfrak{d}) \leqslant R_1(u^{R_f(\mathfrak{d})})$ and $R_1(u^{R_f(\mathfrak{d})}) = |u|_1 - i$ the entry of c in S is at most $|u|_1 - i + 1$. By analogous arguments it follows that $T(f) \in G_{v,v^+}$. $\square$

To conclude the proof of Theorem 34 it remains to check that Φ indeed is a bijection. This is done by showing that Φ is invertible.

5.2 The map Ψ

Let u, u^+, v, v^+ and w be words of length N that satisfy $\mathrm{exc}(u,v;w) = 2$, $u^+ \geqslant u$ and $v^+ \geqslant v$ and let $S \in G_{u,u^+}$, $R+1$ be the largest entry of S, $g \in S^w_{u^+,v^+}$, $T \in G_{v,v^+}$ and $L+1$ the largest entry of T. Furthermore, let u^0, u^1, $\ldots$, u^R, u^{R+1} be words of length N that satisfy that $u^0 = u$, $u^{R+1} = u^+$, $|u^r|_1 = |u|_1$ and $u^{r-1} \xrightarrow{\mathrm{h}} u^r$ for every $1 \leqslant r \leqslant R+1$ and that the sequence

$$\lambda(u) = \lambda(u^0) \subseteq \lambda(u^1) \subseteq \cdots \subseteq \lambda(u^R) \subseteq \lambda(u^{R+1}) = \lambda(u^+)$$

corresponds to S. Finally, let v^0, v^1, $\ldots$, v^L, v^{L+1} be words of length N that satisfy that $v^0 = v$, $v^{L+1} = v^+$, $|v^\ell|_1 = |v|_1$ and $v^{\ell-1} \xrightarrow{\mathrm{h}} v^\ell$ for every $1 \leqslant \ell \leqslant L+1$ and that the sequence

$$\lambda(v) = \lambda(v^0) \subseteq \lambda(v^1) \subseteq \cdots \subseteq \lambda(v^L) \subseteq \lambda(v^{L+1}) = \lambda(v^+)$$

corresponds to T. The TFPL $\Psi(S,g,T)$ then is determined as follows:

1. If $u^+ > u$ and $v^+ = v$, set $\Psi(S,g,T) = (\mathrm{WL}_{u^0} \circ \mathrm{WL}_{u^1} \circ \cdots \circ \mathrm{WL}_{u^{R-1}} \circ \mathrm{WL}_{u^R})(g)$.
2. If $u^+ = u$ and $v^+ > v$, set $\Psi(S,g,T) = (\mathrm{WR}_{v^0} \circ \mathrm{WR}_{v^1} \circ \cdots \circ \mathrm{WR}_{v^{L-1}} \circ \mathrm{WR}_{v^L})(g)$.

3. If $u^+ > u$ and $v^+ > v$, set $\Psi(S,g,T)$ the TFPL that is obtained from g as follows: since $u^+ > u$, $v^+ > v$ and $\mathrm{exc}(u^+,v^+;w) \geqslant 0$ both $\lambda(u^+)/\lambda(u)$ and $\lambda(v^+)/\lambda(v)$ consist of precisely one cell. Thus, let j (*resp.* j') be the index of the column of the cell in $\lambda(u^+)/\lambda(u)$ (*resp.* $\lambda(v^+)/\lambda(v)$) and i_j (*resp.* i'_j) the index of the j-th one in u^+ (*resp.* v^+). To begin with, replace the horizontal edge incident to L_{i_j} by a drifter $\mathfrak{d}_\mathfrak{r}$ incident to L_{i_j+1} and move the latter R times by the moves M_1, M_2 and M_3. Thereafter, replace the horizontal edge incident to $R_{i'_j}$ by a drifter $\mathfrak{d}_\mathfrak{l}$ incident to $R_{i'_j+1}$ and move the latter L times by the moves M_1^{-1}, M_2^{-1} and M_3^{-1}.

