Counting outerplanar maps

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Abstract

A map is outerplanar if all its vertices lie in the outer face. We enumerate various classes of rooted outerplanar maps with respect to the number of edges and vertices. The proofs involve several bijections with lattice paths. As a consequence of our results, we obtain an efficient scheme for encoding simple outerplanar maps.

1 Statement of results

A map is a connected planar multigraph with a specific embedding in the 2-sphere, up to oriented homeomorphisms [5]. All maps we consider are rooted: a root edge is selected and oriented in one of the two possible directions. By convention, when represented in the plane the drawing is such that the face to the right of the oriented root edge is the outer face. The root vertex is the tail of the root edge. A rooted map is outerplanar if all its vertices are in the outer face.

Our main goal is to count outerplanar maps according to the number of edges or vertices, subject to various conditions. The main tool is to set up bijections with several kinds of lattice paths. A Dyck path is a lattice path from $(0,0)$ to $(2n,0)$ using steps $U=(1,1)$ and $D=(1,-1)$ that never goes below the x axis. A *peak* in a Dyck path is a sequence UD, and a *valley* is a sequence DU. Considering a Dyck path as a well-formed system of parentheses, every U step has its *matching* D step. It is well known that the number of Dyck paths of length $2n$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. We also deal with Dyck paths in which every down step can either be marked or not (equivalently, can be coloured in one of two colours). Clearly the number of such paths is $2^n C_n$.

A Schröder path is a lattice path from $(0,0)$ to $(2n,0)$ using steps U, D and $H=(2,0)$, that never goes below the x axis. A Schröder path is *small* if it has no H steps at level zero.

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The number of Schröder paths of length $2n$ is the Schröder number r_n. It is well-known that the number s_n of small Schröder paths satisfies $2s_n = r_n$, for $n \geqslant 1$ (see [4]). We also need the following result: it seems to be folklore, but not being able to find a suitable reference we provide a short proof of it.

Lemma 1. *The number of small Schröder paths of length $2n$ with m horizontal steps is equal to*

$$\frac{1}{n}\binom{n}{m}\binom{2n-m}{n+1}, \qquad 0 \leqslant m \leqslant n-1. \tag{1}$$

In this paper we prove the following results, which to the best of our knowledge are new. A map is loopless if it has no loops, and is simple if it is loopless and has no multiple edges.

Theorem 2. *The number of outerplanar maps with n edges is equal to $2^n C_n$, for $n \geqslant 1$. The number of outerplanar maps with n edges and k vertices is equal to*

$$\binom{n}{k-1} C_n = \frac{1}{n+1}\binom{n}{k-1}\binom{2n}{n}, \qquad 1 \leqslant k \leqslant n+1.$$

Theorem 3. *The number of loopless outerplanar maps with n edges is equal to the small Schröder number s_n. The number of loopless outerplanar maps with n edges and k vertices is equal to*[1]

$$\frac{1}{k-1}\binom{n-1}{k-2}\binom{n+k-1}{k-2}, \qquad 2 \leqslant k \leqslant n+1.$$

Theorem 4. *The number of simple outerplanar maps with n edges and k vertices is equal to*[2]

$$\frac{1}{n}\binom{n}{k-1}\binom{2k-2}{n+1}, \qquad k-1 \leqslant n \leqslant 2k-3.$$

The extreme values for n are Catalan numbers and correspond to trees ($n = k-1$) and to polygon triangulations ($n = 2k-3$).

Finally we show a bijection between simple outerplanar maps and certain Dyck paths.

Theorem 5. *There is a bijection between simple outerplanar maps and the set of Dyck paths in which every U step at non-zero level can be marked or not. The bijection takes simple outerplanar maps with n edges and k vertices to Dyck paths of length $2(k-1)$ with $n-k+1$ marked steps.*

As a consequence of the previous result we obtain an optimal encoding for simple outerplanar maps having k vertices with $3k$ bits, $2k$ bits for encoding the Dyck path and k additional bits for marking the U steps. Such an encoding was obtained earlier in [2]

[1]Sequence A033282 in OEIS

[2]Sequence A033282 in OEIS, called Borel's triangle.

using a rather involved argument, based on selecting a special spanning tree of the map and encoding the additional edges. Our encoding is much simpler and easier to compute.

