The cycle descent statistic on permutations

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Abstract

In this paper we study the cycle descent statistic on permutations. Several involutions on permutations and derangements are constructed. Moreover, we construct a bijection between negative cycle descent permutations and Callan perfect matchings.

Keywords: Permutations; Cycle descents; Perfect matchings; Involutions; Eulerian polynomials

1 Introduction

Let $\mathfrak{S}_n$ be the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. We write an element π in $\mathfrak{S}_n$ as $\pi = \pi(1)\pi(2)\cdots\pi(n)$. An *excedance* in π is an index i such that $\pi(i) > i$ and a *fixed point* in π is an index i such that $\pi(i) = i$. A fixed-point-free permutation is called a *derangement*. Denote by $\mathcal{D}_n$ the set of derangements of $[n]$. As usual, let $\operatorname{exc}(\pi)$, $\operatorname{fix}(\pi)$ and $\operatorname{cyc}(\pi)$ denote the number of excedances, fixed points

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and cycles in π respectively. For example, the permutation $\pi = 3142765$ has the cycle decomposition $(1342)(57)(6)$, so $\mathrm{cyc}\,(\pi) = 3$, $\mathrm{exc}\,(\pi) = 3$ and $\mathrm{fix}\,(\pi) = 1$.

The Eulerian polynomials $A_n(x)$ are defined by

$$A_0(x) = 1, \quad A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} \quad \text{for } n \geqslant 1,$$

and have been extensively investigated. Foata and Schützenberger [7] introduced a q-analog of the Eulerian polynomials defined by

$$A_n(x; q) = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)}.$$

Brenti [2, 3] further studied q-Eulerian polynomials and established the link with q-symmetric functions arising from plethysm. Brenti [3, Proposition 7.3] obtained the exponential generating function for $A_n(x; q)$:

$$1 + \sum_{n \geqslant 1} A_n(x; q) \frac{z^n}{n!} = \left(\frac{1 - x}{e^{z(x-1)} - x} \right)^q.$$

Remarkably, Brenti [3, Corollary 7.4] derived the following identity:

$$\sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cyc}\,(\pi)} = -(x - 1)^{n-1}. \tag{1}$$

From then on, there is a large of literature devoted to various generalizations and refinements of the joint distribution of excedances and cycles (see [1, 6, 9, 15] for instance). For example, Ksavrelof and Zeng [9] constructed bijective proofs of (1) and the following formula:

$$\sum_{\pi \in \mathcal{D}_n} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cyc}\,(\pi)} = -x - x^2 - \cdots - x^{n-1}.$$

In particular, their bijection leads to a refinement of the above identity:

$$\sum_{\pi \in \mathcal{D}_{n,i}} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cyc}\,(\pi)} = -x^{n-i},$$

where $\mathcal{D}_{n,i}$ is the set of derangements π of $[n]$ such that $\pi(n) = i$.

A *standard cycle decomposition* of $\pi \in \mathfrak{S}_n$ is defined by requiring that each cycle is written with its smallest element first, and the cycles are written in increasing order of their smallest element. A permutation is said to be cyclic if there is only one cycle in its cycle decomposition. Let $(c_1, c_2, \ldots, c_i)$ be a cycle in the standard cycle decomposition of π. We say that c_j is a *cycle descent* if $c_j > c_{j+1}$, where $1 < j < i$. Denote by $CDES(\pi)$ the set of cycle descents of π and let $\mathrm{cdes}\,(\pi) = |CDES(\pi)|$ be the number of cycle descents of π. For example, for $\pi = (1342)(57)(6)$, we have $CDES(\pi) = \{4\}$ and $\mathrm{cdes}\,(\pi) = 1$. For $\pi \in \mathfrak{S}_n$, it is clear that $\mathrm{exc}\,(\pi) + \mathrm{cyc}\,(\pi) + \mathrm{cdes}\,(\pi) = n$. Thus

$$A_n(x; q) = q^n \sum_{\pi \in \mathfrak{S}_n} \left(\frac{x}{q} \right)^{\mathrm{exc}\,(\pi)} \left(\frac{1}{q} \right)^{\mathrm{cdes}\,(\pi)}.$$

Let $\mathfrak{S}_{n,i}$ be the set of permutations $\pi \in \mathfrak{S}_n$ with $\pi(i) = 1$. For any $\pi \in \mathfrak{S}_n$, let π^{-1} denote the inverse of π, so $\pi^{-1}(1) = i$ if $\pi \in \mathfrak{S}_{n,i}$. For $n \geqslant 2$, we recently observed the following formulas:

$$\begin{aligned}
\sum_{\pi\in\mathfrak{S}_{n,i}} (-1)^{\operatorname{cdes}(\pi)} t^{\pi^{-1}(1)} &= \begin{cases} 2^{n-2}t & \text{if} \quad i=1, \\ 0 & \text{if} \quad i=2,\ldots,n-1, \\ 2^{n-2}t^n & \text{if} \quad i=n, \end{cases} \\
\sum_{\pi\in\mathfrak{S}_n} (-1)^{\operatorname{cdes}(\pi)} t^{\pi^{-1}(1)} &= 2^{n-2}(t+t^n), \\
\sum_{\pi\in\mathcal{D}_n} x^{\operatorname{exc}(\pi)} (-1)^{\operatorname{cdes}(\pi)} t^{\pi^{-1}(1)} &= \sum_{i=2}^{n} (-1)^{n-i} x^{i-1} t^i, \\
\sum_{\pi\in\mathcal{D}_n} (-1)^{\operatorname{cdes}(\pi)} &= \frac{1}{2}[1-(-1)^{n-1}].
\end{aligned}$$

The above formulas can be easily proved by taking $x = 1$ in Theorem 1 of Section 2. Motivated by these formulas, we shall study the cycle descent statistic of permutations. In the next section, we present the main results of this paper and collect some notation and definitions that will be needed in the rest of the paper.

2 Definitions and main results

Consider the following enumerative polynomials

$$P_{n,i}(x,y,q,t) = \sum_{\pi\in\mathfrak{S}_{n,i}} x^{\operatorname{exc}(\pi)} y^{\operatorname{cdes}(\pi)} q^{\operatorname{fix}(\pi)} t^{\pi^{-1}(1)}.$$

It is remarkable that the polynomials $P_{n,i}(x,-1,1,t)$ and $P_{n,i}(x,-1,0,t)$ have simple closed formulas. We state them as the first main result of this paper.

Theorem 1. *For $n \geqslant 2$, we have*

$$P_{n,i}(x,-1,1,t) = \begin{cases} t(1+x)^{n-2} & \textit{if} \quad i=1, \\ 0 & \textit{if} \quad i=2,\ldots,n-1, \\ t^n x(1+x)^{n-2} & \textit{if} \quad i=n, \end{cases} \tag{2}$$

and

$$P_{n,i}(x,-1,0,t) = (-1)^{n-i} x^{i-1} t^i. \tag{3}$$

A signed permutation (π, ϕ) of $[n]$ is a permutation $\pi \in \mathfrak{S}_n$ together with a map $\phi : [n] \mapsto \{+1, -1\}$ and we call $\phi(i)$ the sign of i. For simplicity, we indicate the sign of $\pi(i)$ by writing $\pi(i)^+$ or $\pi(i)^-$. The group, which consists of all the signed permutations of $[n]$ with composition as the group operation, is called the signed permutation group of order n.

Let (π, ϕ) be a signed permutation. Let $NEG(\pi, \phi)$ be the set of numbers $\pi(i)$ with the sign -1, i.e.

$$NEG(\pi, \phi) = \{\pi(i) \mid \phi(\pi(i)) = -1\},$$

and let $\operatorname{neg}(\pi, \phi) = |NEG(\pi, \phi)|$.

Definition 2. *A negative cycle descent permutation (π, ϕ) of $[n]$ is a signed permutation (π, ϕ) such that $NEG(\pi, \phi) \subseteq CDES(\pi)$.*

Let

$$b_n(y, q) = \sum_{i=1}^{n} P_{n,i}(1, y, q, 1) = \sum_{\pi \in \mathfrak{S}_n} y^{\operatorname{cdes}(\pi)} q^{\operatorname{fix}(\pi)}.$$

It is easy to verify that $b_n(2, 1)$ is the number of negative cycle descent permutations of $[n]$ since

$$b_n(2, 1) = \sum_{\pi \in \mathfrak{S}_n} 2^{\operatorname{cdes}(\pi)}$$

and $b_n(2, 0)$ is the number of negative cycle descent derangements of $[n]$ since

$$b_n(2, 0) = \sum_{\pi \in \mathfrak{S}_n} 2^{\operatorname{cdes}(\pi)} 0^{\operatorname{fix}(\pi)} = \sum_{\pi \in \mathcal{D}_n} 2^{\operatorname{cdes}(\pi)}.$$

We present the second main result of this paper as follows.

Theorem 3. *For $n \geqslant 1$, we have*

$$b_{n+1}(y, 1) = b_n(y, 1) + \sum_{i=1}^{n} b_i(y, 1) \binom{n}{i-1} (y-1)^{n-i} \tag{4}$$

with the initial condition $b_1(y, 1) = 1$, and

$$b_{n+1}(y, 0) = \sum_{i=0}^{n-1} \binom{n}{i} \left[b_{i+1}(y, 0) + b_i(y, 0)\right] (y-1)^{n-i-1} \tag{5}$$

with initial conditions $b_0(y, 0) = 1, b_1(y, 0) = 0$.