4. If $u^+ = u$ and $v^+ = v$, then $\Psi(S,g,T) = g$.

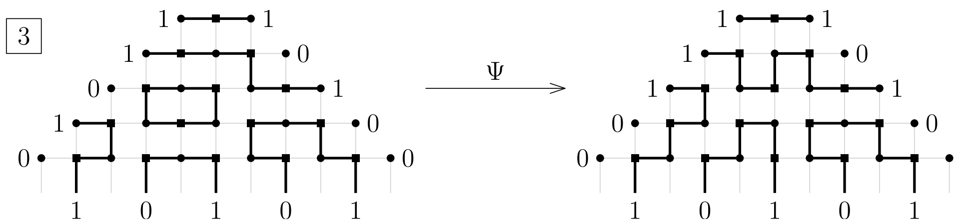


Figure 26: A stable TFPL with boundary $(01011, 00101; 10101)$, a tableau in $G_{00011,01011}$ and their image under Ψ. (Here, $u^0 = u^1 = u^2 = 00111$ and $u^3 = 01011$.)

Proposition 43. *Let u, u^+, v, v^+ and w be words of length N that satisfy $u^+ \geqslant u$, $v^+ \geqslant v$ and $\mathrm{exc}(u,v;w) = 2$. If $u^+ = u$ or $v^+ = v$ then $\Psi(S,g,T) \in T^w_{u,v}$.*

Proof. Without loss of generality, let $v^+ = v$. If $u^+ = u$ then $\Psi(S,g,T) = g \in T^w_{u,v}$. Hence, let $u^+ > u$. To begin with, observe that $\Psi(S,g,T)$ has right boundary v if and only if for every $0 < r \leqslant R$ no drifter in $(\mathrm{WL}_{u^r} \circ \mathrm{WL}_{u^{r+1}} \circ \cdots \circ \mathrm{WL}_{u^R})(g)$ is incident to a vertex in $\mathcal{R}^N$. Thus, let $0 < r \leqslant R$ and let $\mathfrak{d}_r$ be a drifter in $g_r = (\mathrm{WL}_{u^r} \circ \cdots \circ \mathrm{WL}_{u^R})(g)$. Then there exists an $r \leqslant x \leqslant R$ such that in $g_x = (\mathrm{WL}_{u^x} \circ \mathrm{WL}_{u^{x+1}} \circ \cdots \circ \mathrm{WL}_{u^R})(g)$ a drifter $\mathfrak{d}$ is incident to a vertex $L_i \in \mathcal{L}^N$ and $\mathfrak{d}(x-r) = \mathfrak{d}_r$ in g_r. In addition, it holds that $u^{x+1} > u^x$. Now, if u^x_i is the j-th one in u^x then there exists a cell c in the j-th column of $\lambda(u^+)/\lambda(u)$ that has entry $x+1$ in S. Therefore, $r < x+1 \leqslant |u|_1 - j + 1$.

On the other hand, since $\mathfrak{d}$ is incident to a vertex of $\mathcal{L}^N$ in g_x it must satisfy $R_{g_x}(\mathfrak{d}) = 0$. Therefore, it holds that $L_{g_x}(\mathfrak{d}) \geqslant R_1(u^{R_{g_x}(\mathfrak{d})}) + 1 = |u|_1 - j + 1 > r$ by Corollary 29. This is why $\mathfrak{d}_r$ is not incident to a vertex of $\mathcal{R}^N$ in g_r, which concludes the proof. $\square$

Proposition 44. *Let u, v, w, u^+ and v^+ be words of length N that satisfy $u^+ > u$, $v^+ > v$ and $\mathrm{exc}(u,v;w) = 2$. Then for any $S \in G_{u,u^+}$, $g \in S^w_{u^+,v^+}$ and $T \in G_{v,v^+}$ the TFPL $\Psi(S,g,T)$ can be generated in the way described above and is an element of $T^w_{u,v}$.*

Proof. To begin with, let j be the column of the cell in the skew diagram $\lambda(u^+)/\lambda(u)$, i_j the index of the j-th one in u^+, $R+1$ the entry of S and g' the TFPL obtained from g

