

Top Degree Part in b -Conjecture for Unicellular Bipartite Maps

Maciej Dołęga*

Wydział Matematyki i Informatyki
Uniwersytet im. Adama Mickiewicza
Poznań, Poland

Instytut Matematyczny
Uniwersytet Wrocławski
Wrocław, Poland

`maciej.dolega@amu.edu.pl`

Submitted: May 10, 2016; Accepted: May 30, 2017; Published: Aug 11, 2017

Mathematics Subject Classifications: Primary 05C10; Secondary 05E05, 05C30, 20C30

Abstract

Goulden and Jackson (1996) introduced, using Jack symmetric functions, some multivariate generating series $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; 1, 1 + \beta)$ with an additional parameter β that may be interpreted as a continuous deformation of the rooted bipartite maps generating series. Indeed, it has the property that for $\beta \in \{0, 1\}$, it specializes to the rooted, orientable (general, i.e. orientable or not, respectively) bipartite maps generating series. They made the following conjecture: coefficients of ψ are polynomials in β with positive integer coefficients that can be written as a multivariate generating series of rooted, general bipartite maps, where the exponent of β is an integer-valued statistic that in some sense “measures the non-orientability” of the corresponding bipartite map.

We show that except for two special values of $\beta = 0, 1$ for which the combinatorial interpretation of the coefficients of ψ is known, there exists a third special value $\beta = -1$ for which the coefficients of ψ indexed by two partitions μ, ν , and one partition with only one part are given by rooted, orientable bipartite maps with arbitrary face degrees and black/white vertex degrees given by μ/ν , respectively. We show that this evaluation corresponds, up to a sign, to a top-degree part of the coefficients of ψ . As a consequence, we introduce a collection of integer-valued statistics of maps (η) such that the top-degree of the multivariate generating series of rooted bipartite maps with only one face (called *unicellular*) with respect to η gives the top-degree of the appropriate coefficients of ψ . Finally, we show that the b -conjecture holds true for all rooted, unicellular bipartite maps of genus at most 2.

Keywords: Map enumeration; Jack symmetric functions; b -conjecture

*Supported from *NCN*, grant UMO-2015/16/S/ST1/00420.

1 Introduction

1.1 Maps

Roughly speaking, a *map* is a graph drawn on a certain topological surface such as the sphere or the Klein bottle (see Section 2.3 for precise definitions). This simple object carries both combinatorial and geometric informations so it turned out that maps appear naturally in many different contexts. In particular, they have deep connections with various branches of discrete mathematics, algebra, or physics (see e.g. [18, 7] and references therein). One of the major steps in the study of maps is developing methods for their enumeration (either by generating functions, matrix integral techniques, algebraic combinatorics, or bijective methods) which is now a well established domain on its own. Moreover, in many areas in mathematics and physics map enumeration is crucial. We refer to [20] – a great introductory text on the enumeration of maps – which shows how studying enumerative properties of maps is of great importance in many different contexts.

In this paper we will focus on an interesting connection, explored mainly by Goulden and Jackson [9], between map enumeration and symmetric functions theory. This connection led to a twenty years old open problem, called *b-conjecture*, that will be the main subject of this paper.

1.2 Enumeration of bipartite maps in terms of Jack polynomials

We define a *map* as a connected graph embedded into a *surface* (i. e. compact, connected 2-manifold without boundary) in a way that the *faces* (connected components of the complement of the graph) are simply connected. A *hypermap* (and, by duality, *bipartite map*) is a face two-colored map (vertex two-colored map), so that each edge separates faces (vertices) of different colors. A map is rooted by distinguishing *the root*, that is the unique side and the beginning of the selected edge. A rooted hypermap (bipartite map, by duality) has the black root face (vertex, respectively), by convention, where root face (vertex, respectively) is the unique face (vertex, respectively) incident to the root. An example of a bipartite map is illustrated in Fig. 1.

Let $m_{\mu,\nu}^\tau$ ($\tilde{m}_{\mu,\nu}^\tau$, respectively) be the number of rooted hypermaps on orientable (all, respectively) surfaces, such that μ lists the degrees of black faces, ν lists the degrees of white faces and τ lists the degrees divided by two of vertices (since a map is face two-colored, all vertex degrees are even numbers). By duality $m_{\mu,\nu}^\tau$ ($\tilde{m}_{\mu,\nu}^\tau$, respectively) is also the number of rooted bipartite maps on orientable (all, respectively) surfaces, such that μ lists the degrees of black vertices (we say μ is a *black vertex distribution*), ν lists the degrees of white vertices (ν is a *white vertex distribution*) and τ lists the degrees divided by two of faces (τ is a *face distribution*). As standard in enumerative combinatorics, we will consider the multivariate generating series (m.g.s, for short) for these objects:

$$M(\mathbf{x}, \mathbf{y}, \mathbf{z}; t) = \sum_{n \geq 1} t^n \sum_{|\tau|=|\mu|=|\nu|=n} m_{\mu,\nu}^\tau p_\tau(\mathbf{x}) p_\mu(\mathbf{y}) p_\nu(\mathbf{z}), \quad (1)$$

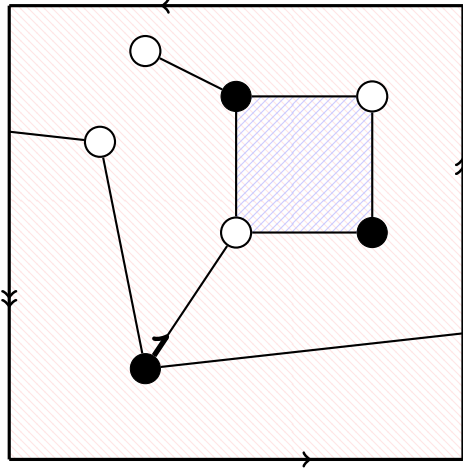


Figure 1: Example of a *rooted, bipartite non-orientable map* with the face distribution $\tau = (12, 4)$, the black vertex distribution $\nu = (3, 3, 2)$, and the white vertex distribution $\mu = (3, 2, 2, 1)$. Faces are indicated by blue and red shaded regions and the root is indicated by a thick beginning of the selected edge with a distinguished side. The map is drawn on a projective plane: the left side of the square should be glued to the right side, as well as bottom to top, as indicated by the arrows.

$$\widetilde{M}(\mathbf{x}, \mathbf{y}, \mathbf{z}; t) = \sum_{n \geq 1} t^n \sum_{|\tau|=|\mu|=|\nu|=n} \widetilde{m}_{\mu, \nu}^{\tau} p_{\tau}(\mathbf{x}) p_{\mu}(\mathbf{y}) p_{\nu}(\mathbf{z}), \quad (2)$$

where $p_{\tau}(\mathbf{x})$ is a *power-sum symmetric function*, i.e.

$$p_{\tau}(\mathbf{x}) = \prod_i p_{\tau_i}(\mathbf{x}), \quad p_k(\mathbf{x}) = x_1^k + x_2^k + \cdots \text{ for } k \geq 1,$$

and $|\mu| := \mu_1 + \mu_2 + \cdots$ denotes the size of the list μ . The use of the power-sum symmetric functions as formal variables in the above m.g.s is justified by the remarkable relation, explored by Goulden and Jackson, between bipartite maps enumeration, and symmetric functions theory. Let $J_{\lambda}^{(\alpha)}(\mathbf{x})$ be the Jack symmetric function indexed by a partition λ in the infinite alphabet \mathbf{x} and let $\langle \cdot, \cdot \rangle_{\alpha}$ be the α -deformation of the Hall scalar product on the space of symmetric functions (see Section 2.2 for a precise definition). We also use \mathcal{Y} for the set of all integer partitions. Goulden and Jackson defined in their article [9] a family of coefficients $(h_{\mu, \nu}^{\tau}(\alpha - 1))_{\mu, \nu, \tau}$ by the following equation:

$$\begin{aligned} \psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, \alpha) &:= \alpha t \frac{\partial}{\partial t} \log \left(\sum_{\lambda \in \mathcal{Y}} \frac{J_{\lambda}^{(\alpha)}(\mathbf{x}) J_{\lambda}^{(\alpha)}(\mathbf{y}) J_{\lambda}^{(\alpha)}(\mathbf{z}) t^{|\lambda|}}{\langle J_{\lambda}, J_{\lambda} \rangle_{\alpha}} \right) \\ &= \sum_{n \geq 1} t^n \left(\sum_{\mu, \nu, \tau \vdash n} h_{\mu, \nu}^{\tau}(\alpha - 1) p_{\mu}(\mathbf{x}) p_{\nu}(\mathbf{y}) p_{\tau}(\mathbf{z}) \right), \quad (3) \end{aligned}$$

where $\mu, \nu, \tau \vdash n$ means that μ, ν and τ are three partitions of size n .

This rather involved definition is motivated by the below described combinatorial interpretations of ψ for two particular values of α .

Theorem 1.1 ([11, 10]). *The following equalities of the m.g.s. hold true:*

$$M(\mathbf{x}, \mathbf{y}, \mathbf{z}; t) = \psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, 1), \quad \widetilde{M}(\mathbf{x}, \mathbf{y}, \mathbf{z}; t) = \psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, 2).$$

In other words, Theorem 1.1 says that for any partitions $\mu, \nu, \tau \vdash n$ the coefficient $h_{\mu, \nu}^{\tau}(0) = m_{\mu, \nu}^{\tau}$ counts some rooted, orientable, bipartite maps and $h_{\mu, \nu}^{\tau}(1) = \widetilde{m}_{\mu, \nu}^{\tau}$ counts some rooted, general (i.e. orientable or not), bipartite maps. Thus we may wonder whether, for a general $\beta := \alpha - 1$, the quantity $h_{\mu, \nu}^{\tau}(\beta)$ also admits a nice combinatorial description. Note that $h_{\mu, \nu}^{\tau}(\beta)$ is *a priori* a quantity depending on a parameter α , and describing it as a quantity depending on a different parameter $\beta := \alpha - 1$ might seem be artificial. However, it turned out that this shift seems to be a right one for finding a combinatorial interpretation of $h_{\mu, \nu}^{\tau}(\beta)$, as suggested by Goulden and Jackson [9] in the following conjecture.

Conjecture 1.2 (b-conjecture). *For all partitions $\tau, \mu, \nu \vdash n \geq 1$ the quantity $h_{\mu, \nu}^{\tau}(\beta)$ can be expressed as:*

$$h_{\mu, \nu}^{\tau}(\beta) = \sum_M \beta^{\eta(M)}, \quad (4)$$

where the summation index runs over all rooted, bipartite maps M with the face distribution τ , the black vertex distribution μ , the white vertex distribution ν , and where $\eta(M)$ is a nonnegative integer which is equal to 0 if and only if M is orientable.

1.3 b-conjecture and main result

The above conjecture is still to be resolved, but some progress towards determining a suitable statistic η , based on the combinatorial interpretation of the so-called *marginal sums for maps*, has been made in the last two decades. Note that there is a natural bijection between the set of rooted maps with n edges, which are not necessarily bipartite, and the set of rooted, bipartite maps with the white vertex distribution given by $\nu = (2^n)$. In particular, the following m.g.s

$$\Psi(\mathbf{x}, \mathbf{y}; t, 1 + \beta) := \sum_{n \geq 1} t^n \sum_{\mu, \tau \vdash 2n} h_{\mu, (2^n)}^{\tau}(\beta) p_{\tau}(\mathbf{x}) p_{\mu}(\mathbf{y})$$

is of special interest, as $\Psi(\mathbf{x}, \mathbf{y}; t, 1)$ is the m.g.s. for rooted orientable maps, and $\Psi(\mathbf{x}, \mathbf{y}; t, 2)$ is the m.g.s. for all rooted maps. A formula for $\Psi(\mathbf{x}, \mathbf{y}; t, 1 + \beta)$ involving the Selberg integral was found by Goulden, Harer and Jackson [8], who suggested that using their formula it is possible to find a combinatorial interpretation of the following marginal sum

$$l_{\mu}^r(\beta) = \sum_{\ell(\tau)=r} h_{\mu, (2^n)}^{\tau}(\beta)$$

in terms of a map statistic as in Eq. (4) (here $\mu, \tau \vdash 2n$, and the summation is taken over all partitions τ which have precisely r nonnegative parts; see Section 2.1 for a precise

definition). A weaker result was first established by Brown and Jackson [1], who found some statistics η of maps that describe the total marginal sum

$$k_\mu(\beta) = \sum_{r \geq 1} l_\mu^r(\beta) = \sum_{\tau \vdash 2n} h_{\mu, (2^n)}^\tau(\beta).$$

A simpler description of η was found by La Croix [14], who used it to give a combinatorial description of $l_\mu^r(\beta)$, as suggested by Goulden, Harer and Jackson.