By taking $y = 2$ in the identity (4), we obtain Klazar's recurrence for $w_{12}(n)$ (see [8, Eq. (39)] for details), which can be written as follows:

$$b_{n+1}(2, 1) = b_n(2, 1) + \sum_{i=1}^{n} b_{n+1-i}(2, 1) \binom{n}{i}. \tag{6}$$

In [4, 8, 12], the sets of some combinatorial objects, which have cardinality $b_n(2, 1)$, were studied. We list some of them as follows:

(i) The set of drawings of rooted plane trees with $n + 1$ vertices (see [8]);

(ii) The set of Klazar trees with $n+1$ vertices (see [4]);

(iii) The set of perfect matchings on the set $[2n]$ in which no even number is matched to a larger odd number (see [4]).

(iv) The set of ordered partitions of $[n]$ all of whose left-to-right minima occur at odd locations (see [12]).

Now we begin to introduce the concept of perfect matchings (see [5, 11] for instance). Let $\mathbb{P}_A = A \times \{0,1\}$, where $A = \{i_1, \ldots, i_k\}$ is a finite set of positive integers with $i_1 < i_2 < \cdots < i_k$. When $A = [n]$, we write $\mathbb{P}_A$ as $\mathbb{P}_n$. A perfect matching is a partition of $\mathbb{P}_A$ into 2-element subsets or matches. For any match $\{(i,x),(j,y)\}$ in a perfect matching, we say that (i,x) is the *partner* of (j,y). For convenience, we represent a perfect matching as a dot diagram with vertices arranged in two rows.

Example 4. *We give a dot diagram of a perfect matching M of $\mathbb{P}_8$ as follows:*

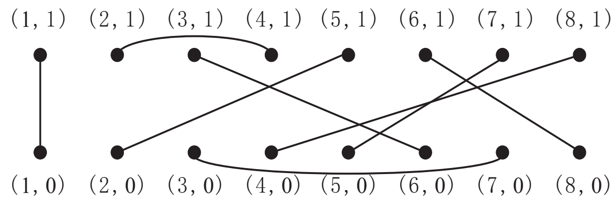


Fig.1. A perfect matching M of $\mathbb{P}_8$

Thus, for any perfect matching M of $\mathbb{P}_A$, we say that $\mathbb{P}_A$ is the vertex set of M and every match is an edge of M. We use $V(M)$ and $E(M)$ to denote vertices set and edges set in M respectively. Moreover, an edge is called an *arc* if it joins two dots in the same row; otherwise, this edge is called a *line*. For any line $\{(i,0),(j,1)\}$, it is said to be a *upline* if $i < j$, a *downline* if $i > j$ and a *vertical* line if $i = j$. For any perfect matching M, let $\operatorname{arc}(M), \operatorname{down}(M)$ and $\operatorname{ver}(M)$ be the numbers of arc, down lines and vertical lines in M, respectively.

Example 5. *In the perfect matching of Example 4, the edge $\{(1,1),(1,0)\}$ is a vertical line, the edges $\{(5,1),(2,0)\}$, $\{(7,1),(5,0)\}$ and $\{(8,1),(4,0)\}$ are three uplines, the edges $\{(3,1),(6,0)\}$ and $\{(6,1),(8,0)\}$ are two downlines, and the edges $\{(2,1),(4,1)\}$ and $\{(3,0),(7,0)\}$ are two arcs; finally, $arc(M) = 2, down(M) = 2, ver(M) = 1$.*

Definition 6. *A perfect matching M of $\mathbb{P}_n$ is a Callan perfect matching if M has no uplines.*

Example 7. *We give a dot diagram of a Callan perfect matching M of $\mathbb{P}_8$ as follows:*

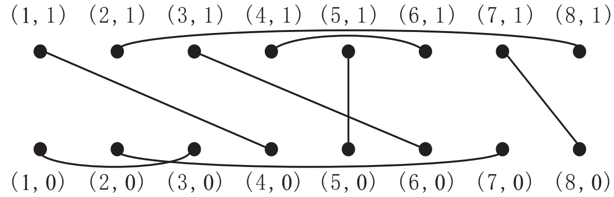


Fig.2. A Callan perfect matching M of $\mathbb{P}_8$

Let m_n be the number of Callan perfect matchings of $\mathbb{P}_n$. Callan [4] proved that m_n satisfies the recurrence (6). So the number of negative cycle descent permutations of $[n]$ equals to the number of Callan perfect matching of $\mathbb{P}_n$.

Let M be a perfect matching of $\mathbb{P}_n$. We say that M' is a *sub-perfect matching* of M if M' is a perfect matching such that $V(M') \subseteq V(M)$ and $E(M') \subseteq E(M)$. For any $V \subseteq [n]$, if there is a sub-perfect matching M' of M with $V(M') = V \times \{0, 1\}$, then M' is said to be the sub-perfect matching induced by V and is denoted by $M[V]$.

Denote by $\mathcal{G}(M)$ a graph which is obtained from M by identifying each two vertices $(i, 0)$ and $(i, 1)$ as a new vertex i for any $i \in [n]$. It is easy to see that the graph $\mathcal{G}(M)$ is the union of some disjoint cycles. For a cycle C in $\mathcal{G}(M)$, suppose C has the vertices set V. Note that there is a sub-perfect matching of M induced by V. We say that $M[V]$ is a connected component of M. Let $\mathrm{com}\,(M)$ be the number of connected components in a perfect matching M. If a perfect matching M has exactly one connected component, i.e., $\mathrm{com}\,(M) = 1$, then we say that M is a connected perfect matching.

Example 8. *For the perfect matching M of Example 7, we draw the graph $\mathcal{G}(M)$ as follows:*

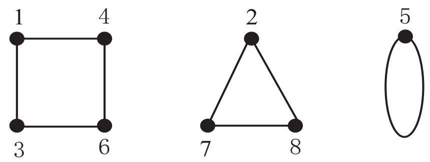


Fig.3. A graph $\mathcal{G}(M)$.

So we have $\mathrm{com}\,(M) = 3$.

We state the third main result of this paper as follows.

Theorem 9. *There is a bijection Γ_n between the set of negative cycle descent permutations of $[n]$ and the set of Callan perfect matchings of $\mathbb{P}_n$. Moreover, for any negative cycle descent permutation (π, ϕ) of $[n]$, we have*

$$\mathrm{com}\,(\Gamma_n(\pi, \phi)) = \mathrm{cyc}\,(\pi), \;\; \mathrm{ver}\,(\Gamma_n(\pi, \phi)) = \mathrm{fix}\,(\pi),$$

and

$$\mathrm{down}\,(\Gamma_n(\pi, \phi)) = \begin{cases} \mathrm{neg}\,(\pi, \phi) & \textit{if } (1,1) \textit{ and its partner are in the same row,} \\ \mathrm{neg}\,(\pi, \phi) + 1 & \textit{otherwise.} \end{cases}$$

Let $\Gamma|_{\mathcal{D}_n}$ denote the restriction of Γ_n on the set of negative cycle descent derangements of $[n]$. So the following corollary is immediate.

Corollary 10. $\Gamma|_{\mathcal{D}_n}$ *is a bijection between the set of negative cycle descent derangements of $[n]$ and the set of Callan perfect matchings of $\mathbb{P}_n$ which have no vertical lines.*

The rest of this paper is organized as follows. In Section 3 and Section 4, we respectively prove (2) and (3) in Theorem 1. In Section 5 and Section 6, we respectively prove (4) and (5) in Theorem 3. In Section 7, we construct the bijection Γ_n in Theorem 9.

3 Proof of the explicit formula (2) in Theorem 1

Suppose that $\pi = \pi(i_1)\ldots\pi(i_k)$ is a permutation on the set $\{i_1, \ldots, i_k\}$ of positive integers with $i_1 < i_2 < \cdots < i_k$. Throughout this paper, we always let

$$\mathrm{red}\,(\pi) := \mathrm{red}\,(\pi(i_1)) \cdots \mathrm{red}\,(\pi(i_k)) \in \mathfrak{S}_k,$$

where red is a increasing map from $\{i_1, \ldots, i_k\}$ to $\{1, 2, \ldots, k\}$ defined by $\mathrm{red}\,(i_j) = j$ for any $j = 1, 2, \ldots, k$.

Let $P_n(x, y, 1, 1) = \sum_{i=1}^{n} P_{n,i}(x, y, 1, 1)$. We give a recurrence for $P_{n,i}(x, y, 1, 1)$ in the following lemma.