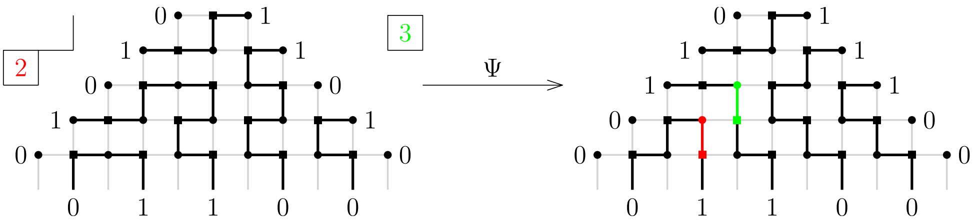


Figure 27: A semi-standard Young tableau in $G_{00110,01010}$, a stable TFPL with boundary $(01010, 01011; 01100)$, a semi-standard Young tableau in $G_{00111,01011}$ and their image under Ψ.

by deleting the horizontal edge incident to L_{i_j} and adding a drifter $\mathfrak{d}_\mathfrak{r}$ incident to L_{i_j+1}. Then $R_{g'}(\mathfrak{d}_\mathfrak{r}) = 0$ and therefore by Corollary 29

$$L_{g'}(\mathfrak{d}_\mathfrak{r}) = R_1(u^{R_{g'}(\mathfrak{d}_\mathfrak{r})}) + R_1(v^{L_{g'}(\mathfrak{d}_\mathfrak{r})}) + 1 \geqslant R_1(u^{R_{g'}(\mathfrak{d}_\mathfrak{r})}) + 1 = |u|_1 - j + 1 > R. \tag{10}$$

This is why $\mathfrak{d}_\mathfrak{r}(R)$ is part of $\mathrm{WL}^R(g')$ and is not incident to a vertex in $\mathcal{R}^N$. On the other hand, since $\mathrm{exc}(u, v^+; w) = 1$ by left-Wieland drift neither M_4 nor M_5 is applied to $\mathrm{WL}^r(g')$ for every $0 \leqslant r < R$. Thus, starting with g' the drifter $\mathfrak{d}_\mathfrak{r}$ can be moved R many times by M_1, M_2 or M_3 and the thereby obtained TFPL equals $\mathrm{WL}^R(g')$. In the following, denote $\mathrm{WL}^R(g')$ by g''.

Now, let j' be the column of the cell in the skew diagram $\lambda(v^+)/\lambda(v)$, i'_j the index of the j'-th one in v^+ and $L+1$ the entry of T. Since $\mathfrak{d}_\mathfrak{r}$ is not incident to a vertex in $\mathcal{R}^N$ in g'' the edge incident to $R_{i'_j}$ in g'' is horizontal and thus can be replaced by a drifter $\mathfrak{d}_\mathfrak{l}$ that is incident to $R_{i'_j+1}$. It is shown next that $\mathfrak{d}_\mathfrak{l}$ then can be moved L times by M_1^{-1}, M_2^{-1} and M_3^{-1}. To this end, assume the contrary, that is, assume that $\mathfrak{d}_\mathfrak{l}$ either is incident to a vertex in $\mathcal{L}^N$ or is prevented by $\mathfrak{d}_\mathfrak{r}$ from being moved after it has been moved ℓ many times by M_1^{-1}, M_2^{-1} or M_3^{-1} for an $\ell < L$.

First, suppose that $\mathfrak{d}_\mathfrak{l}$ is incident to a vertex L_i in $\mathcal{L}^N$ after it has been moved ℓ many times by M_1^{-1}, M_2^{-1} or M_3^{-1} for an $\ell < L$. Hence, suppose that i is the j^*-th one in u. Then it follows from Proposition 28 (or [5, Lemma 6.14]) that $\ell = |u|_1 - j^* + |v|_1 - j' + 1$. Since $|v|_1 - j' \geqslant L$ this implies $\ell > L$, which is a contradiction. In conclusion, $\mathfrak{d}_\mathfrak{l}$ is prevented by $\mathfrak{d}_\mathfrak{r}$ from being moved L many times by M_1^{-1}, M_2^{-1} or M_3^{-1}.