However, not much was known about an algebraic or combinatorial structure of $h_{\mu, \nu}^\tau(\beta)$ for arbitrary partitions $\tau, \mu, \nu \vdash n$, until very recently we have proved in a joint paper with Féray [5] the following theorem:

Theorem 1.3. *For all partitions $\tau, \mu, \nu \vdash n \geq 1$ the quantity $h_{\mu, \nu}^\tau(\beta)$ is a polynomial in β of degree $2 + n - \ell(\tau) - \ell(\mu) - \ell(\nu)$ with rational coefficients.*

In this paper we are focused on a combinatorial part of b -conjecture, especially in the case of rooted, bipartite maps, with only one face (called *unicellular*). Let us fix a positive integer n , and two partitions $\mu, \nu \vdash n$. According to Conjecture 1.2, there exists some statistic η on the set of all rooted bipartite maps, such that the quantity $h_{\mu, \nu}^{(n)}(\beta)$ is given by the m.g.s. of rooted, unicellular, bipartite maps with the black (white, respectively) vertex distribution μ (ν , respectively). We show that except two special values of $\beta = 0, 1$ for which the combinatorial interpretation of $h_{\mu, \nu}^{(n)}(\beta)$ was known, there exists a third special value $\beta = -1$ for which we provide a combinatorial interpretation of $h_{\mu, \nu}^{(n)}(\beta)$. As a result we prove the following:

Theorem 1.4. *For all partitions $\mu, \nu \vdash n \geq 1$*

$$h_{\mu, \nu}^{(n)}(\beta) = \sum_{\mathcal{M}} \beta^{\eta(\mathcal{M})},$$

holds true for $\beta \in \{-1, 0, 1\}$, where the summation index runs over all rooted, bipartite unicellular maps \mathcal{M} with the black vertex distribution μ , the white vertex distribution ν , and where $\eta(\mathcal{M})$ is a nonnegative integer which is equal to 0 if and only if \mathcal{M} is orientable.

We show that the top-degree part of the polynomial $h_{\mu, \nu}^{(n)}(\beta)$ is equal, up to a sign, to its evaluation at $\beta = -1$, thus we show that it is given by some rooted, bipartite, unicellular maps with the black (white, respectively) vertex distribution μ (ν , respectively), which are called “unhandled” (the origin of this terminology will be clear later, after we define an appropriate statistic η ; see Section 3). We also show that these maps are in a bijection with rooted, *orientable*, bipartite maps with the black (white, respectively) vertex distribution μ (ν , respectively) and with the arbitrary face distribution.

Finally, we show that b -conjecture holds true for an infinite family of rooted, unicellular bipartite maps of genus at most 2:

Theorem 1.5. *For all partitions $\mu, \nu \vdash n \geq 1$ satisfying $\ell(\mu) + \ell(\nu) \geq n - 3$ and $\tau = (n)$ the b -conjecture holds true, i.e.*

$$h_{\mu, \nu}^{(n)}(\beta) = \sum_{\mathcal{M}} \beta^{\eta(\mathcal{M})},$$

where the summation index runs over all rooted, bipartite unicellular maps \mathcal{M} with the black vertex distribution μ , the white vertex distribution ν , and $\eta(\mathcal{M})$ is a nonnegative integer which is equal to 0 if and only if \mathcal{M} is orientable.

1.4 Related problems

We finish this section by mentioning two very similar problems. First, a very similar conjecture to Conjecture 1.2 was also stated by Goulden and Jackson [9] for power-sum expansion of the m.g.s. that is inside the paranthesis of \log in Eq. (3). The latter is conjecturally a m.g.s. of *matchings*, where the exponent of β is some combinatorial integer-valued statistic. Some special cases of the conjecture have been solved by Goulden and Jackson in their original article [9] and recently by Kanunnikov and Vassilieva [13, 24], but in general the conjecture is still open. The polynomiality of the coefficients of the studied m.g.s. was proven by the author of this paper together with Féray [4].

A second related problem is the investigation of *Jack characters* – suitably normalized coefficients of the power-sum symmetric function expansion of Jack polynomials. It was suggested by Lassalle that a combinatorial description of these objects might exist. This combinatorial setup was indicated by some polynomiality and positivity conjectures that he stated in a series of papers [15, 16]. Although these conjectures are not fully resolved, it was proven by us together with Śniady [6] that in some special cases bipartite maps together with some statistics that “measures their non-orientability” give the desired combinatorial setup. Even more, Śniady [22] found the top-degree part of the Jack character indexed by a single partition with respect to some gradation. His result states that this top-degree part can be written as a linear combination of certain functionals, where the index set is the set of rooted, *orientable*, bipartite maps with the arbitrary face distribution. While conjecturally it should be expressed as a linear combination of the same functionals, where the index set is a set of some special rooted, unicellular, bipartite maps. Śniady was able to find a bijection between these two index sets [21], which inspired us to investigate the combinatorial side of b -conjecture in the case of unicellular maps, presented in this paper.

Note added in revision: After submission of the current paper, the aforementioned result of Śniady appeared in a joint paper with Czyżewska-Jankowska [3].

We cannot resist stating that there must be a deep connection between all these problems, and understanding it would be of great interest.

1.5 Organization of the paper

In Section 2 we describe all necessary definitions and background. Then, we introduce a family of statistics of the maps and we study their properties in Section 3. Section 4 is

devoted to the proof of Theorem 1.4 and its consequence, which says that the family of statistics presented in the previous section describes the top-degree part of the polynomial $h_{\mu\nu}^{(n)}(\beta)$ associated with unicellular maps. In Section 5 we introduce some special subfamily of the statistics presented in Section 3, we study their properties and we give a proof of Theorem 1.5. We finish this paper by stating some concluding remarks and questions in Section 6.

2 Preliminaries

2.1 Partitions

We call $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_l)$ a *partition* of n if it is a weakly decreasing sequence of positive integers such that $\lambda_1 + \lambda_2 + \dots + \lambda_l = n$. Then n is called *the size* of λ while l is *its length*. As usual we use the notation $\lambda \vdash n$, or $|\lambda| = n$, and $\ell(\lambda) = l$. We denote the set of partitions of n by \mathcal{Y}_n and we define a partial order on \mathcal{Y}_n , called the *dominance order*, in the following way:

$$\lambda \leq \mu \iff \sum_{i \leq j} \lambda_i \leq \sum_{i \leq j} \mu_i \text{ for any positive integer } j.$$

Given two partitions $\lambda \in \mathcal{Y}_n$ and $\mu \in \mathcal{Y}_m$ we can construct a new partition $\lambda \cup \mu \in \mathcal{Y}_{n+m}$ obtained by merging parts of λ and μ and ordering them in a decreasing fashion.

2.2 Jack polynomials

In this section we recall the definition of Jack polynomials and present several known results about them. Since they are well-established (mostly in a seminal work of Stanley [23]), we do not give any proof, but explicit references.

Consider the vector space Sym of the symmetric functions Λ over the field of rational functions $\mathbb{Q}(\alpha)$ and endow it with a scalar product $\langle \cdot, \cdot \rangle_\alpha$ defined on the power-sum symmetric functions basis by the following formula (and then extended by bilinearity):

$$\langle p_\lambda, p_\mu \rangle_\alpha = z_\lambda \alpha^{\ell(\lambda)} \delta_{\lambda, \mu},$$

where

$$z_\lambda := \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!. \quad (5)$$

Here, $m_i(\lambda)$ denotes the number of parts of λ equal to i . This is a classical deformation of the *Hall scalar-product* (which corresponds to $\alpha = 1$).

Now, Jack polynomials $J_\lambda^{(\alpha)}$ are symmetric functions with an additional parameter α uniquely determined (see [19, Section VI,10]) by the following conditions:

$$(C1) \quad J_\lambda^{(\alpha)} = \sum_{\mu \leq \lambda} a_\mu^\lambda m_\mu, \text{ where } a_\mu^\lambda \in \mathbb{Q}(\alpha);$$

$$(C2) \quad [m_{1^{|\lambda|}}] J_{\lambda}^{(\alpha)} := \alpha_{1^{|\lambda|}}^{\lambda} = |\lambda|!;$$

$$(C3) \quad \langle J_{\lambda}^{(\alpha)}, J_{\mu}^{(\alpha)} \rangle_{\alpha} = 0 \text{ for } \lambda \neq \mu;$$

where m_{λ} denotes the monomial symmetric function.

2.2.1 Basic properties

We present here several well-known identities for Jack polynomials that will be useful for us later.

$$\langle J_{(n)}, J_{(n)} \rangle_{\alpha} = (1 + \alpha)(1 + 2\alpha) \cdots (1 + (n - 1)\alpha) \alpha^n n!, \quad (6)$$

$$J_{(n)}^{(\alpha)} = \sum_{\mu \vdash n} \frac{n! \alpha^{n - \ell(\mu)}}{z_{\lambda}} p_{\mu}, \quad (7)$$

$$J_{\lambda}^{(\alpha)}((t, 0, 0, \dots)) = \begin{cases} (1 + \alpha)(1 + 2\alpha) \cdots (1 + (n - 1)\alpha) t^n & \text{for } \lambda = (n), \\ 0 & \text{for } \ell(\lambda) > 1. \end{cases} \quad (8)$$

Eq. (6) and Eq. (7) are proved in [23, Proposition 2.2], and Eq. (8) is a consequence of the monomial basis expansion given in [12].

2.3 Surfaces, graphs, and maps

A *map* is an embedding of a connected graph G into a *surface* S (i.e. compact, connected, 2-dimensional manifold) in a way that the connected components of $S \setminus G$, called *faces*, are simply connected. Our *graphs* may have loops and multiple edges. Maps are always considered up to homeomorphisms. A map is *unicellular* if it has a single face. Unicellular maps are also called *one-face maps*. We will call a map *orientable* if the underlying surface is orientable; otherwise we will call it *non-orientable*. In this paper we will be mostly focused on non-orientable maps.

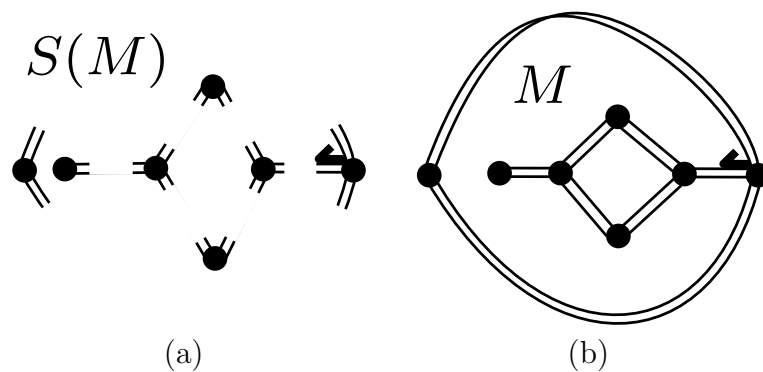


Figure 2: The map M from Fig. 1 is depicted on (b) as a ribbon graph. (a) shows stars $S(M)$ associated with M .

For the purposes of the current paper it is sometimes convenient to represent a map as a *ribbon graph* as follows: each vertex is represented as a small disc and each edge is represented by a thin strip connecting two discs in a way that a walk along the boundary of the ribbons corresponds to the walk along the boundary of the faces of a given map. To draw such a picture it is helpful to do it in the following steps: we first draw vertices (as small discs) together with thin strips attached around them, which represent associated *half-edges*. By definition *half-edges* are obtained by removing middle-points of all the edges (each edge consists of exactly two half-edges). We call this data the *stars of the map* M and we denote it by $S(M)$. See Fig. 2a for an example of $S(M)$ associated with the map M from Fig. 1. Next, for each edge e of M , we can connect the borders of the strips representing two half-edges belonging to e in two possible ways (see Fig. 5). We connect them in a way so that after connecting all the strips from $S(M)$ the walk along the boundary of the ribbons corresponds to the walk along the boundary of the faces of a given map. Fig. 2b presents the map M from Fig. 1 represented as a ribbon graph.

A map is *rooted* if it is equipped with a distinguished half-edge called the *root*, together with a distinguished side of this half-edge. The vertex incident to the root is called the *root vertex*, and the edge containing the root is called the *root edge*. There is an equivalent way to root a map by choosing a corner (called the *root corner*) and its orientation, where a *corner* in a map is an angular sector determined by a vertex, and two half-edges which are consecutive around it. One can then define the root half-edge as the one lying to the left of the root corner (viewed from the root vertex and according to the root corner orientation). In this paper we will use both conventions depending on the situation, and we will represent rooted maps by shading the root corner and/or by indicating the root that is incident to it.

The *degree of a vertex* is the number of incident half-edges, or equivalently the number of incident corners, while the *degree of a face* is the number of corners lying in that face, or equivalently the number of edges incident to it, with the convention that an edge incident to the same face on both sides counts for two.