It is known [1] that the number of (not embedded) connected unlabelled outerplanar graphs with k vertices grows (up to a polynomial term) like γ^k, where $\gamma \approx 7.5$. It follows that $(\log_2 \gamma)k \approx 2.91 \cdot k$ bits are needed in any case to encode such a graph. Hence our scheme provides an encoding with $3k$ bits of unlabelled connected outerplanar graphs by fixing a particular embedding, which is close from being optimal.

We notice that outerplanar graphs have been studied too from a metric point point of view: it is shown in [3] that, scaled by a factor $1/\sqrt{n}$, a random planar map converges to Aldous' Continuum Random Tree.

2 Main bijection

Proof of Theorem 2. We set up a bijection between outerplanar maps with n edges and Dyck paths of length $2n$, in which every down step can be marked or not. Moreover, it will follow from the bijection that maps with k vertices correspond to paths with $k-1$ marks, hence their number is $\binom{n}{k-1}C_n$.

Given a map with n edges we start at the root vertex and move counterclockwise around its incident edges starting at the root edge (refer to Figure 1). Every time we meet a new edge we draw a U step, and every time we meet a previously visited edge we draw a D step. When we finish visiting all the edges incident to the first vertex we move to the last of the new visited vertices v. We repeat the process using as initial edge e the last one previously visited incident to v; in order to record the information that we are visiting a new vertex we mark the D step associated to the second visit to e. We then continue to similarly process the remaining vertices, always choosing the last visited vertex that has not previously been processed and using the last visited edge into that vertex. Since the map is connected all vertices will get processed. Since every edge appears twice (once for each incident vertex, which may be the same if the edge is a loop), the path has $2n$ edges and it cannot go below the x axis, otherwise we would have visited more edges for the second time than for the first time. Notice also that the number of vertices is one (the root vertex) plus the number of marks in D steps. This gives a one-to-one mapping from outerplanar maps to Dyck paths with marked D steps. We define next the inverse mapping.

Given a marked Dyck path we construct an outerplanar map as follows. We start by drawing open half-edges at the root vertex (meaning that we still do not know their endpoints, except those closed by D steps) in counterclockwise order. Whenever we find a marked D step we draw a new vertex in the last open edge and repeat the process in this new vertex. The only potential ambiguity in this process is when there is more than one possibility for closing an edge (more than one open edge). In this case we must close the last one that has been opened: otherwise, we would create a cycle with open edges inside producing a non-outerplanar map since these edges have to end in some new vertices still undiscovered. Thus, we can reconstruct the map from its marked Dyck path. □