In the following, let f' be the TFPL in which $\mathfrak{d}_\mathfrak{l}$ is prevented by $\mathfrak{d}_\mathfrak{r}$ from being moved by M_1^{-1}, M_2^{-1} or M_3^{-1}. Then by vertical symmetry and Lemma 38 it holds that

$$R_{f'}(\mathfrak{d}_\mathfrak{l}) - R_{f'}(\mathfrak{d}_\mathfrak{r}) = R_1(u^{R_{f'}(\mathfrak{d}_\mathfrak{l})}) - R_1(u^{R_{f'}(\mathfrak{d}_\mathfrak{r})}) + 1. \tag{11}$$

On the other hand, starting with f' the drifter $\mathfrak{d}_\mathfrak{r}$ can be moved by M_1^{-1}, M_2^{-1} and M_3^{-1} until it is incident to a vertex in $\mathcal{L}^N$. Now, assume that the r-th such move of $\mathfrak{d}_\mathfrak{r}$ does not coincide with the move applied to $\mathfrak{d}_\mathfrak{r}(r-1)$ in $\mathrm{WR}^{r-1}(f')$ by right-Wieland drift for an $r \leqslant R_{f'}(\mathfrak{d}_\mathfrak{r})$. (This includes the cases when the r-th move of $\mathfrak{d}_\mathfrak{r}$ is M_2^{-1} or M_3^{-1}, while to $\mathfrak{d}_\mathfrak{r}(r-1)$ in $\mathrm{WR}^{r-1}(f')$ the move M_5^{-1} is applied.) Now, if r is chosen minimal then it follows by analogous arguments as in the proof of Proposition 39 that there exists an $r' < r$ such that in $\mathrm{WR}^{r'}(f')$ the drifter $\mathfrak{d}_{\mathfrak{r}}(r')$ is at some point prevented by $\mathfrak{d}_{\mathfrak{l}}(r')$ from being moved by M_1^{-1}, M_2^{-1} or M_3^{-1} until it is incident to a vertex in $\mathcal{L}^N$. Thus, by vertical symmetry and Lemma 38 it must hold that

$$R_{f'}(\mathfrak{d}_{\mathfrak{r}}) - R_{f'}(\mathfrak{d}_{\mathfrak{l}}) \geqslant R_1(u^{R_{f'}(\mathfrak{d}_{\mathfrak{r}})}) - R_1(u^{R_{f'}(\mathfrak{d}_{\mathfrak{l}})}). \tag{12}$$

This is a contradiction to (11), which is why the r-th move of $\mathfrak{d}_{\mathfrak{r}}$ coincides with the move applied to $\mathfrak{d}_{\mathfrak{r}}(r-1)$ in $\mathrm{WR}^{r-1}(f')$ by right-Wieland drift for every $r \leqslant R_{f'}(\mathfrak{d}_{\mathfrak{r}})$. In particular,

$$R_{f'}(\mathfrak{d}_{\mathfrak{r}}) = R \leqslant |u|_1 - j' = R_1(u^{R_{f'}(\mathfrak{d}_{\mathfrak{r}})}). \tag{13}$$

A similar argumentation shows that $L_{f'}(\mathfrak{d}_{\mathfrak{l}}) = \ell$. In summary, from (11) and (13) it follows that $R_{f'}(\mathfrak{d}_{\mathfrak{l}}) \leqslant R_1(u^{R_{f'}(\mathfrak{d}_{\mathfrak{l}})}) + 1$ and thus $\ell \geqslant R_1(v^{L_{f'}(\mathfrak{d}_{\mathfrak{l}})})$ by Corollary 29, which is a contradiction to $\ell < L \leqslant R_1(v^{L_{f'}(\mathfrak{d}_{\mathfrak{l}})})$. The drifter $\mathfrak{d}_{\mathfrak{l}}$ therefore can be moved L many times by M_1^{-1}, M_2^{-1} or M_3^{-1}. ☐

5.3 Proof of Theorem 34

Let u, v and w be words of length N that satisfy $\mathrm{exc}(u, v; w) = 2$. To conclude the proof of Theorem 34 it remains to show that, on the one hand, $\Psi(\Phi(f)) = f$ for every TFPL $f \in T^w_{u,v}$ and, on the other hand, that $\Phi(\Psi(S, g, T)) = (S, g, T)$ for every triple $(S, g, T) \in \bigcup_{u^+ \geqslant u,\, v^+ \geqslant v} G_{u,u^+} \times S^w_{u^+,v^+} \times G_{v,v^+}$.