If M is a map, we let $V(M)$, $E(M)$ and $F(M)$ be its sets of vertices, edges and faces. Their cardinalities $v(M)$, $e(M)$ and $f(M)$ satisfy the *Euler formula*:

$$e(M) = v(M) + f(M) - 2 + 2g(M), \quad (9)$$

where $g(M)$ denotes the *genus* of a map M , that is the genus of the underlying surface. We recall that the Euler characteristic of the surface is $2 - 2g(M)$, thus $g(M)$ is a nonnegative integer when M is orientable or half-integer when M is non-orientable. We also denote by $C(M)$ the set whose elements are indexed by faces of M . For a fixed face $f \in F(M)$ the associated element in $C(M)$ is the set of all corners belonging to f .

A map is *bipartite* if its vertices can be colored in two colors in such a way that adjacent vertices have different colors (say black and white). For a rooted, bipartite map, the color of its root vertex is always taken to be black, by convention.

Let μ, ν, τ be integer partitions. We say that a rooted bipartite map M has *type* $(\mu, \nu; \tau)$ if μ lists the degrees of black vertices (we say μ is the *black vertex distribution*), ν lists the degrees of white vertices (ν is the *white vertex distribution*) and τ lists the

degrees of faces divided by two (τ is the face distribution). We denote the set of rooted bipartite maps of type $(\mu, \nu; \tau)$ on orientable (all, respectively) surfaces by $\mathcal{M}_{\mu, \nu}^{\tau}$ ($\widetilde{\mathcal{M}}_{\mu, \nu}^{\tau}$, respectively). Note that all three partitions μ, ν, τ have necessarily the same size n , which is equal to the number of edges of the corresponding map, while its lengths correspond to the number of black and white vertices and the number of faces, respectively.

From now on, all the maps are rooted, and bipartite, thus by saying a “map”, what we really mean is a “rooted, bipartite map”.

3 Measure of non-orientability in b -conjecture

In this section we are going to construct a function that associates with a map M a nonnegative integer $\eta(M)$ which, in some sense, measures its non-orientability. This “measure of non-orientability” gives, in some special cases, a correct answer to b -conjecture, i.e. Eq. (4) holds true.

The construction presented in this section is due to La Croix [14] who used it to prove that the following marginal sum

$$l_{\mu}^r(\beta) = \sum_{\ell(\tau)=r} h_{\mu, (2^n)}^{\tau}(\beta)$$

can be expressed in the same form as the right hand side of Eq. (4), where the maps in the summation index are not necessarily bipartite, have r faces and the vertex distribution μ . The construction of La Croix was originally defined for all (not necessarily bipartite) maps, but in this paper we are dealing with the case of bipartite maps, thus in the following all maps will be *rooted and bipartite*.

3.1 Root-deletion procedure as measure of non-orientability

Let e be the root edge of the map M . Note that by deleting e from M we create a new map or, possibly, two new maps and we canonically choose how to root them. Recall that rooting a map is the same as distinguishing an oriented corner (called the root corner), see Section 2.3. The root corner of M is contained in the unique corner c of $M \setminus \{e\}$ and we set it as the root corner of the connected component of $M \setminus \{e\}$ containing c with an orientation inherited from the root corner of M . In the case where deleting e from M decomposes it into two connected components, we additionally distinguish the first corner in the root face of M following the root corner and we notice that it is contained in the unique corner c' of $M \setminus \{e\}$ that belongs to the different connected component of $M \setminus \{e\}$ than the corner c . We equip it with the same orientation as the root face of M and we define it as the root corner of the second component of $M \setminus \{e\}$, see Fig. 3.

Now, we can classify the root edges e of the map M in the following manner:

- if e disconnects M (i.e. $M \setminus \{e\}$ has two connected components), e is called a *bridge*;
- otherwise $M \setminus \{e\}$ is connected and there are following possibilities:

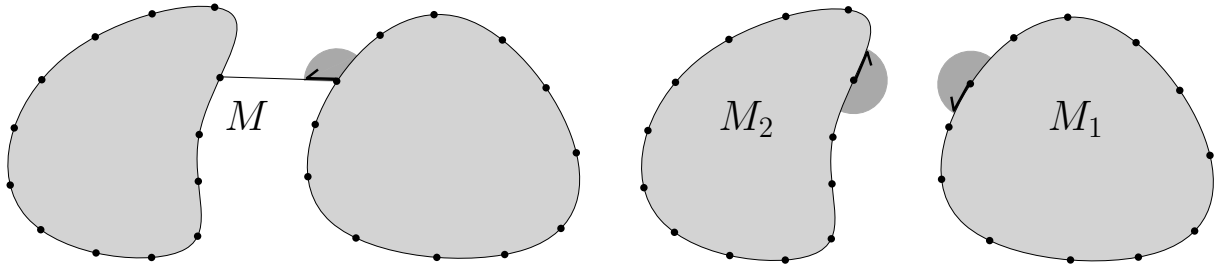


Figure 3: A rooted map M on the left hand side and two rooted maps M_1 and M_2 obtained from M by removing its root edge. Root corners are indicated by dark grey areas.

- the number of faces of $M \setminus \{e\}$ is smaller by 1 than the number of faces of M – in that case e is called a *border*;
- the number of faces of $M \setminus \{e\}$ is equal to the number of faces of M – in that case e is called a *twisted edge*;
- the number of faces of $M \setminus \{e\}$ is greater by 1 than the number of faces of M – in that case e is called a *handle*.

We are now ready to define a statistic η introduced by La Croix.

Definition 3.1. [14, Definition 4.1] A *measure of non-orientability* is an invariant $\eta(M)$ defined for all rooted maps M such that for any map M the invariant $\eta(M)$ associated with it satisfies the following properties:

- If M has no edges, then $\eta(M) = 0$;
- Otherwise, let e be the root edge of M . We have following possibilities:
 - e is a *bridge*. Then $\eta(M) = \eta(M_1) + \eta(M_2)$, where $M \setminus \{e\} = M_1 \cup M_2$;
 - e is a *border*. Then $\eta(M) = \eta(M \setminus \{e\})$;
 - e is *twisted*. Then $\eta(M) = \eta(M \setminus \{e\}) + 1$;
 - e is a *handle*. Then there exists a unique map M' with the root edge e' constructed by twisting the edge e in M such that e' is a handle and such that $M \setminus \{e\} = M' \setminus \{e'\}$. See Fig. 4. In this case we have

$$\{\eta(M), \eta(M')\} = \{\eta(M \setminus \{e\}), \eta(M \setminus \{e\}) + 1\}.$$

Moreover, at most one of M and M' is orientable, and its measure of non-orientability is equal to 0 while a measure of non-orientability of the other (nonorientable) map is equal to 1.

Remark. Note that there are many function η satisfying all the conditions given by Definition 3.1. Thus, the above definition gives a whole class of functions, and any such function is called a measure of non-orientability.

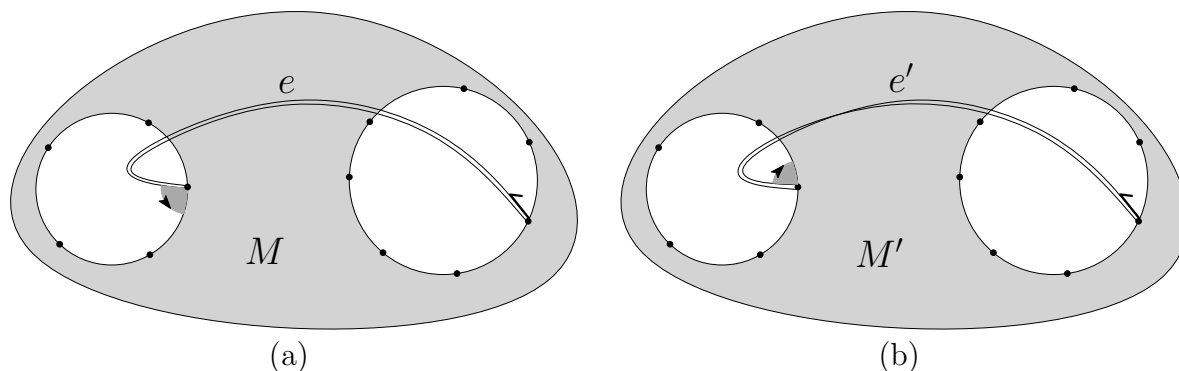


Figure 4: (a) represents diagrammatically a map M , where the root edge e is a handle, and (b) represents diagrammatically a map M' obtained from M by twisting its root edge e , i.e. the unique map M' different from M such that $M \setminus \{e\} = M' \setminus \{e'\}$, where e' is the root edge of M' and such that two distinct corners of $M \setminus \{e\}$ containing two half-edges of e are the same as two distinct corners of $M' \setminus \{e'\}$ containing two half-edges of e' . Two white areas on (a) ((b), respectively) represents two different faces of $M \setminus \{e\} = M' \setminus \{e'\}$ merged by a root e (e' , respectively). To help the reader noticing the difference between M and M' , we shade in dark grey and orient according with the root orientation the first visited corner after the root corner in both maps.

Let M be a rooted map. We label all its edges according to their appearance in the root-deletion procedure. That is, the root edge of M has label 1, the root edge of $M \setminus \{e\}$ has label 2, etc. Here, there must be a convention chosen in which connected component should be treated first, after removing a bridge. Our convention is that we first decompose the connected component with the root corner that contained the root corner of the previous map. That is, we first decompose the map M_1 from Fig. 3.

From now on, we are going to use the following notation: for the rooted map M , and for any $1 \leq i \leq e(M)$ we denote by $e_i(M)$ the edge with the label i and we set M^{i+1} for the rooted map, which is the connected component of $M^i \setminus \{e_i(M)\}$ containing $e_{i+1}(M)$. $M^1 := M$ and $M^{e(M)+1}$ is the unique map with no edges, by convention.

For a given positive integer n and partitions $\mu, \nu, \tau \vdash n$ we can decompose the set $\widetilde{\mathcal{M}}_{\mu, \nu}^{\tau}$ of maps of type $(\mu, \nu; \tau)$ in the following manner:

$$\widetilde{\mathcal{M}}_{\mu, \nu}^{\tau} = \bigcup_{i \geq 0} \widetilde{\mathcal{M}}_{\mu, \nu; i}^{\tau}, \quad (10)$$

where $\widetilde{\mathcal{M}}_{\mu, \nu; i}^{\tau}$ is the set of maps of type $(\mu, \nu; \tau)$ such that exactly i handles appeared during their root-deletion process. In other words, it is the set of rooted maps M of type $(\mu, \nu; \tau)$ such that for all natural numbers $k \in \mathbb{N}$, except i , the root of M^k is not a handle. We call maps from the set $\widetilde{\mathcal{M}}_{\mu, \nu; 0}^{\tau}$ *unhandled*. Finally, we denote the finite set $\{1, 2, \dots, n\}$ by $[n]$. Here, we present a classical, but important for us, relation between a genus of a given map and its root-deletion procedure.

Lemma 3.2. Let $M \in \widetilde{\mathcal{M}}_{\mu,\nu;i}^\tau$ be a rooted map with n edges, and let $j(M)$ denotes the number of positive integers $j \in [n]$ for which the root of M^j is twisted. Then the following equality holds true:

$$j(M) + 2i = 2g(M), \quad (11)$$

where $g(M)$ is a genus of the map M .

Proof. We are going to prove that Eq. (11) holds true for all maps by an induction on the number of edges of M .