Lemma 11. *For any $n \geqslant 2$, we have*

$$P_{n+1,i}(x,y,1,1) = \begin{cases} P_n(x,y,1,1) & \textit{if} \quad i = 1, \\ xP_{n-1}(x,y,1,1) + x\sum_{j=2}^{i-1} P_{n,j}(x,y,1,1) + & \\ y\sum_{j=i}^{n} P_{n,j}(x,y,1,1) & \textit{if} \quad i = 2, \ldots, n+1, \end{cases}$$

with initial conditions

$$P_{1,1}(x,y,1,1) = 1, P_{2,1}(x,y,1,1) = 1, P_{2,2}(x,y,1,1) = x.$$

Proof. For any $\pi = \pi(1)\pi(2)\ldots\pi(n+1) \in \mathfrak{S}_{n+1,1}$, we have $\pi(1) = 1$. Let $\tilde{\pi} = \pi(2)\ldots\pi(n+1)$. Then $\tilde{\pi}$ is a permutation on the set $\{2, 3, \ldots, n\}$ and $\mathrm{red}\,(\tilde{\pi}) \in \mathfrak{S}_n$. Obviously,

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})) \text{ and } \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})).$$

So we have $P_{n+1,1}(x, y, 1, 1) = P_n(x, y, 1, 1)$.

For any $i \geqslant 2$, let $\pi = \pi(1)\pi(2)\ldots\pi(n+1) \in \mathfrak{S}_{n+1,i}$. Let $\sigma = (1, c_1, c_2, \ldots, c_m)$ be the cycle in the standard cycle decomposition of π which contains the entry 1. So π can be split into the cycle σ and a permutation τ on the set $\{1, 2, \ldots, n+1\} \setminus \{1, c_1, \ldots, c_m\}$, i.e., $\pi = \sigma \cdot \tau$. Clearly, $m \geqslant 1$, $i \geqslant 2$ and $c_m = i$ since $\pi \in \mathfrak{S}_{n+1,i}$. We distinguish between the following two cases:

Case 1. $m = 1$.

Deleting the cycle $(1, c_1) = (1, i)$ from the standard cycle decomposition of π, we obtain the permutation

$$\tau = \pi(2) \cdots \pi(i-1)\pi(i+1) \cdots \pi(n+1)$$

which is defined on the set $\{1, 2, \ldots, n+1\} \setminus \{1, i\}$. Note that $\mathrm{red}\,(\tau) \in \mathfrak{S}_{n-1}$ and

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tau)) + 1, \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tau)).$$

This provides the term

$$xP_{n-1}(x, y, 1, 1).$$

Case 2. $m \geqslant 2$.

Suppose that $c_{m-1} = j$ for some $2 \leqslant j \leqslant n+1$. Deleting the number $c_m = i$ from the standard cycle decomposition of π, we obtain a permutation

$$\tilde{\pi} = (1, c_1, \ldots, c_{m-1}) \cdot \tau$$

which is defined on the set $\{1, \ldots, i-1, i+1, \ldots, n+1\}$. Hence $\mathrm{red}\,(\tilde{\pi}) \in \mathfrak{S}_n$. Moreover, if $c_{m-1} = j \leqslant i-1$, then

$$\mathrm{red}\,(\tilde{\pi}) \in \mathfrak{S}_{n,j}, \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})) + 1, \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})).$$

This provides the term

$$x \sum_{j=2}^{i-1} P_{n,j}(x, y, 1, 1).$$

If $c_{m-1} = j \geqslant i+1$, then

$$\mathrm{red}\,(\tilde{\pi}) \in \mathfrak{S}_{n,j-1}, \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})), \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})) + 1.$$

This provides the term

$$y \sum_{j=i}^{n} P_{n,j}(x, y, 1, 1).$$

In conclusion, for any $i \geqslant 2$ we have

$$P_{n+1,i}(x, y, 1, 1) = xP_{n-1}(x, y, 1, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, y, 1, 1) + y \sum_{j=i}^{n} P_{n,j}(x, y, 1, 1). \qquad \square$$

A proof of the identity (2) in Theorem 1:
Note that

$$\sum_{\pi \in \mathfrak{S}_{n,i}} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cdes}\,(\pi)} t^{\pi^{-1}(1)} = t^i \sum_{\pi \in \mathfrak{S}_{n,i}} x^{\mathrm{exc}\,(\pi)} (-1)^{\mathrm{cdes}\,(\pi)} = t^i P_{n,i}(x, -1, 1, 1).$$

In order to prove the identity (2) in Theorem 1, it is sufficient to show that

$$P_{n,i}(x, -1, 1, 1) = \begin{cases} (1+x)^{n-2} & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i = 2, \ldots, n-1, \\ x(1+x)^{n-2}, & \text{if} \quad i = n. \end{cases} \tag{7}$$

(i) An inductive proof of the explicit formula (7).

Proof. It is easy to verify that $P_{2,1}(x,-1,1,1)=1, P_{2,2}(x,-1,1,1)=x$. Assume that the explicit formula (7) holds for any $2 \leqslant k \leqslant n$. By Lemma 11, we have

$$\begin{aligned} P_{n+1,1}(x,-1,1,1) &= P_n(x,-1,1,1) \\ &= P_{n,1}(x,-1,1,1)+P_{n,n}(x,-1,1,1) \\ &= (1+x)^{n-2}+x(1+x)^{n-2}=(1+x)^{n-1}, \end{aligned}$$

$$\begin{aligned} P_{n+1,n+1}(x,-1,1,1) &= xP_{n-1}(x,-1,1,1)+x\sum_{j=2}^{n} P_{n,j}(x,-1,1,1) \\ &= xP_{n-1,1}(x,-1,,1,1)+xP_{n-1,n-1}(x,-1,1,1)+xP_{n,n}(x,-1,1,1) \\ &= x(1+x)^{n-2}+x^2(1+x)^{n-2}=x(1+x)^{n-1} \end{aligned}$$

and

$$\begin{aligned} P_{n+1,i}(x,-1,1,1) &= xP_{n-1}(x,-1,1,1)+x\sum_{j=2}^{i-1} P_{n,j}(x,-1,1,1)-\sum_{j=i}^{n} P_{n,j}(x,-1,1,1) \\ &= xP_{n-1,1}(x,-1,1,1)+xP_{n-1,n-1}(x,-1,1,1)-P_{n,n}(x,-1,1,1) \\ &= x(1+x)^{n-2}-P_{n,n}(x,-1,1,1)=0 \end{aligned}$$

for any $2 \leqslant i \leqslant n$.

(ii) A bijective proof of the explicit formula (7).
Now we give a bijective proof of (7) by establishing an involution $\psi_{n,i}$ on $\mathfrak{S}_{n,i}$.

For any $\pi \in \mathfrak{S}_n$, suppose that $\pi = C_1 \dots C_k$ is the standard cycle decomposition of π. Let

$$\hat{\pi} = a_1 a_2 \cdots a_n$$

be the permutation obtained from π by erasing the parentheses in its standard cycle decomposition. Furthermore, for any $i = 1,2,\dots,n-1$, the number a_i is said to be a value-descent of π if $a_i > a_{i+1}$ in the sequence $\hat{\pi}$, and let q_π be the *last value-descent* which appears in the sequence $\hat{\pi}$. For example, the permutation $\pi = 1472365$ in $\mathfrak{S}_7$ has the standard cycle decomposition (1)(24)(375)(6), so $\hat{\pi} = 1243756$, it has exactly two value-descents 4 and 7, and $q_\pi = 7$. The process of erasing the parentheses from the standard cycle decomposition of a permutation is well-known bijection of Foata and Schützenberger, the “fundamental transformation”, see also [7].

We define a map $\Phi : \mathfrak{S}_n \mapsto \mathfrak{S}_n$ as follows:

For any $\pi \in \mathfrak{S}_n$, if q_π is the last element of a cycle C_i for some i, then let $\Phi(\pi)$ be the permutation obtained from π by erasing the right and left parentheses “)(” after the number q_π in the standard cycle decomposition of π; otherwise, let $\Phi(\pi)$ be the permutation obtained from π by inserting a right parenthesis “)” and a left parenthesis

"(" after the number q_π in the standard cycle decomposition of π. For example, if $\pi = (1)(24)(375)(6)$, then $\hat{\pi} = 1243756$ and $q_\pi = 7$, and so $\Phi(\pi) = (1)(24)(37)(5)(6)$. If $\sigma = (1)(24)(37)(5)(6)$, then $\Phi(\sigma) = (1)(24)(375)(6)$. Clearly, we have

$$\hat{\pi} = \widehat{\Phi(\pi)},\ q_\pi = q_{\Phi(\pi)},\ \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\Phi(\pi)),\ \mathrm{cdes}\,(\pi) - \mathrm{cdes}\,(\Phi(\pi)) = \pm 1.$$

Denote by $\Omega_{n,1}$ the set of the permutations $\pi \in \mathfrak{S}_{n,1}$ such that $\hat{\pi} = 123\cdots n$. For any $\pi \in \Omega_{n,1}$, suppose that

$$\pi = (1)C_1C_2\cdots C_{k-1}C_k$$

is the standard cycle decomposition of π. Let i_s be the largest number in the cycle C_s for every $s = 1, 2, \ldots, k-1$. Then the set $\{i_1, i_2, \ldots, i_{k-1}\}$ is a subset of the set $\{2, 3, \ldots, n-1\}$ and $i_1 < i_2 < \cdots < i_{k-1}$.