Proposition 45. *Let u, v and w be words of length N that satisfy* $\mathrm{exc}(u, v; w) = 2$. *Then $\Psi(\Phi(f)) = f$ for every TFPL $f \in T^w_{u,v}$.*

Proof. Let $f \in T^w_{u,v}$. If f is stable then $\Psi(\Phi(f)) = f$. Thus, let f be instable, $\Phi(f) = (S(f), g(f), T(f))$ and $(u^+, v^+; w)$ the boundary of $g(f)$. If every drifter $\mathfrak{d}$ in f satisfies $R_f(\mathfrak{d}) \leqslant R_1(u^{R_f(\mathfrak{d})})$ then $g(f) = \mathrm{Right}(f) = \mathrm{WR}^{R(f)+1}(f)$. Furthermore, when u^r denotes the left boundary of $\mathrm{WR}^r(f)$ for every $0 \leqslant r \leqslant R(f) + 1$ then $S(f)$ is the semi-standard Young tableau corresponding to the sequence

$$\lambda(u) = \lambda(u^0) \subseteq \lambda(u^1) \subseteq \cdots \subseteq \lambda(u^{R(f)}) \subseteq \lambda(u^{R(f)+1}) = \lambda(u^+)$$

and has largest entry $R(f) + 1$. Since $g(f)$ has boundary $(u^+, v; w)$ for an $u^+ > u$ by the definition of Ψ it holds that

$$\Psi(S(f), g(f), T(f)) = (\mathrm{WL}_{u^0} \circ \mathrm{WL}_{u^1} \circ \cdots \circ \mathrm{WL}_{u^{R(f)-1}} \circ \mathrm{WL}_{u^{R(f)}})(\mathrm{WR}^{R(f)+1}(f)) = f.$$

That latter equality is valid because $\mathrm{WL}_{u^r}(\mathrm{WR}(\mathrm{WR}^r(f))) = \mathrm{WR}^r(f)$ for every $0 \leqslant r \leqslant R(f)$ by Proposition 14. It also holds that $\Psi(S(f), g(f), T(f)) = f$ if every drifter $\mathfrak{d}$ in f satisfies $L_f(\mathfrak{d}) \leqslant R_1(v^{L_f(\mathfrak{d})})$.

Finally, if in f there are two drifters $\mathfrak{d}_{\mathfrak{r}}$ and $\mathfrak{d}_{\mathfrak{l}}$ such that $R_f(\mathfrak{d}_{\mathfrak{r}}) \leqslant R_1(u^{R_f(\mathfrak{d}_{\mathfrak{r}})})$ and $L_f(\mathfrak{d}_{\mathfrak{l}}) \leqslant R_1(v^{L_f(\mathfrak{d}_{\mathfrak{l}})})$ then $g(f)$ is the stable TFPL with boundary $(u^+, v^+; w)$ for words $u^+ > u$ and $v^+ > v$ that is obtained from f as follows: the drifter $\mathfrak{d}_{\mathfrak{l}}$ is moved by the moves M_1, M_2, M_3 until it is incident to a vertex $R_{i'+1}$ in $\mathcal{R}^N$ and thereafter is replaced by a horizontal edge incident to $R_{i'}$; then, the drifter $\mathfrak{d}_\mathfrak{r}$ is moved by the moves M_1^{-1}, M_2^{-1}, M_3^{-1} until it is incident to a vertex L_{i+1} in $\mathcal{L}^N$ and thereafter is replaced by a horizontal edge incident to L_i. Furthermore, $S(f)$ is the semi-standard Young tableau of skew shape $\lambda(u^+)/\lambda(u)$ with entry $R_f(\mathfrak{d}_\mathfrak{r})+1$ and $T(f)$ is the semi-standard Young tableau of skew shape $\lambda(v^+)/\lambda(v)$ with entry $L_f(\mathfrak{d}_\mathfrak{l})+1$.