It is straightforward to check that there is only one rooted, bipartite map with one edge. Its root edge is a bridge and it is planar (i.e. its genus is equal to 0). Thus, Eq. (11) holds true in this case. Now, we fix $n \geq 2$ and we assume that Eq. (11) holds true for all maps with at most $n - 1$ edges. Let $M \in \widetilde{\mathcal{M}}_{\mu,\nu;i}^\tau$ be a map with n edges and let $i(M)$ denotes the number of handles appearing in the root-deletion process of M , i.e. $i(M) = i$. We are going to analyze how the Euler characteristic varies during the root-deletion process. It is straightforward from the classification of root edges and from Eq. (9) that we have following possibilities:

- e is a *bridge*. Then $g(M) = g(M_1) + g(M_2)$, $i(M) = i(M_1) + i(M_2)$, and $j(M) = j(M_1) + j(M_2)$, where $M \setminus \{e\} = M_1 \cup M_2$. Thus, by an inductive hypothesis

$$j(M) + 2i(M) = j(M_1) + j(M_2) + 2(i(M_1) + i(M_2)) = 2g(M_1) + 2g(M_2) = 2g(M);$$

- e is a *border*. Then $g(M) = g(M \setminus \{e\})$, $i(M) = i(M \setminus \{e\})$, and $j(M) = j(M \setminus \{e\})$. Thus, by an inductive hypothesis

$$j(M) + 2i(M) = j(M \setminus \{e\}) + 2i(M \setminus \{e\}) = 2g(M \setminus \{e\}) = 2g(M);$$

- e is *twisted*. Then $g(M) = g(M \setminus \{e\}) + 1/2$, $i(M) = i(M \setminus \{e\})$, and $j(M) = j(M \setminus \{e\}) + 1$. Thus, by an inductive hypothesis

$$j(M) + 2i(M) = j(M \setminus \{e\}) + 2i(M \setminus \{e\}) + 1 = 2g(M \setminus \{e\}) + 1 = 2g(M);$$

- e is a *handle*. Then $g(M) = g(M \setminus \{e\}) + 1$, $i(M) = i(M \setminus \{e\}) + 1$, and $j(M) = j(M \setminus \{e\}) + 1$. Thus, by an inductive hypothesis

$$j(M) + 2i(M) = j(M \setminus \{e\}) + 2i(M \setminus \{e\}) + 2 = 2g(M \setminus \{e\}) + 2 = 2g(M).$$

Since these are all possible cases, we proved by induction that Eq. (11) holds true for any rooted, bipartite map M , which finishes the proof. \square

Corollary 3.3. Let $M \in \widetilde{\mathcal{M}}_{\mu,\nu;i}^\tau$ be a rooted map with n edges. Then

$$0 \leq n + 2 - (\ell(\mu) + \ell(\nu) + \ell(\tau)) - 2i \leq \eta(M) \leq n + 2 - (\ell(\mu) + \ell(\nu) + \ell(\tau)) - i.$$

Proof. The Euler formula given by Eq. (9) yields

$$2g(M) = 2 + e(M) - f(M) - v(M) = n + 2 - (\ell(\mu) + \ell(\nu) + \ell(\tau)).$$

Combining it with Eq. (11) we have the following formula:

$$0 \leq j(M) = n + 2 - (\ell(\mu) + \ell(\nu) + \ell(\tau)) - 2i(M).$$

It is now enough to notice an obvious inequality which comes strictly from the definition Definition 3.1 of η :

$$\begin{aligned} 0 &\leq n + 2 - (\ell(\mu) + \ell(\nu) + \ell(\tau)) - 2i(M) = j(M) \\ &\leq \eta(M) \leq j(M) + i(M) = n + 2 - (\ell(\mu) + \ell(\nu) + \ell(\tau)) - i(M), \end{aligned}$$

which finishes the proof. \square

3.2 Twist involution

Let M be a rooted map such that its root edge e is a handle. We recall that there exists the unique rooted map M' different from M with the root edge e' which is a handle, too, and such that the half-edges belonging to e are lying in the same corners of $M \setminus \{e\} = M' \setminus \{e'\}$, as the half-edges belonging to e' . Notice that the map M' is, roughly speaking, obtained from M by “twisting” its root. In this section we are going to formalize and generalize the concept of “twisting edges”.

Definition 3.4. Let M be a rooted map with n edges and let us fix an integer $i \in [n]$. We denote by $h_i(M)$ the root of M^i and by $h_i(M)'$ the second half-edge belonging to $e_i(M)$. Let c_1 and \tilde{c}_1 be two corners adjacent to $h_i(M)$ and oriented towards $h_i(M)'$. We denote by $\tau_i M$ the map whose ribbon graph is obtained from the ribbon graph of M by “twisting” the edge $e_i(M)$. That is, by connecting the half-edges $h_i(M)$ and $h_i(M)'$ in the (unique!) different way than they are connected in M . One can describe this construction in a more formal way as follows. Let c_2 (\tilde{c}_2 , respectively) be the unique oriented corner adjacent to $h_i(M)'$, which is the first corner visited after c_1 (\tilde{c}_1 , respectively) – see Fig. 5a. There exists a unique map $\tau_i M$ obtained from M by replacing the edge $e_i(M)$ by the edge e'_i connecting $h_i(M)$ with $h_i(M)'$ in $\tau_i M$ such that the oriented corner adjacent to $h_i(M)'$ and visited after c_1 (\tilde{c}_1 , respectively) is the corner \tilde{c}_2 (c_2 , respectively) – see Fig. 5b. We call the operator τ_i *twisting* of i -th edge of M .

Remark. Note that M and $\tau_i M$ are the same graphs (thus the sets $E(M)$, and $E(\tau_i M)$ are the same, and it makes sense to compare properties of an edge e in M to its properties in $\tau_i M$), but it is not true in general that $\tau_i \tau_j M$ is the same map as $\tau_j \tau_i M$. The following proposition resolves when the twisting operators commute.

Proposition 3.5. *Fix a positive integer n , partitions $\mu, \nu \vdash n$, and a map M with n edges. Then*

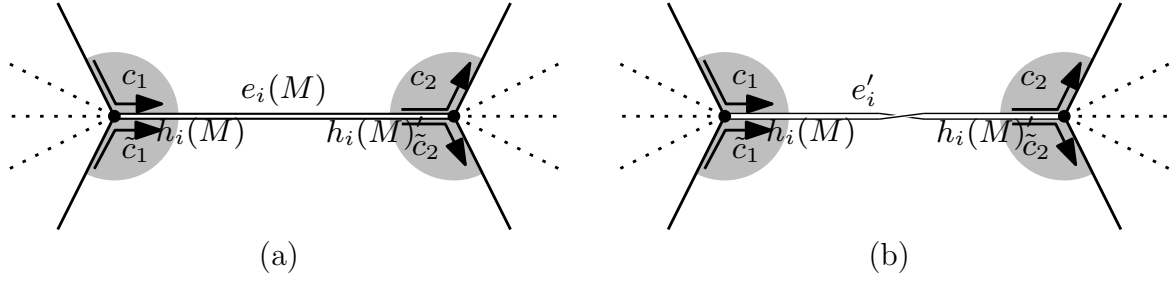


Figure 5

1. for any $i \in [n]$ the operator τ_i is an involution on the set of maps with black (white, respectively) vertex distribution μ (ν , respectively),
2. let $I = \{i_1 < \dots < i_k\}$ be a non-empty subset of $[n]$ such that for all $i \in I$ the root edge of M^i is not a bridge. Then, for any permutation $\sigma \in \mathfrak{S}_I$ the map $\tau_{\sigma(i_1)} \dots \tau_{\sigma(i_k)} M$ is the same, the labels of the edges in M and in $\tau_{\sigma(i_1)} \dots \tau_{\sigma(i_k)} M$ coincide, and for any $j \in [n]$ the root edge of M^j is a bridge iff the root edge of $(\tau_{i_1} \dots \tau_{i_k} M)^j$ is a bridge.

Proof. Let us fix $i \in [n]$. It is clear from the construction that the operator τ_i preserves white and black vertex distributions. Now, notice that twisting an i -th edge in any map M does not change the labels of the first i edges and it may change the labels of the other edges only if the edge $e_i(M)$ is a bridge in M^i . Thus, the first i labels of the edges in both maps M and $\tau_i M$ are the same, so $\tau_i^2 M = M$.

We are going to prove the second item by an induction on size of the set I . We have already proved above that the inductive assertion holds true for $|I| = 1$, so let us fix an integer $1 < k \leq n$. We assume that the inductive assertion holds true for all subsets $I \subset [n]$ of size smaller than k . Let $I = \{i_1 < \dots < i_k\}$ be a non-empty subset of $[n]$ such that for all $i \in I$ the root edge of M^i is not a bridge and let $\sigma \in \mathfrak{S}_I$. Then, there exists an integer $l \in [k]$ and a permutation $\pi \in \mathfrak{S}_{[k] \setminus \{l\}}$ such that

$$\tau_{\sigma(i_1)} \dots \tau_{\sigma(i_k)} M = \tau_{i_l} \left(\tau_{i_{\pi(1)}} \dots \widehat{\tau_{i_{\pi(l)}}} \dots \tau_{i_{\pi(k)}} M \right),$$

where we use a standard notation that the word $a_1 \dots a_{i-1} \widehat{a_i} a_{i+1} \dots a_n$ is obtained from the word $a_1 \dots a_{i-1} a_i a_{i+1} \dots a_n$ by removing the letter a_i . If $l = 1$, then the labels in M and $\left(\tau_{i_{\pi(1)}} \dots \widehat{\tau_{i_{\pi(l)}}} \dots \tau_{i_{\pi(k)}} M \right)$ coincide, and the root edge of $\left(\tau_{i_{\pi(1)}} \dots \widehat{\tau_{i_{\pi(l)}}} \dots \tau_{i_{\pi(k)}} M \right)^{i_1}$ is not a bridge since the root edge of M^{i_1} is not a bridge, by the inductive assertion. Thus, the labels in M , and in $\tau_{\sigma(i_1)} \dots \tau_{\sigma(i_k)} M$ coincide, too. If $l > 1$, then the inductive assertion says that the labels of M and $\tau_{j_1} \dots \tau_{j_m} M$ are the same for all subsets $\{j_1, \dots, j_m\} \subset I$ of size $m < k$, so

$$\begin{aligned} \tau_{i_l} \left(\tau_{i_{\pi(1)}} \dots \widehat{\tau_{i_{\pi(l)}}} \dots \tau_{i_{\pi(k)}} M \right) &= \tau_{i_l} (\tau_{i_1} \dots \widehat{\tau_{i_l}} \dots \tau_{i_k} M) \\ &= \tau_{i_1} (\tau_{i_l} \tau_{i_2} \dots \widehat{\tau_{i_l}} \dots \tau_{i_k} M) = \tau_{i_1} \dots \tau_{i_k} M \end{aligned} \quad (12)$$

by the inductive hypothesis. Moreover, again by the inductive hypothesis, the labels in the maps were not changed when we were swapped operators τ with different indices. Finally, for any $j \in [n]$, the graphs M^j and $(\tau_{i_1} \cdots \tau_{i_k} M)^j$ are the same. Since being a bridge is the same as being a disconnecting edge of the graph, the proof is finished. \square

Lemma 3.6. *Let η be a measure of non-orientability. Then, for all positive integers n, i and partitions $\mu, \nu \vdash n$, there exists an involution σ_η on the set $\widetilde{\mathcal{M}}_{\mu, \nu; i}^{(n)}$, which has the property that*

$$(-1)^{\eta(\sigma_\eta(M))} = (-1)^{\eta(M)+1}.$$

Moreover, for each $M \in \widetilde{\mathcal{M}}_{\mu, \nu; i}^{(n)}$ there exist natural numbers $1 \leq i_1 < \cdots < i_k \leq n$ such that $\sigma_\eta(M) = \tau_{i_k} \cdots \tau_{i_1} M$ and such that for each $j \in [k]$ the root of M^{i_j} is not a bridge.

Proof. We are going to construct σ_η by induction on n . For $n = 1$ all rooted bipartite maps with n edges are unhandled so we set σ_η as an empty map.

We denote by $i(M)$ the number of handles appearing in the root-deletion process of M . We fix $n \geq 1$ and we assume that the involution σ_η is already defined for all unicellular maps with at most n edges which are not unhandled. Let M be a unicellular map with $n + 1$ edges. Since M has a unique face, $e_1(M)$ cannot be a border so there are the following possibilities:

- $e_1(M)$ is a handle. In this case we set $\sigma_\eta(M) = \tau_1 M$. Clearly $(-1)^{\eta(M)} = (-1)^{\eta(\sigma_\eta(M))+1}$ and $i(M) = i(\sigma_\eta(M))$;
- $e_1(M)$ is twisted. In this case M^2 is a unicellular map with n edges. Thus, by the inductive hypothesis there exist natural numbers $1 \leq i_1 < \cdots < i_k \leq n$ such that $i(M^2) = i(\tau_{i_k} \cdots \tau_{i_1} M^2)$ and such that

$$(-1)^{\eta(\tau_{i_k} \cdots \tau_{i_1} M^2)} = (-1)^{\eta(M^2)+1}.$$

Let us consider two rooted maps $M_1 = \tau_{i_k+1} \cdots \tau_{i_1+1} M$ and $M_2 = \tau_{i_k+1} \cdots \tau_{i_1+1} \tau_1 M$. Both maps M_1 and M_2 have a property that after deletion of its root we obtain the same rooted map $M_1^2 = M_2^2 = \tau_{i_k} \cdots \tau_{i_1} M^2$ (Proposition 3.5 asserts that the labels in M_1^2 and in M_2^2 , respectively, correspond to the labels of M_1 and M_2 , respectively, shifted by 1). Thus, exactly one map from M_1 and M_2 is a map M' with the unique face (and its root is twisted) while the second one has two faces (and its root is a border) and we set $\sigma_\eta(M) = M'$. Strictly from the construction, one has

$$(-1)^{\eta(\sigma_\eta(M))} = (-1)^{\eta(\sigma_\eta(M^2))+1} = (-1)^{\eta(M^2)} = (-1)^{\eta(M)+1}$$

and

$$i(M) = i(M^2) = i(\sigma_\eta(M^2)) = i(\sigma_\eta(M)^2) = i(\sigma_\eta(M));$$

- $e_1(M)$ is a bridge. In this case $M^2 = M_1 \cup M_2$ is a disjoint sum of two unicellular maps (and we recall the convention for labeling: the edge $e_2(M)$ belongs to M_1).