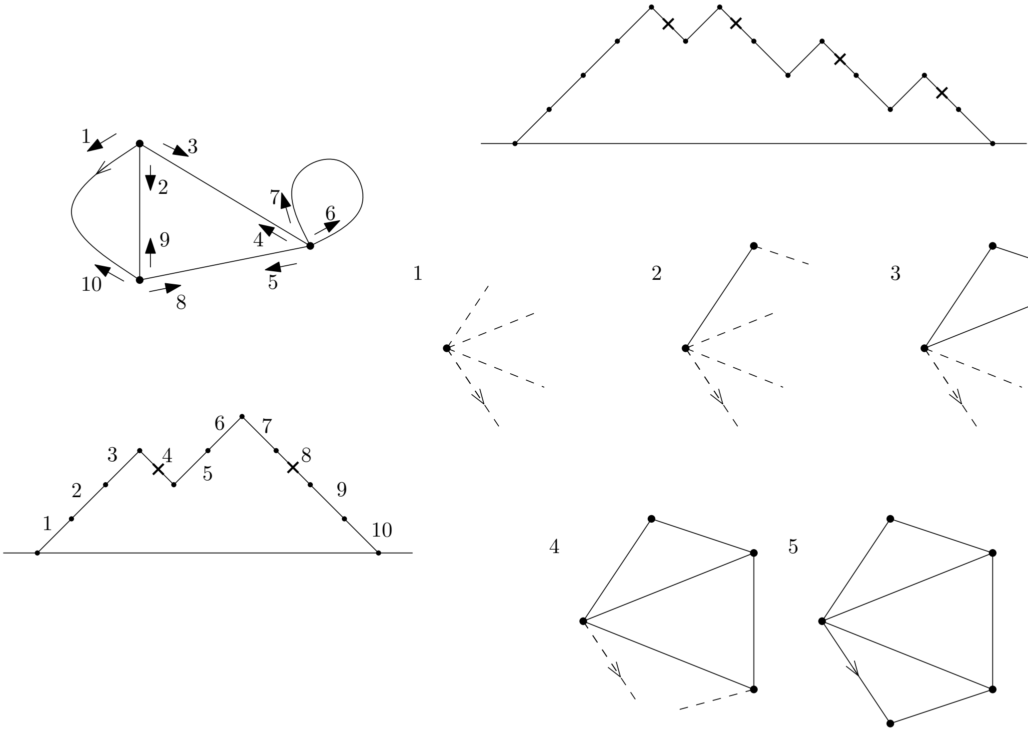

Figure 1: Illustrating the proof of Theorem 2. On the left a map is transformed into a Dyck path. On the right is the reverse mapping.

3 Remaining proofs

Proof of Lemma 1. Let $S(z)$ be the generating function of Schröder paths, where z marks the half-length of the path. Smilarly, let $R(z)$ be the generating function of small Schröder paths, that is, without H steps at level 0. Let u mark H steps. The standard decomposition of Schröder paths according to wether the first step is H or U gives the equation

$$S(z) = uz(1 + S(z)) + z(1 + S(z))^2.$$

Smilarly

$$R(z) = z(1 + S(z))(1 + T(z)).$$

Eliminating S gives

$$R(z) = z(1 + R(z))(1 + R(z) + uR(z)).$$

Lagrange inversion gives directly

$$[z^n]R(z) = \frac{1}{n}[t^{n-1}] \sum_m \binom{n}{m} u^m t^m (1 + t)^{2n-m}.$$

Hence the number of small Schöreder paths of length $2n$ with m H steps (at positive levels) is

$$\frac{1}{n}\binom{n}{m}\binom{2n-m}{n-1-m} = \frac{1}{n}\binom{n}{m}\binom{2n-m}{n+1}.$$

□

Proof of Theorem 3. We use the bijection in the proof of Theorem 2. A loop is created when a vertex first opens and then closes an edge without moving to another vertex. Hence, an outerplanar map is loopless if and only if in the associated path all the D steps belonging to peaks are marked. We construct a bijection between marked Dyck paths with this property and small Schröder paths.

Given a Dyck path in which all peaks are marked, we apply a *reversion* to each maximal descending interval (maximal sequence of contiguous D steps). If we label the steps of a maximal descending interval as $a_0, a_1, \ldots, a_l$, a reversion consists in swapping a_m with a_{l-m} for all $m < l/2$, as illustrated in Figure 3. Then, for every pair of matching U and D steps, if the D step is marked do nothing, and if it is not remove the D step and replace the matching U step by an H step. Since every D step at level zero is marked, no horizontal steps are produced at level zero and the result is a small Schröder path. This gives a one-to-one mapping from loopless outerplanar maps to small Schröder paths.

We construct the inverse mapping as follows. Given a small Schröder path we first mark all the D steps. Then for every H step we find the first D step σ that goes below the level of H, which exists because the H step is not at zero level. We remove the H step, insert a U step where the H step was, and a D step without mark just before σ. In other words, if the path is of the form $\alpha H \beta D^* \gamma$, where β is a small Schröder path and D^* is a marked D step, then we replace it by $\alpha U \beta D D^* \gamma$. Finally, we apply a reversion on every maximal descending interval. This completes the bijection. □

Proof of Theorem 4. Suppose a marked Dyck path corresponds to a simple outerplanar map. By Theorem 3, all D steps at peaks are marked. If a D step is not marked then, since it is not a peak, it is preceded by another D step. If we consider their matching U steps in the Dyck path, they must belong to two different ascents, otherwise they would close edges from the same vertex and the map would have a multiple edge. In other words, whenever a new ascent begins, say at level $k > 0$, we may or may not mark the first D step that goes below level k. These are the only D steps which we are allowed not to mark, which we call *special*. Thus a special D step is the second step in any contiguous DD pair whose matching U steps are not contiguous.