It is shown in Proposition 39 and in Proposition 40 respectively that $\mathfrak{d}_\mathfrak{l}$ is moved $L_f(\mathfrak{d}_\mathfrak{l})$ many times by M_1, M_2 and M_3 until it is incident to $R_{i'+1}$, while $\mathfrak{d}_\mathfrak{r}$ is moved $R_f(\mathfrak{d}_\mathfrak{r})$ many times by M_1^{-1}, M_2^{-1} and M_3^{-1} until it is incident to L_{i+1}. Thus, $\Psi(S(f), g(f), T(f)) = f$. □

Proposition 46. *Let u, u^+, v, v^+ and w be words of length N that satisfy $|u^+|_1 = |u|_1$, $|v^+|_1 = |v|_1$, $u^+ \geqslant u$, $v^+ \geqslant v$ and $\operatorname{exc}(u,v;w) = 2$. Then $\Phi(\Psi(S,g,T)) = (S,g,T)$ for every triple (S,g,T) in $G_{u,u^+} \times S^w_{u^+,v^+} \times G_{v,v^+}$.*

Proof. Let $(S,g,T) \in G_{u,u^+} \times S^w_{u^+,v^+} \times G_{v,v^+}$. If $u^+ = u$ and $v^+ = v$ then g is a stable TFPL with boundary $(u,v;w)$ and both S and T are empty Young tableaux. Therefore, $\Phi(\Psi(S,g,T)) = (S,g,T)$. From now on, assume that $u^+ > u$ or $v^+ > v$.

First, it is checked that $\Phi(\Psi(S,g,T)) = (S,g,T)$ if $v^+ = v$. For this purpose, let $R+1$ be the largest entry of S and u^0, u^1, $\ldots, u^R$, u^{R+1} be the words of length N that satisfy $u^0 = u$, $u^{R+1} = u^+$, $|u^r|_1 = |u|_1$ and $u^{r-1} \xrightarrow{\mathrm{h}} u^r$ for every $1 \leqslant r \leqslant R+1$ and that the sequence

$$\lambda(u) = \lambda(u^0) \subseteq \lambda(u^1) \subseteq \cdots \subseteq \lambda(u^R) \subseteq \lambda(u^{R+1}) \tag{14}$$

corresponds to S. Then $\Psi(S,g,T) = (\mathrm{WL}_{u^0} \circ \mathrm{WL}_{u^1} \circ \cdots \circ \mathrm{WL}_{u^{R-1}} \circ \mathrm{WL}_{u^R})(g)$. Now, let $\mathfrak{d}$ be a drifter in $\Psi(S,g,T)$. Since g is stable there exists an $x \leqslant R$ such that in $(\mathrm{WL}_{u^x} \circ \cdots \circ \mathrm{WL}_{u^R})(g)$ there is a drifter $\mathfrak{d}_x$ that is incident to a vertex L_i in $\mathcal{L}^N$ and that satisfies $\mathfrak{d}_x(x) = \mathfrak{d}$ in $\Psi(S,g,T)$. In particular, $R_{\Psi(S,g,T)}(\mathfrak{d}) = x$ and $R(\Psi(S,g,T)) = R$.

On the other hand, if u_i^x is the j-th one in u^x then u_{i-1}^{x+1} is the j-th one in u^{x+1}. This is why in S there is a cell in the j-th column that has entry $x+1$. Since S is an element of G_{u,u^+} this implies that $x \leqslant |u|_1 - j = R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d})})$. In summary, $R_{\Psi(S,g,T)}(\mathfrak{d}) \leqslant R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d})})$ and

$$g(\Psi(S,g,T)) = \mathrm{Right}(\Psi(S,g,T)) = \mathrm{WR}^{R+1}\left((\mathrm{WL}_{u^0} \circ \mathrm{WL}_{u^1} \circ \cdots \circ \mathrm{WL}_{u^R})(g)\right) = g \tag{15}$$