If M_1 is not unhandled, then there exists a positive integer k and natural numbers $1 \leq i_1 < \cdots < i_k \leq e(M_1)$ such that $i(\tau_{i_k} \cdots \tau_{i_1} M_1) = i(M_1)$ and such that

$$(-1)^{\eta(\tau_{i_k} \cdots \tau_{i_1} M_1)} = (-1)^{\eta(M_1)+1}.$$

In this case we set $\sigma_\eta(M) := \tau_{i_k+1} \cdots \tau_{i_1+1} M$ and since $\sigma_\eta(M)^2 = \sigma_\eta(M_1) \cup M_2$ it is clear that $(-1)^{\eta(\sigma_\eta(M))} = (-1)^{\eta(M)+1}$ and

$$i(M) = i(M_1) + i(M_2) = i(\sigma_\eta(M_1)) + i(M_2) = i(\sigma_\eta(M)).$$

If the map M_1 is unhandled, then the map M_2 is not unhandled and there exist a positive integer k and natural numbers $1 \leq i_1 < \cdots < i_k \leq e(M_2)$ such that $i(\tau_{i_k} \cdots \tau_{i_1} M_2) = i(M_2)$ and such that

$$(-1)^{\eta(\tau_{i_k} \cdots \tau_{i_1} M_2)} = (-1)^{\eta(M_2)+1}.$$

In this case we set $\sigma_\eta(M) := \tau_{i_k+e(M_1)+1} \cdots \tau_{i_1+e(M_1)+1} M$. Since $(\sigma_\eta(M))^2 = M_1 \cup \sigma_\eta(M_2)$ it is clear that $(-1)^{\eta(\sigma_\eta(M))} = (-1)^{\eta(M)+1}$ and

$$i(M) = i(M_2) = i(\sigma_\eta(M_2)) = i(\sigma_\eta(M)).$$

Now it is straightforward from the construction and from the inductive hypothesis that if $\sigma_\eta(M)$ associated with the rooted map M is of the form $\tau_{i_k} \cdots \tau_{i_1} M$, then $\sigma_\eta(\sigma_\eta(M))$ is of the same form, i.e. $\sigma_\eta(\sigma_\eta(M)) = \tau_{i_k} \cdots \tau_{i_1} \sigma_\eta(M)$. But for each $j \in [k]$ the root edge of M^{i_j} is not a bridge. Thus

$$\sigma_\eta(\sigma_\eta(M)) = \tau_{i_k} \cdots \tau_{i_1} (\tau_{i_k} \cdots \tau_{i_1} M) = \tau_{i_k} \cdots \tau_{i_1} (\tau_{i_1} \cdots \tau_{i_k} M) = M,$$

where the last equalities come from Proposition 3.5, which finishes the proof. \square

3.3 Algebraic properties of a measure of non-orientability

Let η be a measure of non-orientability and let μ, ν, τ be partitions of a positive integer n . We define the following statistic associated with η :

$$(H_\eta)_{\mu, \nu}^\tau(\beta) := \sum_{M \in \widetilde{\mathcal{M}}_{\mu, \nu}^\tau} \beta^{\eta(M)}. \quad (13)$$

The main purpose of this section is to investigate algebraic properties of $(H_\eta)_{\mu, \nu}^\tau$ that will be of the great importance in the proof of Theorem 1.4. From now on, we fix a positive integer n , partitions $\mu, \nu, \tau \vdash n$, and a measure of non-orientability η .

Proposition 3.7. *Let $g := n + 2 - (\ell(\mu) + \ell(\nu) + \ell(\tau))$. Then, for any nonnegative integer i , the following quantity*

$$(a_\eta)_{\mu, \nu; i}^\tau(\beta) := \sum_{M \in \widetilde{\mathcal{M}}_{\mu, \nu; i}^\tau} \beta^{\eta(M) + 2i - g} \quad (14)$$

is a polynomial in β of degree at most i .

Proof. It is a direct consequence of Corollary 3.3, which says that for any map $M \in \widetilde{\mathcal{M}}_{\mu,\nu;i}^\tau$ the following inequalities hold:

$$0 \leq \eta(M) + 2i - g \leq i.$$

□

Corollary 3.8. *The quantity $(H_\eta)_{\mu,\nu}^\tau(\beta)$ is a polynomial in β with positive integer coefficients. Moreover, it has the following form:*

$$(H_\eta)_{\mu,\nu}^\tau(\beta) = \sum_{0 \leq i \leq [g/2]} (a_\eta)_{\mu,\nu;i}^\tau(\beta) \beta^{g-2i}, \quad (15)$$

where $g := n + 2 - (\ell(\mu) + \ell(\nu) + \ell(\tau))$.

Proof. Strictly from the definition of $(H_\eta)_{\mu,\nu}^\tau(\beta)$ given by Eq. (13), one has the following formula:

$$(H_\eta)_{\mu,\nu}^\tau(\beta) = \sum_{i \geq 0} \sum_{M \in \widetilde{\mathcal{M}}_{\mu,\nu;i}^\tau} \beta^{\eta(M)} = \sum_{0 \leq i \leq [g/2]} (a_\eta)_{\mu,\nu;i}^\tau(\beta) \beta^{g-2i},$$

where the last equality is simply a definition of $a_{\mu,\nu;i}^\tau(\beta)$ given by Eq. (14). □

Proposition 3.9. *For any positive integer $i \geq 1$ one has*

$$(a_\eta)_{\mu,\nu;i}^{(n)}(-1) = 0.$$

Proof. Plugging $\beta = -1$ into Eq. (14) one has

$$(-1)^{n+1-\ell(\mu)-\ell(\nu)} (a_\eta)_{\mu,\nu;i}^{(n)}(-1) = \sum_{M \in \widetilde{\mathcal{M}}_{\mu,\nu;i}^{(n)}} (-1)^{\eta(M)}.$$

Lemma 3.6 says that for each $i \geq 1$ there exists an involution σ_η on the set $\widetilde{\mathcal{M}}_{\mu,\nu;i}^{(n)}$ which has a property that $(-1)^{\eta(\sigma_\eta(M))} = (-1)^{\eta(M)+1}$. This means that

$$\sum_{M \in \widetilde{\mathcal{M}}_{\mu,\nu;i}^{(n)}} (-1)^{\eta(M)} = \sum_{M \in \widetilde{\mathcal{M}}_{\mu,\nu;i}^{(n)}} (-1)^{\eta(\sigma_\eta(M))} = - \sum_{M \in \widetilde{\mathcal{M}}_{\mu,\nu;i}^{(n)}} (-1)^{\eta(M)} = 0.$$

Thus, $(a_\eta)_{\mu,\nu;i}^{(n)}(-1) = 0$ which finishes the proof. □

Corollary 3.10. *The following equality holds true:*

$$(-1)^{n+1-\ell(\mu)-\ell(\nu)} (H_\eta)_{\mu,\nu}^{(n)}(-1) = (a_\eta)_{\mu,\nu;0}^{(n)}(-1) = \#\widetilde{\mathcal{M}}_{\mu,\nu;0}^{(n)}. \quad (16)$$

Proof. It is enough to plug $\beta = -1$ into Eq. (15) to obtain

$$(-1)^g (H_\eta)_{\mu,\nu}^{(n)}(-1) = \sum_{i \geq 0} (a_\eta)_{\mu,\nu;i}^{(n)}(-1) = (a_\eta)_{\mu,\nu;0}^{(n)}(-1),$$

where $g := n + 1 - \ell(\mu) - \ell(\nu)$ and the last equality is a consequence of Proposition 3.9. An equality

$$(a_\eta)_{\mu,\nu;0}^{(n)}(-1) = \#\widetilde{\mathcal{M}}_{\mu,\nu;0}^{(n)}$$

is obvious from Corollary 3.3. □

4 b -conjecture for unicellular maps and measure of non-orientability

4.1 Marginal sum

We are going to prove that fixing white and black vertex distributions and allowing any face distribution, the corresponding sum of coefficients in $\psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, 1 + \beta)$ is given by a measure of non-orientability of the appropriate maps. The developments in this section are similar to that of [1, Section 3.5] except that here we work in a more general setup (in [1, Section 3.5] $\nu = (2^{n/2})$ with n even, while here ν is an arbitrary partition) and with a slightly different function η . We start with the following proposition.

Proposition 4.1. *For any positive integer n and for any partitions $\mu, \nu \vdash n$, the following identity holds true:*

$$\sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(\beta) = (1 + \beta)^{n+1-\ell(\mu)-\ell(\nu)} \sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(0).$$

Proof. We know that

$$\sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(\beta) = [t^n p_{\mu}(\mathbf{x}) p_{\nu}(\mathbf{y})] \psi(\mathbf{x}, \mathbf{y}, \mathbf{z}; t, \alpha) \Big|_{\mathbf{z}=(1,0,0\dots)}$$

because of the trivial identity

$$J_{\lambda}^{(\alpha)}(1, 0, 0, \dots) = J_{\lambda}^{(\alpha)}(\mathbf{x}) \Big|_{p_1(\mathbf{x})=p_2(\mathbf{x})=\dots=1}.$$

Using Eq. (8) and replacing the scalar product by its expression given in Eq. (6), we obtain

$$\sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(\beta) = [t^n p_{\mu}(\mathbf{x}) p_{\nu}(\mathbf{y})] (1 + \beta) t \frac{\partial}{\partial t} \log \left(\sum_{n \geq 0} t^n \frac{J_{(n)}^{(\alpha)}(\mathbf{x}) J_{(n)}^{(\alpha)}(\mathbf{y})}{\alpha^n n!} \right). \quad (17)$$

The formula for Jack polynomials indexed by one-part partitions given in Eq. (7) leads to the following equality:

$$\begin{aligned} \sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(\beta) &= (1 + \beta) [t^n p_{\mu}(\mathbf{x}) p_{\nu}(\mathbf{y})] t \frac{\partial}{\partial t} \log \left(\sum_{n \geq 0} t^n \sum_{\lambda^1, \lambda^2 \vdash n} \alpha^{n-\ell(\lambda^1)-\ell(\lambda^2)} \frac{n! p_{\lambda^1}(\mathbf{x}) p_{\lambda^2}(\mathbf{y})}{z_{\lambda^1} z_{\lambda^2}} \right) \\ &= (1 + \beta)^{n+1-\ell(\mu)-\ell(\nu)} [t^n p_{\mu}(\mathbf{x}) p_{\nu}(\mathbf{y})] t \frac{\partial}{\partial t} \log \left(\sum_{n \geq 0} t^n \sum_{\lambda^1, \lambda^2 \vdash n} \frac{n! p_{\lambda^1}(\mathbf{x}) p_{\lambda^2}(\mathbf{y})}{z_{\lambda^1} z_{\lambda^2}} \right). \end{aligned}$$

But the last expression is simply equal to

$$\begin{aligned} (1 + \beta)^{n+1-\ell(\mu)-\ell(\nu)} [t^n p_{\mu}(\mathbf{x}) p_{\nu}(\mathbf{y})] \psi(\mathbf{x}, \mathbf{y}, \mathbf{z} = (1, 0, 0, \dots); t, 1) \\ (1 + \beta)^{n+1-\ell(\mu)-\ell(\nu)} \sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(0), \end{aligned}$$

which finishes the proof. \square

We can prove now that the following marginal sum is given by a measure of non-orientability:

Theorem 4.2. *For any measure of non-orientability η , for any positive integer n , and for any partitions $\mu, \nu \vdash n$, the following identity holds true:*

$$\sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(\beta) = \sum_{\tau \vdash n} (H_{\eta})_{\mu, \nu}^{\tau}(\beta), \quad (18)$$

where $(H_{\eta})_{\mu, \nu}^{\tau}(\beta)$ are given by Eq. (13).