Conversely, suppose that $\{i_1, i_2, \ldots, i_{k-1}\}$ is a subset of the set $\{2, 3, \ldots, n-1\}$ and $i_1 < i_2 < \cdots < i_{k-1}$. Let

$$\pi = (1)(2, 3, \ldots, i_1)(i_1+1, i_1+2, \ldots, i_2)\cdots(i_{k-2}+1, i_{k-2}+2, \ldots, i_{k-1})(i_{k-1}+1, i_{k-1}+2, \ldots, n).$$

We have $\pi \in \Omega_{n,1}$ and $\mathrm{exc}\,(\pi) = n-k-1$. Thus, the weight of $\Omega_{n,1}$ is

$$\sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = \sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)} = \sum_{k=1}^{n-1}\binom{n-2}{k-1}x^{n-k-1} = (x+1)^{n-2}.$$

For any permutation $\pi \in \mathfrak{S}_{n,1}\setminus\Omega_{n,1}$, we have $\Phi(\pi) \in \mathfrak{S}_{n,1}\setminus\Omega_{n,1}$. So for any $\pi \in \mathfrak{S}_{n,1}$, let

$$\psi_{n,1}(\pi) = \begin{cases} \Phi(\pi) & \text{if} \quad \pi \in \mathfrak{S}_{n,1}\setminus\Omega_{n,1}, \\ \pi & \text{if} \quad \pi \in \Omega_{n,1}. \end{cases}$$

Note that

$$\begin{aligned} \sum_{\pi\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} &= -\sum_{\pi\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\Phi(\pi))}(-1)^{\mathrm{cdes}\,(\Phi(\pi))} \\ &= -\sum_{\pi'\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1},\pi=\Phi(\pi')} x^{\mathrm{exc}\,(\Phi(\pi))}(-1)^{\mathrm{cdes}\,(\Phi(\pi))} \\ &= -\sum_{\pi'\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\pi')}(-1)^{\mathrm{cdes}\,(\pi')}. \end{aligned}$$

This implies that

$$\sum_{\pi\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = 0.$$

Hence,

$$\begin{aligned} \sum_{\pi\in\mathfrak{S}_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} &= \sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} + \sum_{\pi\in\mathfrak{S}_{n,1}\setminus\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} \\ &= \sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = \sum_{\pi\in\Omega_{n,1}} x^{\mathrm{exc}\,(\pi)} = (1+x)^{n-2}. \end{aligned}$$

For example, we list all $\pi \in \mathfrak{S}_{4,1}$ and $\psi_{4,1}(\pi)$ in Table 1.

$\pi \in \mathfrak{S}_{4,1}$	$x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)}$	$\hat{\pi}$	q_π	$\psi_{4,1}(\pi)$
(1)(2)(3)(4)	1	1234		(1)(2)(3)(4)
(1)(23)(4)	x	1234		(1)(23)(4)
(1)(2)(34)	x	1234		(1)(2)(34)
(1)(234)	x^2	1234		(1)(234)
(1)(24)(3)	x	1243	4	(1)(243)
(1)(243)	$-x$	1243	4	(1)(24)(3)

Table.1. Involution $\psi_{4,1}$

For $2 \leqslant i \leqslant n$, denote by $\mathcal{A}_{n,i}$ the set of permutations $\pi \in \mathfrak{S}_{n,i}$ such that the number q_π isn't in the first cycle in the standard cycle decomposition of π. Let $\mathcal{A}^*_{n,i} = \mathfrak{S}_{n,i} \setminus \mathcal{A}_{n,i}$ for short. For any $\pi \in \mathcal{A}^*_{n,i}$, let $\pi = C_1 \dots C_k$ be the standard cycle decomposition of π, suppose the length of the cycle C_1 is $l+1$ and

$$\hat{\pi} = 1, \dots, i, a_1, \dots, a_{n-l-1}.$$

Then we have

$$a_1 < a_2 < \cdots < a_{n-l-1},$$

since q_π is an element in the cycle C_1.

Now suppose that $C_1 = (1, c_{11}, \dots, c_{1l})$. Let Q be the set of indices $j \in \{1, 2, \dots, l\}$ such that c_{1j} is not the largest number in the set $\{1, 2, \dots, n\} \setminus \{c_{1,j+1}, \dots, c_{1l}\}$, i.e.,

$$Q = \{j \mid 1 \leqslant j \leqslant l \text{ and } c_{1j} < \max\{1, 2, \dots, n\} \setminus \{c_{1,j+1}, \dots, c_{1l}\}\}.$$

Let $\Omega_{n,i}$ be the set of permutations $\pi \in \mathcal{A}^*_{n,i}$ such that $Q = \emptyset$. For any $i = 2, \dots, n-1$, we have $l \in Q$ since $i < n$, and so $\Omega_{n,i} = \emptyset$. Moreover, $\pi \in \Omega_{n,n}$ if and only if

$$\hat{\pi} = 1, k, k+1, \dots n-1, n, 2, 3, \dots, k-2, k-1$$

for some $k > 1$.

We define a map Ψ from $\mathcal{A}^*_{n,i} \setminus \Omega_{n,i}$ to itself as follows:

For any $\pi \in \mathcal{A}^*_{n,i} \setminus \Omega_{n,i}$, let $m = m_\pi = \min Q$ since $Q \neq \emptyset$. If $m = 1$, then there are at least two cycles in the standard cycle decomposition of π. If $m \geqslant 2$, then we have

$$c_{11} < \cdots < c_{1,m-1} \text{ and } c_{1,m-1} > c_{1m}.$$

Furthermore, we distinguish between the following two cases:

Case 1. $2 \leqslant m \leqslant l$.

Let

$$\Psi(\pi) = (1, c_{1m}, \dots, c_{1l}) \cdot C_2 \dots C_k \cdot (c_{11}, \dots, c_{1,m-1}).$$

Then $\Psi(\pi)$ has at least two cycles, $m_{\Phi(\pi)} = 1$ since $c_{1m} < c_{1,m-1}$, and so $\Psi(\pi) \in \mathcal{A}^*_{n,i} \setminus \Omega_{n,i}$. Moreover, we have $\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\Psi(\pi))$ and $\mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\Psi(\pi)) + 1$

Case 2. $m = 1$.

Suppose that $C_1 = (1, c_{11}, \ldots, c_{1l})$ and $C_k = (c_{k1}, \ldots, c_{ks})$ are the first cycle and the last cycle in the standard cycle decomposition of π respectively, where s is the length of the cycle C_k. Let

$$\Psi(\pi) = (1, c_{k1}, \ldots, c_{ks}, c_{11}, \ldots, c_{1l}) \cdot C_2 \ldots C_{k-1}.$$

Then

$$m_{\Psi(\pi)} = s + 1 \geqslant 2,$$

and so $\Psi(\pi) \in \mathcal{A}^*_{n,i} \setminus \Omega_{n,i}$. Moreover, we have $\operatorname{exc}(\pi) = \operatorname{exc}(\Psi(\pi))$ and $\operatorname{cdes}(\pi) = \operatorname{cdes}(\Psi(\pi)) - 1$

When $2 \leqslant i \leqslant n-1$, for any $\pi \in \mathfrak{S}_{n,i}$, let

$$\psi_{n,i}(\pi) = \begin{cases} \Phi(\pi) & \text{if} \quad \pi \in \mathcal{A}_{n,i}, \\ \Psi(\pi) & \text{if} \quad \pi \in \mathfrak{S}_{n,i} \setminus \mathcal{A}_{n,i}. \end{cases}$$

For example, we list all $\pi \in \mathfrak{S}_{4,2}$ and $\psi_{4,2}(\pi)$ in Table 2, and $\pi \in \mathfrak{S}_{4,3}$ and $\psi_{4,3}(\pi)$ in Table 3.

$\pi \in \mathfrak{S}_{4,2}$	$x^{\operatorname{exc}(\pi)}(-1)^{\operatorname{cdes}(\pi)}$	$\hat{\pi}$	q_π	m_π	$\psi_{4,2}(\pi)$
(12)(3)(4)	x	1234		1	(142)(3)
(142)(3)	$-x$	1423	4	2	(12)(3)(4)
(12)(34)	x^2	1234		1	(1342)
(1342)	$-x^2$	1342	4	3	(12)(34)
(1432)	x	1432	3	2	(132)(4)
(132)(4)	$-x$	1324	3	1	(1432)

Table.2. Involution $\psi_{4,2}$

$\pi \in \mathfrak{S}_{4,3}$	$x^{\operatorname{exc}(\pi)}(-1)^{\operatorname{cdes}(\pi)}$	$\hat{\pi}$	q_π	m_π	$\psi_{4,3}(\pi)$
(13)(2)(4)	x	1324	3	1	(143)(2)
(143)(2)	$-x$	1432	3	2	(13)(2)(4)
(13)(24)	x^2	1324	3	1	(1243)
(1243)	$-x^2$	1243	4	3	(13)(24)
(1423)	$-x^2$	1423	4	2	(123)(4)
(123)(4)	x^2	1234		1	(1423)