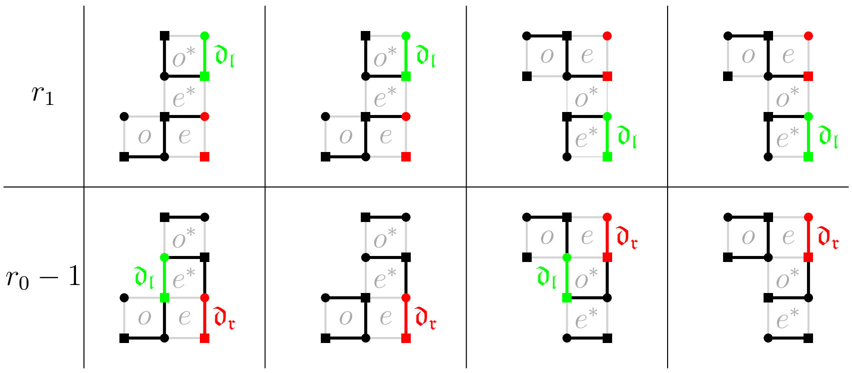
by Proposition 14. Finally, $S(\Psi(S,g,T)) = S$ because $\mathrm{WR}^r(\Psi(S,g,T)) = (\mathrm{WL}_{u^r} \circ \cdots \circ \mathrm{WL}_{u^{R-1}} \circ \mathrm{WL}_{u^R})(g)$ and therefore has left boundary u^r for every $0 \leqslant r \leqslant R+1$. (That $T((\Psi(S,g,T)) = T$ is trivial since T is the empty semi-standard Young tableau in $G_{v,v}$.) In summary, $\Phi(\Psi(S,g,T)) = (S,g,T)$. The same is true if $u^+ = u$.

It remains to check that $\Phi(\Psi(S,g,T)) = (S,g,T)$ if $u^+ > u$ and $v^+ > v$. To this end, let $\mathfrak{d}_\mathfrak{r}$ and $\mathfrak{d}_\mathfrak{l}$ be the two drifters in $\Psi(S,g,T)$. (The drifter $\mathfrak{d}_\mathfrak{r}$ is inserted on the left boundary, while the drifter $\mathfrak{d}_\mathfrak{l}$ is inserted on the right boundary in the course of the generation of $\Psi(S,g,T)$.) First, let j (*resp.* j') be the index of the column of the cell in $\lambda(u^+)/\lambda(u)$ (*resp.* $\lambda(v^+)/\lambda(v)$) and let $R+1$ (*resp.* $L+1$) be the entry of S (*resp.* T). It is shown below that if $(m_1, \ldots, m_R) \in \{M_1, M_2, M_3\}$ is the sequence of the moves that are applied to $\mathfrak{d}_\mathfrak{r}$ in the course of the generation of $\Psi(S,g,T)$ then the move m_{R-r+1}^{-1} is applied to $\mathfrak{d}_\mathfrak{r}(r-1)$ in $\mathrm{WR}^{r-1}(\Psi(S,g,T))$ for every $1 \leqslant r \leqslant R$. This then implies that $R = R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})$ and $|u|_1 - j = R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})})$, which in summary shows that

$$R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})}). \tag{16}$$

Assume now that an $1 \leqslant r_0 \leqslant R$ exists such that the move that is applied to $\mathfrak{d}_\mathfrak{r}(r_0 - 1)$ in $\mathrm{WR}^{r_0-1}(\Psi(S,g,T))$ by right-Wieland drift is not $m_{R-r_0+1}^{-1}$ and choose r_0 minimal. First, if to $\mathfrak{d}_\mathfrak{r}(r_0-1)$ in $\mathrm{WR}^{r_0-1}(\Psi(S,g,T))$ the move M_5^{-1} is applied by WR then $\mathfrak{d}_\mathfrak{r}(r) = \mathfrak{d}_\mathfrak{l}(r)$ for every $r_0 \leqslant r \leqslant R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) = R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})$. Furthermore, $R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r}) = R$ and to $\mathfrak{d}_\mathfrak{l}(r)$ in $\mathrm{WR}^{r}(\Psi(S,g,T))$ the move m_{R-r}^{-1} is applied by right-Wieland drift for every $r_0 \leqslant r < R$. In summary, it must hold that $|u|_1 - j \geqslant R = R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l})$.