Proof. We recall that $(H_{\eta})_{\mu, \nu}^{\tau}(\beta)$ is defined as a weighted sum of some rooted, bipartite maps (see Eq. (13)). Thus, one can define a statistic $(H_{\eta})_{\mu, \nu; i}(\beta)$ as the right hand side of (18) with a summation restricted to the maps with the root vertex of degree i . We are going to prove a stronger result, namely

$$(H_{\eta})_{\mu, \nu; i}(\beta) = (1 + \beta)^{n+1-\ell(\mu)-\ell(\nu)} \tilde{H}_{\mu, \nu; i}, \quad (19)$$

where $\tilde{H}_{\mu, \nu; i}$ is the number of *orientable* maps with the root vertex of degree i , the black vertex distribution μ , and the white vertex distribution ν . If Eq. (19) holds true, then

$$\begin{aligned} \sum_{\tau \vdash n} (H_{\eta})_{\mu, \nu}^{\tau}(\beta) &= \sum_{i \geq 1} (H_{\eta})_{\mu, \nu; i}(\beta) = (1 + \beta)^{n+1-\ell(\mu)-\ell(\nu)} \sum_{i \geq 1} \tilde{H}_{\mu, \nu; i} \\ &= (1 + \beta)^{n+1-\ell(\mu)-\ell(\nu)} \sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(0) = \sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(\beta). \end{aligned}$$

The third equality uses the combinatorial interpretation of $h_{\mu, \nu}^{\tau}(0)$ (see Theorem 1.1) while the last equality comes from Proposition 4.1. In this way we have shown that Eq. (19) implies Eq. (18). Thus, it is sufficient to prove Eq. (19).

Before we start a proof we introduce some notation. Let r_1, \dots, r_k be some positive integers such that $r_1 + \dots + r_k = n$. Let us fix a partition $\mu \vdash n$. We define $\text{Sp}^{(r_1, \dots, r_k)}(\mu)$ as the set of sequences of partitions (μ^1, \dots, μ^k) such that $\mu^1 \vdash r_1, \dots, \mu^k \vdash r_k$ and such that their sum gives the fixed partition μ . That is, $\bigcup_{1 \leq i \leq k} \mu^i = \mu$. Moreover, for any positive integer $i \geq 1$ and partition μ containing a part equal to i , we set

$$\mu_{\downarrow(i)} = (\mu \setminus (i)) \cup (i-1).$$

We are going to prove Eq. (19) by induction on n . Let $n = 1$; there exists only one partition of size n . That is, $\mu = \nu = (1)$. Moreover, there is only one map with one edge, and it is planar, so clearly $H_{(1), (1); 1}(\beta) = 1 = \tilde{H}_{(1), (1); 1}$. Let us fix $n \geq 2$ and assume now that the inductive assertion holds true for all partitions of size smaller than n and all integers $i \geq 1$. Let us fix two partitions $\mu, \nu \vdash n$ and an integer $i \geq 1$. Let M be a map with the root vertex of degree i , and the black (white, respectively) vertex distribution μ (ν , respectively). We are going to understand the structure of $M \setminus \{e\}$, where e is the root edge of M . There are two possibilities:

- $M \setminus \{e\}$ is a disjoint sum of two maps M_1 and M_2 (they are ordered, i.e. their indices matter) with root vertices of degrees $i - 1$, and $j - 1$, respectively, the black vertex distributions μ^1 , and μ^2 , respectively, and the white vertex distributions ν^1 , and ν^2 , respectively, where

$$\begin{aligned}(\mu^1, \mu^2) &\in \text{Sp}^{(l, n-l-1)}(\mu_{\downarrow(i)}), \\ (\nu^1, \nu^2) &\in \text{Sp}^{(l, n-l-1)}(\nu_{\downarrow(j)}),\end{aligned}$$

and $1 \leq j, l + 1 \leq n$ are some integers;

- $M \setminus \{e\}$ is a single map M' with the root vertex of degree $i - 1$, the black vertex distribution $\mu_{\downarrow(i)}$, and the white vertex distribution $\nu_{\downarrow(j)}$, where $2 \leq j \leq n$ is some integer.

Moreover,

- for any ordered pair of maps M_1 , and M_2 with root vertices of degrees $i - 1$, and $j - 1$, respectively, the black vertex distribution μ^1 , and μ^2 , respectively, and the white vertex distribution ν^1 , and ν^2 , respectively, where

$$\begin{aligned}(\mu^1, \mu^2) &\in \text{Sp}^{(l, n-l-1)}(\mu_{\downarrow(i)}), \\ (\nu^1, \nu^2) &\in \text{Sp}^{(l, n-l-1)}(\nu_{\downarrow(j)}),\end{aligned}$$

and $1 \leq j, l + 1 \leq n$ are some integers, there exists the unique map M with the root vertex of degree i and the black (white, respectively) vertex distribution μ (ν , respectively), such that $M^1 = M_1 \cup M_2$. In this case $\eta(M) = \eta(M_1) + \eta(M_2)$;

- for any map M' with the root vertex of degree $i - 1$, the black vertex distribution $\mu \setminus (i) \cup (i - 1)$ and the white vertex distribution $\nu \setminus (j) \cup (j - 1)$, where $2 \leq j \leq n$ is some integer, there exists

$$2(j - 1)m_{j-1}(\nu) + 2(j - 1)$$

maps with the root vertex of degree i and the black (white, respectively) vertex distribution μ (ν , respectively), such that removing its root edge gives a map M' . Indeed, each map with n edges and these properties is obtained by adding an edge to M' , which connects the root corner r of M' to some corner c of M' incident to a white vertex of degree $j - 1$. There are $(j - 1)m_{j-1}(\nu) + (j - 1)$ such corners (since there are $m_{j-1}(\nu) + 1$ white vertices of degree $j - 1$ in the map M') and for each chosen corner there are exactly two ways to connect it with the root corner of M' by an edge (these two ways correspond to construction of maps M and $\tau_1 M$ – we recall that $\tau_1 M$ is a map obtained from M by twisting its root edge; see Definition 3.4). Now, notice that there are following possibilities:

- if the root corner r of M' and the corner c belong to the same face of M' then exactly one rooted bipartite map from $\{M, \tau_1 M\}$ has the twisted root edge, while the second one has the root edge which is a border. Thus

$$\{\eta(M), \eta(\tau_1 M)\} = \{\eta(M'), \eta(M') + 1\};$$

- if the root corner r of M' and the corner c belong to different faces of M' then both root edges of M and $\tau_1 M$ are handles. Thus, strictly from the definition of η , one has

$$\{\eta(M), \eta(\tau_1 M)\} = \{\eta(M'), \eta(M') + 1\}.$$

Above analysis leads us to the following recursion obtained by removing root edges from the maps appearing in the summation index in the definition of $(H_\eta)_{\mu, \nu; i}(\beta)$ given by Eq. (13):

$$\begin{aligned} (H_\eta)_{\mu, \nu; i}(\beta) = & \sum_{1 \leq j, l \leq n} \sum_{\substack{(\mu^1, \mu^2) \in \text{Sp}^{(l-1, n-l)}(\mu_{\downarrow(i)}), \\ (\nu^1, \nu^2) \in \text{Sp}^{(l-1, n-l)}(\nu_{\downarrow(j)})}} (H_\eta)_{\mu^1, \nu^1; i-1}(\beta) (H_\eta)_{\nu^2, \mu^2; j-1}(\beta) \\ & + (1 + \beta) \sum_{2 \leq j \leq n} (j-1)(m_{j-1}(\nu) + 1) (H_\eta)_{\mu_{\downarrow(i)}, \nu_{\downarrow(j)}; i-1}(\beta). \end{aligned}$$

Using the inductive assertion, we obtain:

$$\begin{aligned} (H_\eta)_{\mu, \nu; i}(\beta) = (1 + \beta)^{n+1-\ell(\mu)-\ell(\nu)} & \left(\sum_{1 \leq j, l \leq n} \sum_{\substack{(\mu^1, \mu^2) \in \text{Sp}^{(l-1, n-l)}(\mu_{\downarrow(i)}), \\ (\nu^1, \nu^2) \in \text{Sp}^{(l-1, n-l)}(\nu_{\downarrow(j)})}} \tilde{H}_{\mu^1, \nu^1; i-1} \tilde{H}_{\nu^2, \mu^2; j-1} \right. \\ & \left. + \sum_{2 \leq j \leq n} (j-1)(m_{j-1}(\nu) + 1) \tilde{H}_{\mu_{\downarrow(i)}, \nu_{\downarrow(j)}; i-1} \right). \end{aligned}$$

To finish the proof, it is enough to notice that the following recursion holds true:

$$\begin{aligned} \tilde{H}_{\mu, \nu; i} = & \left(\sum_{1 \leq j, l \leq n} \sum_{\substack{(\mu^1, \mu^2) \in \text{Sp}^{(l-1, n-l)}(\mu_{\downarrow(i)}), \\ (\nu^1, \nu^2) \in \text{Sp}^{(l-1, n-l)}(\nu_{\downarrow(j)})}} \tilde{H}_{\mu^1, \nu^1; i-1} \tilde{H}_{\nu^2, \mu^2; j-1} \right. \\ & \left. + \sum_{2 \leq j \leq n} (j-1)(m_{j-1}(\nu) + 1) \tilde{H}_{\mu_{\downarrow(i)}, \nu_{\downarrow(j)}; i-1} \right). \end{aligned}$$

Above relation comes from the analysis of the process of removing the root edge from an orientable map with the root vertex of degree i , the black vertex distribution μ , and the white vertex distribution ν . Such analysis is almost identical to the analysis we did in the general case, and we leave it as an easy exercise. \square

4.2 Some consequences of the polynomiality and the marginal sum results

We start with an observation that polynomials $h_{\mu, \nu}^\tau(\beta)$ have some specific form:

Lemma 4.3. *For any positive integer n and any partitions $\mu, \nu, \tau \vdash n$ one has the following expansion:*

$$h_{\mu, \nu}^{\tau}(\beta) = \begin{cases} \sum_{0 \leq i \leq [g/2]} a_{\mu, \nu; i}^{\tau} \beta^{g-2i} (\beta + 1)^i & \text{for } \ell(\mu) + \ell(\nu) + \ell(\tau) \leq 2 + n, \\ 0 & \text{otherwise ;} \end{cases}$$

where $g := 2 + n - (\ell(\mu) + \ell(\nu) + \ell(\tau))$ and $a_{\mu, \nu; i}^{\tau} \in \mathbb{Q}$.

Proof. La Croix proved in [14, Corollary 5.22] that Lemma 4.3 holds true, assuming polynomiality of $h_{\mu, \nu}^{\tau}(\beta)$ (that he was not able to prove). Theorem 1.3 completes the proof. \square

We are now ready to prove Theorem 1.4

Proof of Theorem 1.4. Theorem 1.1 already says that the cases $\beta = 0, 1$ correspond to counting maps on orientable, and all (orientable or non-orientable), respectively, surfaces. Thus, we need to establish the remaining identity:

$$h_{\mu, \nu}^{(n)}(-1) = (H_{\eta})_{\mu, \nu}^{(n)}(-1).$$

Note now that for fixed partitions $\mu, \nu \vdash n$ the variable $g := 2 + n - (\ell(\mu) + \ell(\nu) + \ell(\tau))$ taken over all partitions $\tau \vdash n$ realizes a maximum for $\tau = (n)$. Hence

$$[\beta^{1+n-(\ell(\mu)+\ell(\nu))}] \sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(\beta) = [\beta^{1+n-(\ell(\mu)+\ell(\nu))}] h_{\mu, \nu}^{(n)}(\beta) = (-1)^{1+n-(\ell(\mu)+\ell(\nu))} h_{\mu, \nu}^{(n)}(-1)$$

by Lemma 4.3 and, similarly,

$$\begin{aligned} [\beta^{1+n-(\ell(\mu)+\ell(\nu))}] \sum_{\tau \vdash n} (H_{\eta})_{\mu, \nu}^{\tau}(\beta) &= [\beta^{1+n-(\ell(\mu)+\ell(\nu))}] (H_{\eta})_{\mu, \nu}^{(n)}(\beta) \\ &= (-1)^{1+n-(\ell(\mu)+\ell(\nu))} (H_{\eta})_{\mu, \nu}^{(n)}(-1) \end{aligned}$$

by Eq. (15) and Proposition 3.9. By Theorem 4.2 the following equality holds

$$[\beta^{1+n-(\ell(\mu)+\ell(\nu))}] \sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(\beta) = [\beta^{1+n-(\ell(\mu)+\ell(\nu))}] \sum_{\tau \vdash n} (H_{\eta})_{\mu, \nu}^{\tau}(\beta)$$

which implies the desired result. \square

Remark. The equality

$$[\beta^{1+n-(\ell(\mu)+\ell(\nu))}] \sum_{\tau \vdash n} h_{\mu, \nu}^{\tau}(\beta) = [\beta^{1+n-(\ell(\mu)+\ell(\nu))}] \sum_{\tau \vdash n} (H_{\eta})_{\mu, \nu}^{\tau}(\beta),$$

combined with Proposition 4.1, Proposition 3.9 and Lemma 4.3 says that the top degree coefficient of $h_{\mu, \nu}^{(n)}(\beta)$ is enumerated by *unhandled* maps of type $(\mu, \nu; (n))$, but it is also enumerated by *orientable* maps with the black (white, respectively) vertex distribution μ

(ν , respectively) and the arbitrary face degree. In fact, one can use the proof of Proposition 4.1 to construct a bijection between these two sets recursively. An interesting result of Śniady [22, Corollary 0.5] states that the top-degree part of Jack characters indexed by a one-part partition can be also expressed as a linear combination of certain functions indexed by orientable maps. Śniady informed us in private communication [21] that he can construct a similar bijection, but for different “measure of non-orientability”, which inspired us to initiate a research presented in this section. The connection between striking similarities in both results seems to be far from being understood.