Table.3. Involution $\psi_{4,3}$

Hence,

$$\sum_{\pi \in \mathfrak{S}_{n,i}} x^{\operatorname{exc}(\pi)}(-1)^{\operatorname{cdes}(\pi)} = \sum_{\pi \in \mathcal{A}_{n,i}} x^{\operatorname{exc}(\pi)}(-1)^{\operatorname{cdes}(\pi)} + \sum_{\pi \in \mathfrak{S}_{n,i} \setminus \mathcal{A}_{n,i}} x^{\operatorname{exc}(\pi)}(-1)^{\operatorname{cdes}(\pi)} = 0.$$

When $i = n$, we claim that the weight of $\Omega_{n,n}$ is $x(1+x)^{n-2}$. For any $\pi \in \Omega_{n,n}$, suppose that

$$\pi = (1, c_{11}, c_{12}, \ldots, c_{1s})C_1 C_2 \ldots C_k$$

is the standard cycle decomposition of π, where $c_{1s} = n$. Let

$$\pi' = (1)C_1C_2 \dots C_k(c_{11}, c_{12}, \dots, c_{1s}).$$

Then $\pi' \in \Omega_{n,1}$ and $\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\pi') + 1$. So, the weight of $\Omega_{n,n}$ is

$$\sum_{\pi \in \Omega_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = \sum_{\pi \in \Omega_{n,n}} x^{\mathrm{exc}\,(\pi)} = \sum_{\pi' \in \Omega_{n,1}} x^{\mathrm{exc}\,(\pi')+1} = x(1+x)^{n-2}.$$

For any $\pi \in \mathfrak{S}_{n,n}$, let

$$\psi_{n,n}(\pi) = \begin{cases} \Phi(\pi) & \text{if} \quad \pi \in \mathcal{A}_{n,n}, \\ \Psi(\pi) & \text{if} \quad \pi \in \mathfrak{S}_{n,n} \setminus (\mathcal{A}_{n,n} \cup \Omega_{n,n}), \\ \pi & \text{if} \quad \pi \in \Omega_{n,n}. \end{cases}$$

For example, we list all $\pi \in \mathfrak{S}_{4,4}$ and $\psi_{4,4}(\pi)$ in Table 4.

π	$x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)}$	$\hat{\pi}$	q_π	m_π	$\psi_{4,4}(\pi)$
(14)(2)(3)	x	1423	4		(14)(2)(3)
(14)(23)	x^2	1423	4		(14)(23)
(134)(2)	x^2	1342	4		(134)(2)
(1234)	x^3	1234			(1234)
(124)(3)	x^2	1243	4	1	(1324)
(1324)	$-x^2$	1324	3	2	(124)(3)

Table.4. Involution $\psi_{4,4}$

Hence,

$$\begin{aligned} & \sum_{\pi \in \mathfrak{S}_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} \\ = & \sum_{\pi \in \mathcal{A}_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} + \sum_{\pi \in \mathfrak{S}_{n,n} \setminus (\mathcal{A}_{n,n} \cup \Omega_{n,n})} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} + \sum_{\pi \in \Omega_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} \\ = & \sum_{\pi \in \Omega_{n,n}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = x(1+x)^{n-2}. \end{aligned}$$

□

4 Proof of the explicit formula (3) in Theorem 1

Let $P_n(x, y, 0, 1) = \sum_{i=1}^{n} P_{n,i}(x, y, 0, 1)$. We first give the recurrence for $P_n(x, y, 0, 1)$.

Lemma 12. *For any $n \geqslant 2$ and $2 \leqslant i \leqslant n+1$, we have*

$$P_{n+1,i}(x, y, 0, 1) = xP_{n-1}(x, y, 0, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, y, 0, 1) + y \sum_{j=i}^{n} P_{n,j}(x, y, 0, 1).$$

Proof. For any $\pi = \pi(1)\pi(2)\ldots\pi(n+1) \in \mathcal{D}_{n+1,i}$, let $\sigma = (1, c_1, c_2, \ldots, c_l)$ be the cycle in the standard cycle decomposition of π which contains the number 1. So π can be split into the cycle σ and a permutation τ on the set $\{1, 2, \ldots, n+1\} \setminus \{1, c_1, \ldots, c_l\}$, i.e., $\pi = \sigma \cdot \tau$. Clearly, $l \geqslant 1$, $i \geqslant 2$ and $c_l = i$ since $\pi \in \mathcal{D}_{n+1,i}$. We distinguish between the following two cases:

Case 1. $l = 1$.

Deleting the cycle $(1, c_1) = (1, i)$ from the standard cycle decomposition of π, we obtain the permutation

$$\tau = \pi(2)\ldots\pi(i-1)\pi(i+1)\ldots\pi(n+1)$$

which is defined on the set $\{2, \ldots, i-1, i+1, \ldots, n+1\}$. Note that $\mathrm{red}\,(\tau) \in \mathcal{D}_{n-1}$,

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tau)) + 1 \text{ and } \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tau)).$$

This provides the term $xP_{n-1}(x, y, 0, 1)$.

Case 2. $l \geqslant 2$.

Suppose that $c_{l-1} = j$ for some $2 \leqslant j \leqslant n+1$. Deleting the number $c_l = i$ from the standard cycle decomposition of π, we obtain a permutation

$$\tilde{\pi} = (1, c_1, \ldots, c_{l-1}) \cdot \tau$$

which is defined on the set $\{1, \ldots, i-1, i+1, \ldots, n+1\}$. Note that $\mathrm{red}\,(\tilde{\pi}) \in \mathcal{D}_n$. Moreover, if $c_{l-1} = j \leqslant i-1$, then

$$\mathrm{red}\,(\tilde{\pi}) \in \mathcal{D}_{n,j}, \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})) + 1, \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})).$$

This provides the term

$$x \sum_{j=2}^{i-1} P_{n,j}(x, y, 0, 1).$$

If $c_{l-1} = j \geqslant i+1$, then

$$\mathrm{red}\,(\tilde{\pi}) \in \mathcal{D}_{n,j-1}, \mathrm{exc}\,(\pi) = \mathrm{exc}\,(\mathrm{red}\,(\tilde{\pi})), \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\mathrm{red}\,(\tilde{\pi})) + 1.$$

This provides the term

$$y \sum_{j=i}^{n} P_{n,j}(x, y, 0, 1).$$

Thus, for any $i \geqslant 2$ we have

$$P_{n+1,i}(x, y, 0, 1) = xP_{n-1}(x, y, 0, 1) + x \sum_{j=2}^{i-1} P_{n,j}(x, y, 0, 1) + y \sum_{j=i}^{n} P_{n,j}(x, y, 0, 1). \qquad \square$$

A proof of the identity (3) in Theorem 1:

Proof. Note that

$$\sum_{\pi\in\mathcal{D}_{n,i}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)}t^{\pi^{-1}(1)} = t^i \sum_{\pi\in\mathcal{D}_{n,i}} x^{\mathrm{exc}\,(\pi)}(-1)^{\mathrm{cdes}\,(\pi)} = t^i P_{n,i}(x,-1,0,1).$$

Therefore, it is sufficient to show that

$$P_{n,i}(x,-1,0,1) = (-1)^{n-i}x^{i-1} \tag{8}$$

for any $n \geqslant 2$ and $2 \leqslant i \leqslant n$.

(i) An inductive proof of the explicit formula (8):
It is easy to check that

$$P_{2,2}(x,-1,0,1) = x.$$

Assume that the formula holds for any $2 \leqslant k \leqslant n$. By Lemma 12, we have

$$\begin{aligned} P_{n+1,i}(x,-1,0,1) &= xP_{n-1}(x,-1,0,1) + x\sum_{j=2}^{i-1} P_{n,j}(x,-1,0,1) - \sum_{j=i}^{n} P_{n,j}(x,-1,0,1) \\ &= x\sum_{j=2}^{n-1} P_{n-1,j}(x,-1,0,1) + x\sum_{j=2}^{i-1} P_{n,j}(x,-1,0,1) - \sum_{j=i}^{n} P_{n,j}(x,-1,0,1) \\ &= \sum_{j=2}^{n-1}(-1)^{n-1-j}x^j + \sum_{j=2}^{i-1}(-1)^{n-j}x^j - \sum_{j=i}^{n}(-1)^{n-j}x^{j-1} \\ &= (-1)^{n+1-i}x^{i-1} \end{aligned}$$

for any $2 \leqslant i \leqslant n$.

(ii) A bijective proof of the explicit formula (8):
Next we give a bijective proof of the explicit formula (8) by establishing an involution $\varphi_{n,i}$ on $\mathcal{D}_{n,i}$. Fix $i \in \{2,\ldots,n\}$. By definition, the weight of each $\pi \in \mathcal{D}_{n,i}$ is $(-1)^{\mathrm{cdes}\,(\pi)}x^{\mathrm{exc}\,(\pi)}$, hence the weight of the cyclic permutation

$$\sigma^i = (1,2,\ldots,i-1,n,n-1,\ldots,i) \in \mathcal{D}_{n,i}$$

is $(-1)^{n-i}x^{i-1}$.

For any $\pi \in \mathcal{D}_{n,i}$, suppose that $\pi = C_1\ldots C_k$ is the standard cycle decomposition of π and $C_k = (c_{k,1},\ldots,c_{k,s})$. We distinguish among the following three cases:

Case 1. $k=1$ and $C_k = (1,2,\ldots,i-1,n,n-1,\ldots,i)$.
Then let $\varphi_{n,i}(\pi) = \pi$.