Next, consider the case when to $\mathfrak{d}_\mathfrak{r}(r_0-1)$ in $\mathrm{WR}^{r_0-1}(\Psi(S,g,T))$ the move M_5^{-1} is not applied by WR. In that case there exist an $r_1 < r_0 - 1$ and cells o, e, o^* and e^* of G^N such that o, e, o^* and e^* in $\mathrm{WR}^{r_1}(\Psi(S,g,T))$ exhibit one of the configurations in the first row of the following table and in $\mathrm{WR}^{r_0-1}(\Psi(S,g,T))$ the respective configuration in the second row of the following table:



Therefore, to $\mathfrak{d}_\mathfrak{l}(r)$ in $\mathrm{WR}^{r}(\Psi(S,g,T))$ the move $m_{R-r_0-r+r_1+1}^{-1}$ is applied by WR for every $r_1 < r \leqslant R - r_0 + r_1$. Thus, $R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) = r_1 + R - r_0 < R - 1 < |u|_1 - j$.

In the next step, it is proved that starting from $\Psi(S,g,T)$ the drifter $\mathfrak{d}_\mathfrak{l}$ can be moved by M_1^{-1}, M_2^{-1} and M_3^{-1} until it is incident to L_i in $\mathcal{L}^N$. To this end, note that $\mathfrak{d}_\mathfrak{r}$ and $\mathfrak{d}_\mathfrak{l}$ satisfy

$$R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r}) - R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) = 0 = R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})}) - R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l})}) \tag{17}$$

if in $\mathrm{WR}^{r_0-1}(\Psi(S,g,T))$ to $\mathfrak{d}_\mathfrak{r}(r_0-1)$ the move M_5^{-1} is applied by WR and

$$R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r}) - R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) > R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})}) - R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l})}) \tag{18}$$

otherwise. In general, $\mathfrak{d}_\mathfrak{l}$ and $\mathfrak{d}_\mathfrak{r}$ thus satisfy that

$$R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r}) - R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) \geqslant R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})}) - R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l})}). \tag{19}$$

Now, if starting from $\Psi(S,g,T)$ the drifter $\mathfrak{d}_\mathfrak{l}$ cannot be moved by M_1^{-1}, M_2^{-1} and M_3^{-1} until it is incident to L_i in $\mathcal{L}^N$ then by vertical symmetry and Lemma 38 it must hold that

$$R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) - R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r}) > R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l})}) - R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})}), \tag{20}$$

which is a contradiction to (19). Thus, $\mathfrak{d}_\mathfrak{l}$ can be moved by M_1^{-1}, M_2^{-1} and M_3^{-1} until it is incident to L_i in $\mathcal{L}^N$. This is why

$$R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) + L = |u|_1 - j + |v|_1 - j' + 1 \tag{21}$$

by Proposition 28 (or [5, Lemma 6.14]). Since $R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) \leqslant |u|_1 - j$ this implies that $L \geqslant |v|_1 - j' + 1$, which is a contradiction to $T \in G_{v,v^+}$. Therefore, the move m_{R-r+1}^{-1} is applied to $\mathfrak{d}_\mathfrak{r}(r-1)$ in $\mathrm{WR}^{r-1}(\Psi(S,g,T))$ for every $1 \leqslant r \leqslant R$, $R = R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})$ and $|u|_1 - j = R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})})$.

Finally, it follows by symmetry that $L = L_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l})$ and $|v|_1 - j' = R_1(v^{L_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l})})$. In conclusion, $R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r}) \leqslant R_1(u^{R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r})})$ and $L_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) \leqslant R_1(v^{L_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l})})$. The stable TFPL $g(\Psi(S,g,T))$ thus is the TFPL obtained from $\Psi(S,g,T)$ by moving $\mathfrak{d}_\mathfrak{l}$ by M_1, M_2 and M_3 until it is incident to a vertex in $\mathcal{R}^N$ at which point it is replaced by a horizontal edge and thereafter moving $\mathfrak{d}_\mathfrak{r}$ by M_1^{-1}, M_2^{-1} and M_3^{-1} until it is incident to a vertex in $\mathcal{L}^N$ at which point it is replaced by a horizontal edge. Thus, $g(\Psi(S,g,T)) = g$. Finally, $S(\Psi(S,g,T)) = S$ since $R_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{r}) = R$ and $T(\Psi(S,g,T)) = T$ since $L_{\Psi(S,g,T)}(\mathfrak{d}_\mathfrak{l}) = L$. □

Proof of Theorem 34. Theorem 34 follows immediately from Proposition 45 and Proposition 46. □

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