5 Low genera cases and orientations

In this section we are going to prove Theorem 1.5. In fact, we are going to show that there is an infinite family of measures of non-orientability for which, in low genera cases, b -conjecture holds true.

5.1 A measure of non-orientability given by orientations

Let M be a map. We say that O is an *orientation* of M if it defines an orientation of each face of M such that

- the orientation of the root face given by O is consistent with the orientation given by the root;
- if M is orientable then O is the canonical orientation of M , that is the orientation for which each face of M is oriented clockwise (counterclockwise, respectively) iff the root face is oriented clockwise (counterclockwise, respectively).

Let \mathcal{O} be a *set of orientations of all rooted maps* (*set of orientations*, for short), i.e. for any map M there exists the *unique* orientation O of M such that $O \in \mathcal{O}$. We are going to define a function $\eta_{\mathcal{O}}$ associated with \mathcal{O} that takes as values maps and returns a nonnegative integer that, in some sense, “measures non-orientability” of the given map. This function will be defined recursively using the same procedure of deleting the root edge from the given map, as in Definition 3.1.

Definition 5.1. Let \mathcal{O} be a set of orientations. We set $\eta_{\mathcal{O}}(M) = 0$ for M without edges, we fix a positive integer n , and we assume that $\eta_{\mathcal{O}}(M)$ is already defined for all maps with at most $n - 1$ edges. Let M be a map with n edges and let $O \in \mathcal{O}$ be the orientation associated with M^2 (in the case where M^2 is a disjoint sum of two maps M_1, M_2 , we are taking two associated orientations $O_1, O_2 \in \mathcal{O}$, respectively). Let c' be the unique corner of M^2 containing the first corner c of M visited after the root corner of M and c' inherits an orientation from the corner c . We set $\eta_{\mathcal{O}}(M) := \eta_{\mathcal{O}}(M^2)$ if the orientation of c' is consistent with the orientation given by O and we say that e is of the first kind; otherwise we set $\eta_{\mathcal{O}}(M) := \eta_{\mathcal{O}}(M^2) + 1$ and we say that e is of the second kind (in the case where $M^2 = M_1 \cup M_2$ is a disjoint sum of two rooted maps we set, by convention, $\eta_{\mathcal{O}}(M) := \eta_{\mathcal{O}}(M_1) + \eta_{\mathcal{O}}(M_2)$ and we set e to be of the first kind).

It is easy to see that for each set of orientations \mathcal{O} , the following holds true:

- if the root edge e of M is a *bridge* then $\eta_{\mathcal{O}}(M) = \eta_{\mathcal{O}}(M \setminus \{e\})$,
- if the root edge e of M is a *border* then $\eta_{\mathcal{O}}(M) = \eta_{\mathcal{O}}(M \setminus \{e\})$,
- if the root edge e of M is a *twisted* edge then $\eta_{\mathcal{O}}(M) = \eta_{\mathcal{O}}(M \setminus \{e\}) + 1$,
- if the root edge e of M is a *handle* then $\{\eta_{\mathcal{O}}(M), \eta_{\mathcal{O}}(\tau_1 M)\} = \{\eta_{\mathcal{O}}(M \setminus \{e\}), \eta_{\mathcal{O}}(M \setminus \{e\}) + 1\}$.

In other words, for any orientation \mathcal{O} of all rooted maps, the measure of non-orientability $\eta_{\mathcal{O}}$ associated with \mathcal{O} is also a measure of non-orientability given by Definition 3.1. However, the converse statement is not true, i.e. there are many measures of non-orientability η which are not given by any set of orientations of all rooted maps.

Remark. Note that we use **two different** ways to make a distinction between edges: the first way is by determining their *type* which can be a bridge, a border, a twisted edge, or a handle. The second way is by determining their *kind* which can be the first or the second. Each *type* of edges has uniquely determined *kind* except a handle which can be both of the first and of the second kind. With this notation, invariant $\eta_{\mathcal{O}}(M)$ associated with the rooted map M is equal to the number of edges of the second kind appearing in its root-deletion process.

We are ready to restate Theorem 1.5 in the more general form:

Theorem 5.2. *For a set \mathcal{O} of orientations, for any positive integer n , and for any partitions $\mu, \nu \vdash n$ such that $\ell(\mu) + \ell(\nu) \geq n - 3$ the following equality holds true:*

$$h_{\mu, \nu}^{(n)}(\beta) = (H_{\eta_{\mathcal{O}}})_{\mu, \nu}^{(n)}(\beta).$$

The next subsections are devoted to its proof.

5.2 Unicellular maps of low genera with two handles

In this section we are going to analyze the structure of the unicellular maps of low genera with two handles, which is necessary for the proof of Theorem 5.2. Since this section is very technical, we would like to precede it by a description of the main idea of the proof of Theorem 5.2. We should try to keep it light and a bit informal to motivate the reader to understand all the technicalities that will appear after this introduction and that are necessary to present the formal proof of Theorem 5.2.

5.2.1 General idea

Let \mathcal{O} be a set of orientations and let us fix a positive integer n and partitions $\mu, \nu \vdash n$ such that $\ell(\mu) + \ell(\nu) = n - 3$. The most important construction in this section gives an involution

$$\sigma_{\mathcal{O}} : \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)} \rightarrow \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}$$

such that

$$\eta_{\mathcal{O}}(\sigma_{\mathcal{O}}(M)) = 2 - \eta_{\mathcal{O}}(M).$$

It is an easy exercise (later on it will be explained in details) that having the above mentioned involution we can use a simple polynomial interpolation argument to prove Theorem 5.2, thus in the following we are going to focus on the construction of such involution.

Firstly, we need to understand how a map $M \in \widetilde{\mathcal{M}}_{\mu,\nu;2}^{(n)}$ can look like. This map has a unique face, genus 2, and exactly 2 handles appear during its root-deletion process that correspond to edges $e_i(M)$ and $e_j(M)$ with labels $j > i$. Moreover, Lemma 3.2 asserts that no twisted edges appear during the root-deletion process of M , thus $\eta_{\mathcal{O}}(M) \in \{0, 1, 2\}$ and it depends only on the fact whether the root edges of M^i and M^j are handles of the first kind or of the second kind. The most natural idea of how to construct the map $\sigma_{\mathcal{O}}(M)$ is to reverse somehow the root-deletion process of M in a way that the edges of M and $\sigma_{\mathcal{O}}(M)$ with the same labels have the same types and such that the root edges of $\sigma_{\mathcal{O}}(M)^i$ and M^i are handles of different kinds. Similarly the root edges of $\sigma_{\mathcal{O}}(M)^j$ and M^j are handles of different kinds. This suggests that the map $\sigma_{\mathcal{O}}(M)$ might be constructed by twisting some of the edges of M in the appropriate way, similar to how it was done in Lemma 3.6. However, it turned out that in some cases it is impossible and one needs a deeper analysis of the structure of the map M . In the following section, we present an example where twisting some edges never works, and we show on this specific example how to overcome this problem. We believe that this example is the accurate toy-example of the general case.

5.2.2 Example

Let us consider the following map $M \in \widetilde{\mathcal{M}}_{(5),(5);2}^{(5)}$ presented in Fig. 6a. Types of the root edges of the consecutive maps obtained in the root-deletion process of M are as follows: $e_1(M)$ is a handle (of the first kind), $e_2(M)$ is a border, $e_3(M)$ is a handle (of the first kind), $e_4(M)$ is a border again, and $e_5(M)$ is a bridge. We would like to construct $\sigma_{\mathcal{O}}(M)$ by reversing the root-deletion process of M somehow in a way that:

- the edges of M and $\sigma_{\mathcal{O}}(M)$ with the same labels have same types
- the root edges of $\sigma_{\mathcal{O}}(M)^3$ and M^3 are handles of different kinds. Similarly, the root edges of $\sigma_{\mathcal{O}}(M)^1$ and M^1 are handles of different kinds.

Thus, $\sigma_{\mathcal{O}}(M)^5 = M^5$ and $\sigma_{\mathcal{O}}(M)^4 = M^4$. Since the root edges of $\sigma_{\mathcal{O}}(M)^3$ and M^3 are handles of different kinds, $\sigma_{\mathcal{O}}(M)^3$ is obtained from M^3 by twisting its root edge. Thus, if we would like to build $\sigma(M)$ by twisting edges of M , it has to be of the following form: $\sigma(M) = \tau_1^{\epsilon_1} \tau_2^{\epsilon_2} \tau_3 M$, where $\epsilon_1, \epsilon_2 \in \{0, 1\}$. However, for all these possible choices of ϵ_1, ϵ_2 the map $\tau_1^{\epsilon_1} \tau_2^{\epsilon_2} \tau_3 M$ does not satisfy above required properties. Indeed, the root edge of the map $(\tau_1^{\epsilon_1} \tau_2 M)^2 = \tau_2 M^2$ is twisted. See Fig. 6b, which is a problem since we want this root edge to be a border. If we twist it, that is we consider the map $(\tau_1^{\epsilon_1} \tau_2 \tau_3 M)^2 = \tau_1 \tau_2 M^2$, then its root edge is a border, so we fix previous problem. However, we encounter another

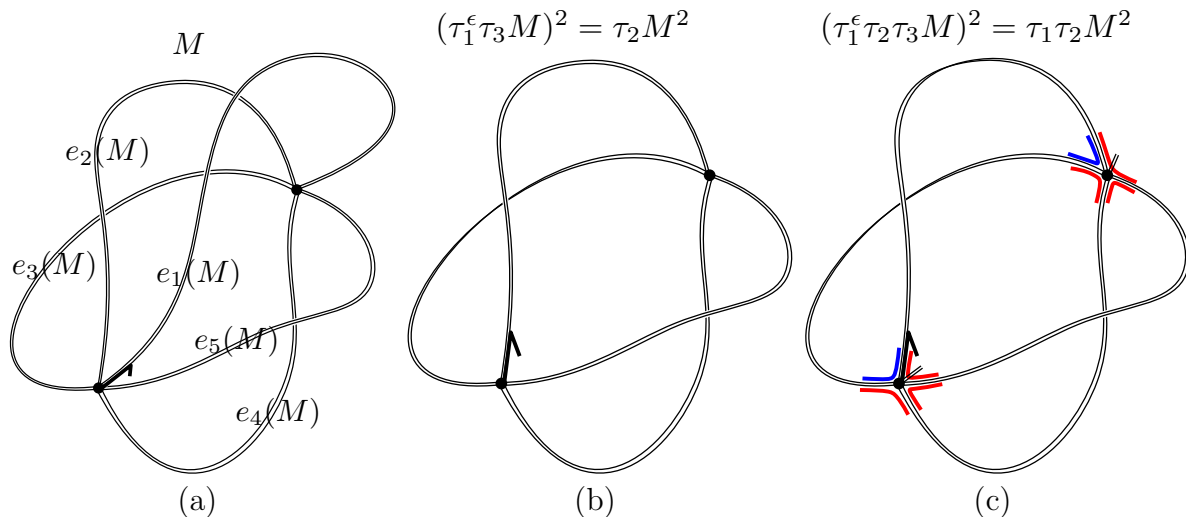


Figure 6: (a) shows the map $M \in \widetilde{\mathcal{M}}_{(5),(5);2}^{(5)}$ analyzed in this section. (b) shows that the map $(\tau_1^\epsilon \tau_3 M)^2$ is unicellular and its root edge is twisted. (c) shows the map $(\tau_1^\epsilon \tau_2 \tau_3 M)^2$, which has two faces and the corners lying in the same face have the same color.

case: half-edges $h_1(\tau_1^{\epsilon_1} \tau_2 \tau_3 M)$ and $h_1(\tau_1^{\epsilon_1} \tau_2 \tau_3 M)'$ lie in the same face of $\tau_1 \tau_2 M^2$, thus the root edge of $\tau_1^{\epsilon_1} \tau_2 \tau_3 M$ cannot be a handle, see Fig. 6b.

A crucial observation which helps to overcome the problem is the following: the set of corners lying in the root face of $\tau_1 \tau_2 M^2$ differs from the set of corners lying in the root face of M^2 (compare Fig. 6c to Fig. 7a). We would like to fix this. One can show that if we erase the edges $e_1(M^2)$ and $e_2(M^2)$ from the map M^2 , but we do not erase the corresponding half-edges $h_1(M^2)$, $h_1(M^2)'$, $h_2(M^2)$, $h_2(M^2)'$, then there is a unique way to choose two pairs from these four half-edges and draw two new edges connecting half-edges in each pair to obtain a map M' different from M^2 such that:

- its root edge is a border,
- the set of corners lying in its root face is exactly the same as the set of corners lying in the root face of M^2 .