Case 2. $k \geqslant 2$ and $\mathrm{red}\,(C_k) = (1, 2, \ldots, r-1, s, s-1, \ldots, r)$ for some $r = 2, 3, \ldots, s$.

Suppose that $C_{k-1} = (c_{k-1,1}, c_{k-1,2}, \ldots, c_{k-1,t})$ and $c_{k,j}$ is the largest number in the set $\{c_{k,1}, c_{k,2}, \ldots, c_{k,s}\}$ for some $j \in \{1, 2, \ldots, s\}$. If $c_{k-1,2} < c_{k,j-1}$, then let

$$\varphi_{n,i}(\pi) = C_1 \cdots C_{k-2} \cdot (c_{k-1,1}, c_{k1}, c_{k2}, \ldots, c_{ks}, c_{k-1,2}, \ldots, c_{k-1,t}),$$

and so we have

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\varphi_{n,i}(\pi)) \text{ and } \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\varphi_{n,i}(\pi)) - 1.$$

For example, we consider a permutation $\pi = (1397)(24586) \in \mathcal{D}_{9,7}$. The largest number in the cycle (24586) is 8, and so $j = 4$. Since $c_{1,2} = 3 < c_{2,3} = 5$, we have

$$\varphi_{9,7}(\pi) = (124586397)$$

and

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\varphi_{9,7}(\pi)) = 5, \mathrm{cdes}\,(\pi) = 2, \mathrm{cdes}\,(\varphi_{9,7}(\pi)) = 3.$$

If $c_{k-1,2} > c_{k,j-1}$, then let

$$\varphi_{n,i}(\pi) = C_1 \cdots C_{k-2} \cdot (c_{k-1,1}, c_{k1}, \ldots, c_{k,j-2}, c_{kj} \ldots, c_{ks}, c_{k,j-1}, c_{k-1,2}, \ldots, c_{k-1,t}),$$

and so we have

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\varphi_{n,i}(\pi)) \text{ and } \mathrm{cdes}\,(\pi) = \mathrm{cdes}\,(\varphi_{n,i}(\pi)) - 1.$$

For example, we consider a permutation $\pi = (1793)(24586) \in \mathcal{D}_{9,3}$. The largest number in the cycle (24586) is 8, and so $j = 4$. Since $c_{1,2} = 7 > c_{2,3} = 5$, we have

$$\varphi_{9,3}(\pi) = (124865793)$$

and

$$\mathrm{exc}\,(\pi) = \mathrm{exc}\,(\varphi_{9,3}(\pi)) = 5, \mathrm{cdes}\,(\pi) = 2, \mathrm{cdes}\,(\varphi_{9,3}(\pi)) = 3.$$

Case 3. $\mathrm{red}\,(C_k) \neq (1, 2, \ldots, r-1, s, s-1, \ldots, r)$ for any $r = 2, 3, \ldots, s$.

There exists a unique index $\tilde{s}$ such that

$$\mathrm{red}\,(c_{k1}, c_{k2}, \ldots, c_{k\tilde{s}}) = 1, 2, \ldots, r-1, \tilde{s}, \tilde{s}-1, \ldots, r$$

for some $r = 2, 3, \ldots, \tilde{s}$ and

$$\mathrm{red}\,(c_{k1}, c_{k2}, \ldots, c_{k,\tilde{s}+1}) \neq 1, 2, \ldots, \tilde{r}-1, \tilde{s}+1, \tilde{s}, \ldots, \tilde{r}$$

for any $\tilde{r} = 2, 3, \ldots, \tilde{s}+1$. It is easy to check $3 \leqslant \tilde{s} \leqslant s-1$. Moreover, suppose that c_{kj} is the largest number in the set $\{c_{k1}, c_{k2}, \ldots, c_{k\tilde{s}}\}$. Then we have

$$c_{k,\tilde{s}+1} < c_{k,j-1} \text{ or } c_{k,\tilde{s}+1} > c_{k\tilde{s}}.$$

If $c_{k,\tilde{s}+1} < c_{k,j-1}$ then

$$\varphi_{n,i}(\pi) = C_1 \cdots C_{k-2} \cdot (c_{k1}, c_{k,\tilde{s}+1}, \ldots, c_{ks}) \cdot (c_{k2}, \ldots, c_{k\tilde{s}}),$$

we have

$$\operatorname{exc}(\pi) = \operatorname{exc}(\varphi_{n,i}(\pi)) \text{ and } \operatorname{cdes}(\pi) = \operatorname{cdes}(\varphi_{n,i}(\pi)) + 1.$$

For example, we consider a permutation $\pi = (124586397) \in \mathcal{D}_{9,7}$. Then $\tilde{s} = 6$ and the largest number in the set $\{1, 2, 4, 5, 8, 6\}$ is 8, and so $j = 5$. Since $c_{1,7} = 3 > c_{1,4} = 5$, we have

$$\varphi_{9,7}(\pi) = (1397)(24586)$$

and

$$\operatorname{exc}(\pi) = \operatorname{exc}(\varphi_{9,7}(\pi)) = 5, \operatorname{cdes}(\pi) = 3, \operatorname{cdes}(\varphi_{9,3}(\pi)) = 2.$$

If $c_{k,\tilde{s}+1} > c_{k,\tilde{s}}$, then let

$$\varphi_{n,i}(\pi) = C_1 \cdots C_{k-2} \cdot (c_{k1}, c_{k,\tilde{s}+1}, \ldots, c_{ks}) \cdot (c_{k2}, \ldots, c_{k,j-1}, c_{k,\tilde{s}}, c_{kj}, \ldots, c_{k,\tilde{s}-1}),$$

we have

$$\operatorname{exc}(\pi) = \operatorname{exc}(\varphi_{n,i}(\pi)) \text{ and } \operatorname{cdes}(\pi) = \operatorname{cdes}(\varphi_{n,i}(\pi)) + 1.$$

For example, we consider a permutation $\pi = (124865793) \in \mathcal{D}_{9,3}$. Then $\tilde{s} = 6$ and the largest number in the set $\{1, 2, 4, 8, 6, 5\}$ is 8, and so $j = 4$. Since $c_{1,7} = 7 > c_{1,6} = 5$, we have $\varphi_{9,3}(\pi) = (1793)(24586)$ and

$$\operatorname{exc}(\pi) = \operatorname{exc}(\varphi_{9,3}(\pi)) = 5, \operatorname{cdes}(\pi) = 3, \operatorname{cdes}(\varphi_{9,3}(\pi)) = 2.$$

For the case with $n = 4$, we list all π and $\varphi_{n,i}(\pi)$ in Table. 5.

$\pi \in \mathcal{D}_{4,2}$	$\varphi_{4,2}(\pi)$	$\pi \in \mathcal{D}_{4,3}$	$\varphi_{4,3}(\pi)$	$\pi \in \mathcal{D}_{4,4}$	$\varphi_{4,4}(\pi)$
(12)(34)	(1342)	(13)(24)	(1423)	(14)(23)	(1324)
(1342)	(12)(34)	(1423)	(13)(24)	(1324)	(14)(23)
(1432)	(1432)	(1243)	(1243)	(1234)	(1234)

Table. 5. Involutions $\varphi_{n,i}(\pi)$ for $n = 4$

Hence,

$$\sum_{\pi \in \mathcal{D}_{n,i}} x^{\operatorname{exc}(\pi)}(-1)^{\operatorname{cdes}(\pi)} = (-1)^{n-i}x^{i-1} + \sum_{\pi \in \mathcal{D}_{n,i} \setminus \{\sigma^i\}} x^{\operatorname{exc}(\pi)}(-1)^{\operatorname{cdes}(\pi)} = (-1)^{n-i}x^{i-1}.$$

$\square$

5 Proof of the recurrence relation (4)

Suppose that y is a positive integer. Let $\mathfrak{S}_n(y)$ denote the set of pairs $[\pi, \phi]$ such that $\pi \in \mathfrak{S}_n$ and ϕ is a map from the set $CDES(\pi)$ to the set $\{0, 1, \ldots, y-1\}$. It is easy to see that $\mathfrak{S}_n(y)$ is a subset of the wreath product of $\mathbb{Z}_y \wr \mathfrak{S}_n$ and $b_n(y, 1) = |\mathfrak{S}_n(y)|$.

For any $[\pi, \phi] \in \mathfrak{S}_{n+1}(y)$, we distinguish the following two cases:

Case 1. $\pi(1) = 1$.

Let $\tau = \pi(2) \ldots \pi(n+1)$. Then τ is a permutation defined on the set $\{2, 3, \ldots, n+1\}$ and

$$\mathrm{red}\,(\tau) \in \mathfrak{S}_n.$$

Define a map $\phi' : [n] \mapsto \{0, 1, \ldots, y-1\}$ by letting $\phi'(i) = \phi(\mathrm{red}^{-1}(i))$ for $i = 1, 2, \ldots, n$. Then

$$[\mathrm{red}\,(\tau), \phi'] \in \mathfrak{S}_n(y),$$

and so this provides the term $b_n(y, 1)$.

Case 2. $\pi(1) \neq 1$.