Thus our task is to count Dyck paths of length $2n$ with $k-1$ marked D steps (so that that the number of vertices is k), such that all D steps are marked, except possibly the special ones. Let $A_{n,k}$ be the number of such paths of length $2n$ with k marked D steps, and let $A(z,u) = \sum A_{n,k} u^k z^n$ be the associated generating function. We claim that $A(z,u)$ satisfies

$$A(z,u) = uz(1 + A(z,u))^2 + z(A(z,u))^2.$$

Let $\pi = U \alpha D \beta$ be the canonical decomposition of a Dyck path, where the distinguished D step if the first one at level zero. If D is marked, then we have no restriction on α and β and this gives the term $uz(1+A)^2$ (we add 1 to A to allow for the empty path). If D is not marked then it is special and we must have $\alpha = \gamma\delta$, where δ starts at the first valley

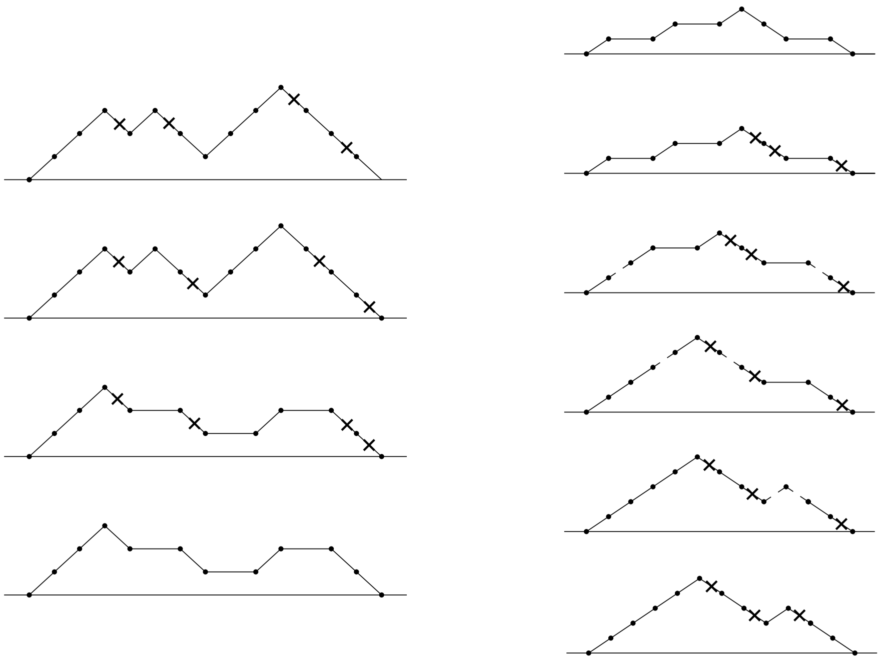

Figure 2: Illustrating the proof of Theorem 3. On the left a Dyck path is transformed into a Schröder path. On the right is the reverse mapping.

at level one before D. Then we can encode $\pi = U\gamma\delta D\beta$ uniquely as the ordered pair of paths $(\gamma, \delta\beta)$, both non-empty; this accounts for the term zA^2.

Rewriting the former equation as $A = z(u(1+A)^2+A^2)$ and applying Lagrange inversion we obtain

$$A_{n,k} = [u^k]\frac{1}{n}[t^{n-1}](u(1+t)^2+t^2)^n = \frac{1}{n}\binom{n}{k}\binom{2k}{n+1}.$$

Replacing k by $k-1$ gives the result as claimed. □

Proof of Theorem 5. Let M be a simple outerplanar map and let π be its associated Dyck path. By Theorems 3 and 4, every non-special D step is marked. Let us recall that every valley at positive level gives rise to exactly one special D step. Several valleys at the same level can produce the same special step: if this is the case then we say that the valley that appears first *corresponds* to the special step.