Let $\sigma = (1, c_1, c_2, \ldots, c_l)$ be the cycle in the standard cycle decomposition of π which contains the number 1. So, π is split into the cycle σ and a permutation τ on the set $\{1, 2, \ldots, n+1\} \setminus \{1, c_1, \ldots, c_l\}$, i.e., $\pi = \sigma \cdot \tau$. Clearly, $l \geqslant 1$ since $\pi(1) \neq 1$.

Note that there is a unique index $k \geqslant 1$ which satisfies $c_{k-1} < c_k$ and $c_k > c_{k+1} > \cdots > c_l$. For the sequence $c_k \ldots c_l$, if $\phi(c_i) = 0$ for some $k \leqslant i \leqslant l-1$ then let k' be the largest index in $\{k, k+1, \ldots, l-1\}$ such that $\phi(c_{k'}) = 0$; otherwise, $k' = k-1$. Let

$$\sigma' = (1, c_1, \ldots, c_{k'}) \text{ and } \pi' = \sigma' \cdot \tau.$$

Then π' is a permutation defined on the set $[n+1] \setminus B$, where

$$B = \{c_{k'+1}, \ldots, c_l\},$$

and

$$\mathrm{red}\,(\pi') \in \mathfrak{S}_{n+1-|B|}.$$

Define a map $\phi' : [n+1-|B|] \mapsto \{0, 1, \ldots, y-1\}$ by letting

$$\phi'(i) = \phi(\mathrm{red}^{-1}(i))$$

for any $1 \leqslant i \leqslant n+1-|B|$. Then

$$[\mathrm{red}\,(\pi'), \phi'] \in \mathfrak{S}_{n+1-|B|}(y).$$

Note that $1 \leqslant |B| \leqslant n$ and $B \setminus \{c_l\} \subseteq CDES_{n+1}(\pi)$. For any $k \leqslant i \leqslant l-1$, let $\theta(c_i) = \phi(c_i)$. Then θ is a map from the set $\{c_{k'+1}, \ldots, c_{l-1}\}$ to $\{1, 2, \ldots, y-1\}$. So there are $\binom{n}{|B|}$ ways to form the set B and $(y-1)^{|B|-1}$ ways to form the map θ. This provides the term

$$\sum_{i=1}^{n} b_{n+1-i}(y, 1) \binom{n}{i} (y-1)^{i-1}$$

Hence we derive the recurrence relation

$$\begin{aligned} b_{n+1}(y,1) &= b_n(y,1) + \sum_{i=1}^{n} b_{n+1-i}(y,1)\binom{n}{i}(y-1)^{i-1} \\ &= b_n(y,1) + \sum_{i=1}^{n} b_i(y,1)\binom{n}{i-1}(y-1)^{n-i}. \end{aligned}$$

6 Proof of the recurrence relation (5)

Clearly, we have $b_0(y,0) = 1$ and $b_1(y,0) = 0$. Suppose that y is a positive integer. Let $\mathcal{D}_n(y)$ denote the set of pairs $[\pi, \phi]$ such that $\pi \in \mathcal{D}_n$ and ϕ is a map from the set $CDES(\pi)$ to the set $\{0, 1, \cdots, y-1\}$. Hence $b_n(y,0) = |\mathcal{D}_n(y)|$.

For any $[\pi, \phi] \in \mathcal{D}_{n+1}(y)$, let $\sigma = (1, c_1, c_2, \ldots, c_l)$ be the cycle in the standard cycle decomposition of π which contains the number 1. So, π is split into the cycle σ and a permutation τ on the set $[n+1] \setminus \{1, c_1, \ldots, c_l\}$, i.e., $\pi = \sigma \cdot \tau$. Clearly, $l \geqslant 1$ since $\pi(1) \neq 1$.

Note that there is a unique index $k \geqslant 1$ which satisfies $c_{k-1} < c_k$ and $c_k > c_{k+1} > \cdots > c_l$. For the sequence $c_k \ldots c_l$, if $\phi(c_i) = 0$ for some $k \leqslant i \leqslant l-1$ then let k' be the largest index in $\{k, k+1, \ldots, l-1\}$ such that $\phi(c_{k'}) = 0$; otherwise, $k' = k-1$.

We distinguish between the following two cases:

Case 1. $k' = 0$.
Let

$$B = \{c_1, \ldots, c_l\}.$$

Note that τ is a permutation defined on the set $[n+1] \setminus \{1, c_1, \ldots, c_l\}$ and

$$\mathrm{red}\,(\tau) \in \mathfrak{S}_{n-|B|}.$$

Define a map $\phi' : [n-|B|] \mapsto \{0, 1, \ldots, y-1\}$ by letting

$$\phi'(i) = \phi(\mathrm{red}^{-1}(i))$$

for any $1 \leqslant i \leqslant n-|B|$. Then

$$[\mathrm{red}\,(\tau), \phi'] \in \mathfrak{S}_{n-|B|}(y)$$

and there are $b_{n-|B|}(y,0)$ ways to form the pairs $[\mathrm{red}\,(\tau), \phi']$.

Note that $1 \leqslant |B| \leqslant n$ and $B \setminus \{c_l\} \subseteq CDES_{n+1}(\pi)$. For any $k \leqslant i \leqslant l-1$, let $\theta(c_i) = \phi(c_i)$. Then θ is a map from the set $\{c_k, \ldots, c_{l-1}\}$ to $\{1, 2, \ldots, y-1\}$. So there are $\binom{n}{|B|}$ ways to form the set B and $(y-1)^{|B|-1}$ ways to form the mapping θ.

This provides the term

$$\sum_{i=1}^{n} b_{n-i}(y,0)\binom{n}{i}(y-1)^{i-1}.$$

Case 2. $k' \geqslant 1$.

Let

$$\sigma' = (1, c_1, \ldots, c_{k'}) \text{ and } \pi' = \sigma' \cdot \tau.$$

Then π' is a permutation defined on the set $[n+1] \setminus B$, where

$$B = \{c_{k'+1}, \ldots, c_l\},$$

and

$$\mathrm{red}\,(\pi') \in \mathfrak{S}_{n+1-|B|}.$$

Define a map $\phi' : [n+1-|B|] \mapsto \{0, 1, \ldots, y-1\}$ by letting

$$\phi'(i) = \phi(\mathrm{red}^{-1}(i))$$

for any $1 \leqslant i \leqslant n+1-|B|$. Then

$$[\mathrm{red}\,(\pi'), \phi'] \in \mathcal{D}_{n+1-|B|}(y)$$

and there are $b_{n+1-|B|}(y, 0)$ ways to form the pairs $(\mathrm{red}\,(\pi'), \phi')$.

Note that $1 \leqslant |B| \leqslant n-1$ and $B \setminus \{c_l\} \subseteq CDES(\pi)$. For any $k \leqslant i \leqslant l-1$, let $\theta(c_i) = \phi(c_i)$. Then θ is a map from the set $\{c_{k'+1}, \ldots, c_{l-1}\}$ to $\{1, 2, \ldots, y-1\}$. So there are $\binom{n}{|B|}$ ways to form the set B and $(y-1)^{|B|-1}$ ways to form the map θ.

This provides the term

$$\sum_{i=1}^{n-1} b_{n+1-i}(y, 0) \binom{n}{i} (y-1)^{i-1}.$$

Hence we have

$$b_{n+1}(y, 0) \;=\; \sum_{i=1}^{n} \binom{n}{i} b_{n-i}(y, 0)(y-1)^{i-1} + \sum_{i=1}^{n-1} \binom{n}{i} b_{n+1-i}(y, 0)(y-1)^{i-1}.$$

7 Proof of Theorem 9

Lemma 13. *There is a bijection Θ_n from the set of cyclic negative cycle descent permutations of $[n]$ to the set of connected Callan perfect matchings of $\mathbb{P}_n$.*

Proof. Let (π, ϕ) be a cyclic negative cycle descent permutation of $[n]$. Then there is exactly one cycle C in the standard cycle decomposition of π. Suppose $C = (c_1, c_2, \ldots, c_n)$, where $c_1 = 1$. Erase the parentheses, draw a bar after each element c_i which has sign $+1$, and add a bar before c_1. Regard the numbers between two consecutive bars as "blocks". So, we decompose (π, ϕ) into a sequence of blocks

$$B_1, B_2, \ldots, B_k.$$

Suppose that the i-th block B_i contains t_i number $b_{i1}, \ldots, b_{it_i}$ with $b_{i1} > \cdots > b_{it_i}$. We construct a perfect matchings M as follows:

- Step 1. For every block B_i, we connect the vertex $(b_{i,j}, 0)$ to the vertex $(b_{i,j+1}, 1)$ as a downline of M for any $1 \leqslant j \leqslant t_i - 1$.
- Step 2. For any odd integer $i \in \{1, 2, \ldots, k-1\}$, we connect the vertex $(b_{i,t_i}, 0)$ to the vertex $(b_{i+1,t_{i+1}}, 0)$ as an arc of M. For any even integer $i \in \{1, 2, \ldots, k-1\}$, we connect the vertex $(b_{i,1}, 1)$ to the vertex $(b_{i+1,1}, 1)$ as an arc of M.
- Step 3. If k is odd, we connect the vertex $(b_{1,1}, 1) = (1, 1)$ to the vertex $(b_{k,t_k}, 0)$ as a downline of M; otherwise, connect the vertex $(b_{1,1}, 1) = (1, 1)$ to the vertex $(b_{k1}, 1)$ as an arc of M.

It is easy to check that M is connected and has no uplines. So, M is a connected Callan perfect matching. Define Θ_n as a map from the set of cyclic negative cycle descent permutations of $[n]$ to the set of connected Callan perfect matchings of $\mathbb{P}_n$ by letting $\Theta_n(\pi, \phi) = M$. Let (π, ϕ) and (π', ϕ') be two different cyclic negative cycle descent permutations of $[n]$. Then the sequence of blocks of (π, ϕ) and (π', ϕ') are different. This implies $\Theta_n(\pi, \phi) \neq \Theta_n(\pi', \phi')$, and so the map Θ_n is an injection.

Conversely, let M be a connected Callan perfect matching of $\mathbb{P}_n$. Delete the edge incident with the vertex $(1, 1)$ from M, identify two vertices $(i, 0)$ and $(i, 1)$ in M as a new vertex i for each $i = 1, 2, \ldots, n$, denote by $\mathcal{G}^*(M)$ the graph obtained from M. Then the graph $\mathcal{G}^*(M)$ is a path on the vertex set $[n]$ and can be written as

$$a_1 a_2 \cdots a_n,$$

where $a_1 = 1$ and the set $\{a_1a_2, a_3a_4, \ldots, a_{n-1}a_n\}$ is the edge set of $\mathcal{G}^*(M)$. Draw a bar after each number a_i which satisfies either (1) $i = n$ or (2) there is an arc of M in

$$\{\{(a_i, 0), (a_{i+1}, 0)\}, \{(a_i, 1), (a_{i+1}, 1)\}\},$$

and add a bar before a_1. Regard the numbers between two consecutive bars as "blocks". So, we obtain a sequence of blocks

$$B'_1, B'_2, \ldots, B'_k.$$

We construct a cyclic negative cycle descent permutations (π, ϕ) of $[n]$ as follows:

- Step 1′. For each block B'_i, we write the numbers in B'_i in decreasing order, denote by τ_i the obtained sequence, and let $\pi = (\tau_1, \tau_2, \ldots, \tau_k)$.
- Step 2′. For any number $j \in [n]$, suppose j is in a block B'_i for some $1 \leqslant i \leqslant k$. If j is the smallest number in B'_i, then let the sign of j be $+1$; otherwise, let the sign of j be -1. In fact, this defines a map ϕ from $[n]$ to $\{+1, -1\}$.

Then (π, ϕ) is a cyclic negative cycle descent permutation of $[n]$. □

Example 14. *Let us consider a cyclic negative cycle descent permutation*

$$(1^+6^-4^-3^+2^+8^-7^-5^+)$$

on the set $\{1, 2, \ldots, 8\}$*. We erase the parentheses, draw a bar after each element which has sign* $+1$*, and add a bar before* 1*. Thus we obtain*

$$|1|643|2|875|$$

and the sequence of blocks

$$B_1 = 1, B_2 = 643, B_3 = 2, B_4 = 875.$$

By Steps 1,2,and 3 in the proof of Lemma 13, we construct the following dot diagram.

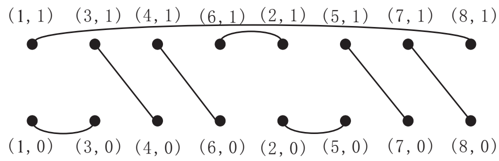


Fig.4. A dot diagram constructed by Step 1,2, and 3 in the proof of Lemma 13

Finally, we obtain a connected Callan perfect matching M *corresponding with*

$$(1^+6^-4^-3^+2^+8^-7^-5^+)$$

as follows:

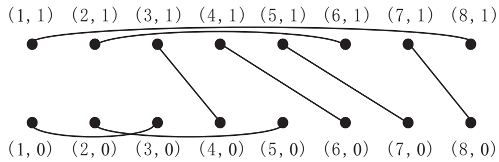


Fig.5. A connected Callan perfect matching M *corresponding with* $(1^+6^-4^-3^+2^+8^-7^-5^+)$

Conversely, let us consider the connected perfect matching M *in Fig.5. After deleting the edge* $\{(1,1),(8,1)\}$*, we can obtain the graph* $\mathcal{G}^*(M) = 13462578$*, which has the edge set* $\{13, 34, 46, 62, 25, 57, 78\}$*. Note that there are* 3 *arcs*

$$\{(1,0),(3,0)\},\{(6,1),(2,1)\},\{(2,0),(5,0)\}$$

in M*. So, we draw bars after the numbers* $1, 6, 2, 8$*, and add a bar before* 1*. Thus we obtain*

$$|1|346|2|578|$$

and the sequence of blocks

$$B'_1 = 1, B'_2 = 346, B'_3 = 2, B'_4 = 578.$$

By Steps $1'$ *and* $2'$ *in the proof of Lemma 13, we construct a cyclic negative cycle descent permutation* $(1^+6^-4^-3^+2^+8^-7^-5^+)$ *on the set* $\{1, 2, \ldots, 8\}$*.*

A bijective proof of Theorem 9

Proof. Let (π, ϕ) be a negative cycle descent permutation of $[n]$. Suppose that $\pi = C_1 \cdots C_k$ is the standard cycle decomposition of π and

$$C_i = (c_{i1}, \ldots, c_{i,l_i})$$

for each $i = 1, 2, \ldots, k$. Then $\mathrm{red}\,(C_i) \in \mathfrak{S}_{l_i}$. Define a map $\phi^i : [l_i] \mapsto \{+1, -1\}$ by letting

$$\phi^i(j) = \phi(\mathrm{red}^{\,-1}(j)),$$

i.e., the sign of $\mathrm{red}\,(c_{ij})$ is the same as that of c_{ij}. Then

$$(\mathrm{red}\,(C_i), \phi^i)$$

is a cyclic negative cycle descent permutation of $[l_i]$. By Lemma 13, $\Theta_{l_i}(\mathrm{red}\,(C_i), \phi^i)$ is a connected Callan perfect matching. For any $1 \leqslant j \leqslant l_i$, we replace the labels $(j, 0)$ and $(j, 1)$ of vertices in $\Theta_{l_i}(\mathrm{red}\,(C_i), \phi^i)$ with $(\mathrm{red}^{\,-1}(j), 0)$ and $(\mathrm{red}^{\,-1}(j), 1)$ respectively and denote by M^i the perfect matching obtained from $\Theta_{l_i}(\mathrm{red}\,(C_i), \phi^i)$. At last, let

$$M = M^1 \cup M^2 \cup \cdots \cup M^k,$$

where the notation $M \cup M'$ denotes the union of two perfect matchings M and M' such that the vertex set of $M \cup M'$ is $V(M) \cup V(M')$ and the edge set of $M \cup M'$ is $E(M) \cup E(M')$. So M is a Callan perfect matching of $\mathbb{P}_n$. Define Γ_n as a map from the set of negative cycle descent permutations of $[n]$ to the set of Callan perfect matchings of $\mathbb{P}_n$ by letting $\Gamma_n(\pi, \phi) = M$. Note that Γ_n is injective, and so it is a bijection.

By the definition of Γ_n, it is easy to see that

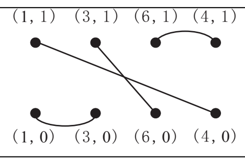
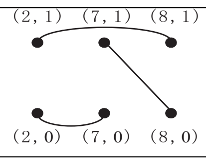
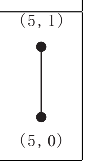
$$\mathrm{com}\,(\Gamma_n(\pi, \phi)) = \mathrm{cyc}\,(\pi) \text{ and } \mathrm{ver}\,(\Gamma_n(\pi, \phi)) = \mathrm{fix}\,(\pi).$$

If the vertices $(1, 1)$ and its partner are in the same row, then $\mathrm{down}\,(\Gamma_n(\pi, \phi)) = \mathrm{neg}\,(\pi, \phi)$; otherwise, $\mathrm{down}\,(\Gamma_n(\pi, \phi)) = \mathrm{neg}\,(\pi, \phi) + 1$. ☐

Example 15. *Let us consider a negative cycle descent permutation*

$$(1^+6^-3^+4^+)(2^+8^-7^+)(5^+)$$

of the set $\{1, 2, \ldots, 8\}$. We draw the perfect matchings M^1, M^2 and M^3 corresponding with the cycles C_1, C_2 and C_3 respectively as follows:

<table>
<tr><td>Cycles</td><td>C_1</td><td>C_2</td><td>C_3</td></tr>
<tr><td></td><td>$(1^+6^-3^+4^+)$</td><td>$(2^+8^-7^+)$</td><td>(5^+)</td></tr>
<tr><td>Perfect matchings</td><td>M^1</td><td>M^2</td><td>M^3</td></tr>
<tr><td></td><td>
</td><td>
</td><td>
</td></tr>
</table>

Finally, we obtain a Callan perfect matching $M = M^1 \cup M^2 \cup M^3$, which is given in Example 7.

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