We read π from the left, and every time we find a non-marked D step, we swap it with the D step of its corresponding valley (refer to Figure 3). We read again π from the left and whenever we find a non-marked D step (necessarily in a valley), we erase it along with its matching U step, and we mark the U step in the corresponding valley. In other words, if the path is of the form $\alpha U\beta DU\gamma$, where D is not marked, we replace it with $\alpha\beta U^*\gamma$, where U^* is marked. After these two operations, we obtain a Dyck path with all the D steps marked, and some marked U steps. Since none of the D steps at level 0 in π was marked, none of the U steps at level 0 in the new paths are marked neither. Hence, the resulting Dyck path will be of length $2k$, where k is the number of marked D steps in π, and none of the U steps at level 0 are marked.

To prove that this is a bijection we show how to reconstruct the original path π. Given a Dyck path with marked U steps at non-zero level, we read it from the left. Whenever we find a marked U step it means that just before it there was a non-marked D step x in π, say at level $k \geqslant 1$. It remains to identify where is its corresponding U step y. Between two consecutive points at level k there is a path with a $\cup$ form, which we call an A-path, or with a $\cap$ form, which we call a B-path. If we start at x and read from right to left, the U step is at the beginning of an A-path, or at the last possible point at level k. In other words, if the path is of the form $\alpha A B_1 B_2 \dots B_n U^* \beta$, then it will remain as $\alpha A U B_1 B_2 \dots B_n D U \beta$. This is because if we place an A-path in between, there will be steps at level k between x and y, contradicting the assumption that they were corresponding; and if we do not place all possible B-paths in between, then y will form another valley at level k previous to the one where x is, contradicting the assumption that if two valleys are at the same level we choose the first one corresponding to the special step. Hence the inverse is uniquely determined and our mapping is a bijection. $\square$

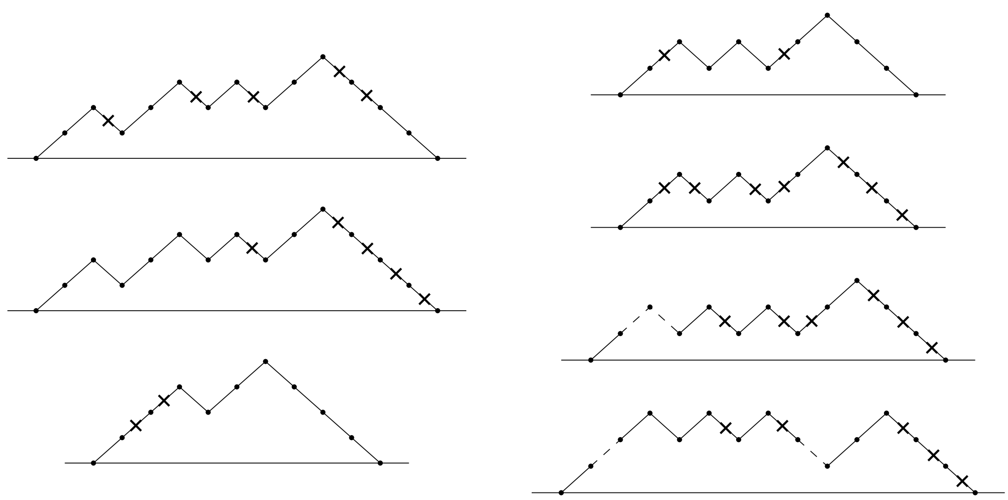

Figure 3: Illustrating the proof of Theorem 5. On the left a Dyck path with marked D steps is transformed into another Dyck path with marked U steps, whose length depends on the number of marks in the original path. On the right the inverse mapping.

To conclude, we provide a table with the numbers of small outerplanar maps with given number of edges or vertices, together with references to the Online Encyclopedia of Integer Sequences.

Numer of edges	1	2	3	4	5	6	OEIS
All maps	2	8	40	224	1344	8448	A052701
Loopless	1	3	11	45	197	903	A001003
Simple	1	2	6	20	72	272	A071356
Number of vertices	1	2	3	4	5	6	
Simple	1	1	3	13	67	381	A064062

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