This map is shown in Fig. 7b. Moreover, the root edge of $(M')^2$ is a handle of the different kind than the root edge of M^3 . Now, we can finish the construction of $\sigma_{\mathcal{O}}(M)$. The set of corners lying in the root face of M' is the same as the set of corners lying in the root face of M^2 . Thus, if we connect the corners of M' corresponding to the root edge of M by a new edge we will always create a handle (of two possible kinds). Thus, it is enough to connect these corners by a handle of different kind than the root edge of M to construct $\sigma_{\mathcal{O}}(M)$. In our case it is a handle of the second kind and this map is shown in Fig. 7c. It is straightforward from this construction that if we repeat this recipe to construct $\sigma_{\mathcal{O}}(\sigma_{\mathcal{O}}(M))$, we construct exactly the map M .

It turned out that the general situation is basically the same. The next section is devoted to the description of the structure of a map M in a general case when we cannot

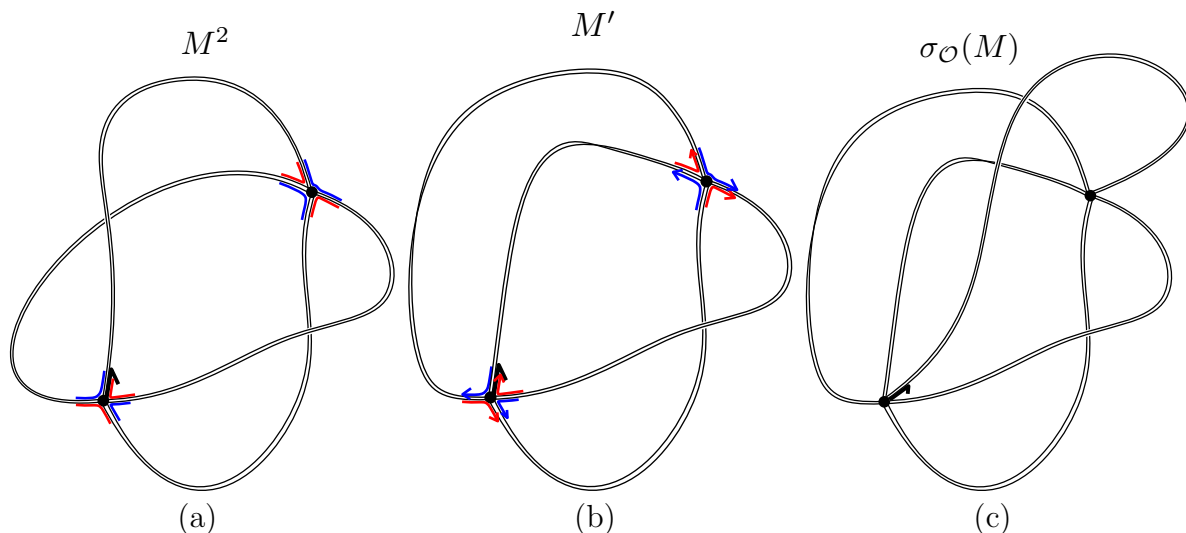


Figure 7: Comparing (a) to (b) we can see that both maps M^2 and M' have two faces, and the corners in the same face have the same colors (red or blue), thus $C(M^2) = C(M')$. (c) shows the map $\sigma_O(M)$ obtained from M' by connecting $h_1(M)$ with $h_1(M')$ in the appropriate way.

construct $\sigma_O(M)$ simply by twisting some edges of M . This description, which is given in Lemma 5.3 and its proof are very technical. However, the reader should think that a general picture looks almost the same as the picture from this section: a map M might have a lot of edges, but only three edges play important role in the construction of $\sigma_O(M)$ and their role is the same as the role of $e_1(M)$, $e_2(M)$, and $e_3(M)$ in the above example.

5.2.3 Details

Let us fix a positive integer n , and partitions $\mu, \nu \vdash n$ such that $\ell(\mu) + \ell(\nu) = n - 3$. Let $M \in \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}$. Then Eq. (11) states that there are no twisted edges in the root-deletion process of M (since $\ell(\mu) + \ell(\nu) = n - 3$). Thus, for all positive integers $k < i$ the root edge of M^k is a bridge and there are two possible situations:

- for all positive integers $i < k < j$ the root edge of M^k is a bridge;
- there exists a unique positive integer $i < k < j$ such that the root edge of M^k is a border. Then there are still two possible cases:
 - $C(M^k) = C((\tau_j M)^k)$;
 - $C(M^k) \neq C((\tau_j M)^k)$.

Note that the map M presented in Section 5.2.2 was exactly the second case of the second case in the above analysis. We start with a technical lemma that treats this case in general. All the symbols n, i, j, μ, ν are as above and we recall that for any $m \in [n]$ the

half-edge $h_m(M)$ is the root of M^m and $h_m(M)'$ is the second half-edge belonging to the root edge $e_m(M)$.

Lemma 5.3. *Let $M \in \widetilde{\mathcal{M}}_{\mu,\nu;2}^{(n)}$. We assume that there exists a positive integer $i < k < j$ such that the root edge of M^k is a border and such that $C(M^m) = C((\tau_j M)^m)$ for all positive integers $m > k$, but $C(M^k) \neq C((\tau_j M)^k)$. Then:*

- (P1) *if M' is the map created from M by erasing edges $e_k(M)$ and $e_j(M)$, and merging pairs of half edges $\{h_k(M), h_j(M)'\}$ and $\{h_j(M), h_k(M)'\}$ into edges e'_k and e'_j , respectively (in arbitrary way), then for all $m \in [n]$ the root edge of M^m is a bridge iff the root edge of $(M')^m$ is a bridge and $h_m(M) = h_m(M')$;*
- (P2) *there is a unique way to construct a map M' as in (P1) such that $C((M')^k) = C(M^k)$; then for all $m \in [n]$ types of the root edges of M^m and $(M')^m$ coincide and for any set \mathcal{O} of orientations of all rooted maps the edges $e_j(M)$ and $e_j(M')$ are handles of different kinds. That is,*

$$\{\eta_{\mathcal{O}}(M^j), \eta_{\mathcal{O}}((M')^j)\} = \{0, 1\}.$$

Proof. Let M' be a map constructed from M as in (P1). Strictly from the construction of root-deletion process – see Section 3.1 – it is clear that if $m \in [n]$, and the root edges of both M^m and $(M')^m$ are not bridges, then the roots of M^m and $(M')^m$ coincide. It is also clear that if the root edges of both M^m , and $(M')^m$ are bridges, and if the second visited corner after the root corner in both maps M^{m+1} and $(M')^{m+1}$ coincide, then the root of M^{m+1} and $(M')^{m+1}$ coincide, too. So in order to prove (P1) it is enough to show that for all $m \in [n]$ the root edge of M^m is a bridge iff the root edge of $(M')^m$ is a bridge. We claim that

- (\star) if one removes the edges containing $h_k(M)$ and $h_j(M)$ from M^k then the resulting object F is connected. Moreover, F is obtained by planting some maps into some corners of M^{j+1} .

This claim easily implies the fact that for all $m \in [n]$ the root edge of M^m is a bridge iff the root edge of $(M')^m$ is a bridge. This may be shown by induction on m . Assume that the root edge of M^m is a bridge iff the root edge of $(M')^m$ is a bridge for all $m < l \leq n$. This implies that for all $m < l$ the root $h_m(M')$ of $(M')^m$ is equal to the root $h_m(M)$ of M^m . Note that (\star) implies that the number of connected components of the graph $M \setminus \{e_1(M), \dots, e_l(M)\}$ is the same as the number of connected components of $M \setminus \{e_k(M), e_j(M)\} \setminus \{e_1(M), \dots, e_l(M)\}$. But strictly from the definition of M' , the last set is equal to $M' \setminus \{e'_k, e'_j\} \setminus \{e_1(M'), \dots, e_l(M')\}$ which has the same number of connected components as $M' \setminus \{e_1(M'), \dots, e_l(M')\}$ by (\star). Thus, the root edge of M^l is a bridge iff the root edge of $(M')^l$ is a bridge, which finishes the proof of (P1).

In order to prove (\star), we need to analyze the structure of the map M^{k+1} . This analysis will be also crucial in proving (P2), (the structure of the map M^{k+1} is shown on Fig. 8). First of all, it is clear from the classification of types of edges – see Section 3.1 – that

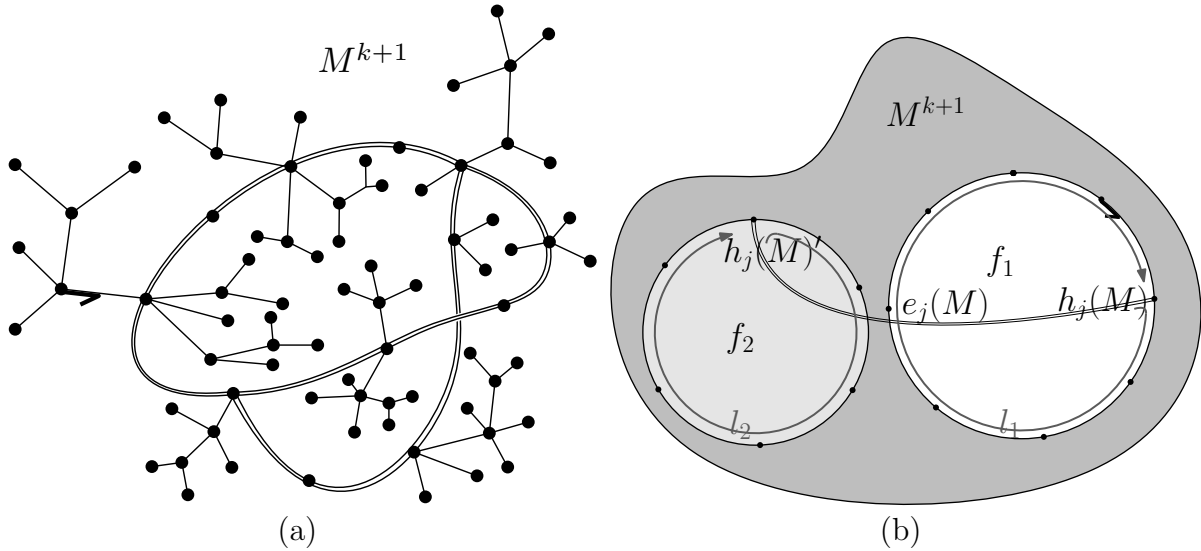


Figure 8: (a) shows that the unicellular map M^{k+1} is obtained from the unicellular map M^j of genus 1 by planting a collection of trees. (b) represents diagrammatically the unicellular map M^{k+1} – the edge $e_j(M)$ is its handle, and two areas f_1, f_2 represents two faces of the map $M^{k+1} \setminus \{e_j(M)\}$, and two oriented arcs l_1, l_2 correspond to the words given by reading consecutive corners visited in the root face.

- for all $m \in [i]$ both maps M^m , and $(\tau_j M)^m$ are unicellular,
- for $i < m \leq k$ both maps M^m , and $(\tau_j M)^m$ have two faces iff they contain a half-edge $h_k(M)$ (otherwise they are unicellular),
- for $k < m \leq j$ both maps M^m , and $(\tau_j M)^m$ are unicellular.

As a result, all of the connected components of $M \setminus \{e_1(M), \dots, e_{j-1}(M)\}$ are planar and unicellular (thus they are trees), except for the component containing the edge $e_j(M)$ which is also unicellular but has genus equal to 1. Indeed, it follows immediately from Euler formula Eq. (9) and from the definition of types of edges given in Section 3.1. Since the root edges of all the maps M^m are bridges for $k < m < j$, the map M^{k+1} is obtained from the unicellular map M^j by planting some trees into some corners of it. In particular, the edge containing $h_j(M)$ has the same type in both maps M^j and M^k . Thus, it is a handle. We conclude that $M^{k+1} \setminus \{e_j(M)\}$ is a map with two faces – in particular it is connected which proves our claim (\star).

Above analysis says that if one removes the edge containing $h_j(M)$ from the map M^{k+1} but does not remove the corresponding half-edges $h_j(M)$ and $h_j(M)'$, the resulting object is a map with two faces f_1 , and f_2 and with two additional half-edges $h_j(M), h_j(M)'$ such that $h_j(M)$ lies in some corner belonging to f_1 , and $h_j(M)'$ lies in some corner belonging to f_2 (see Fig. 8). Let l_1 (l_2 , respectively) be the word given by reading consecutive corners of the face f_1 with additional half-edge $h_j(M)$ (f_2 with additional half-edge $h_j(M)'$, respectively) in a way that the word given by reading consecutive corners of the unique

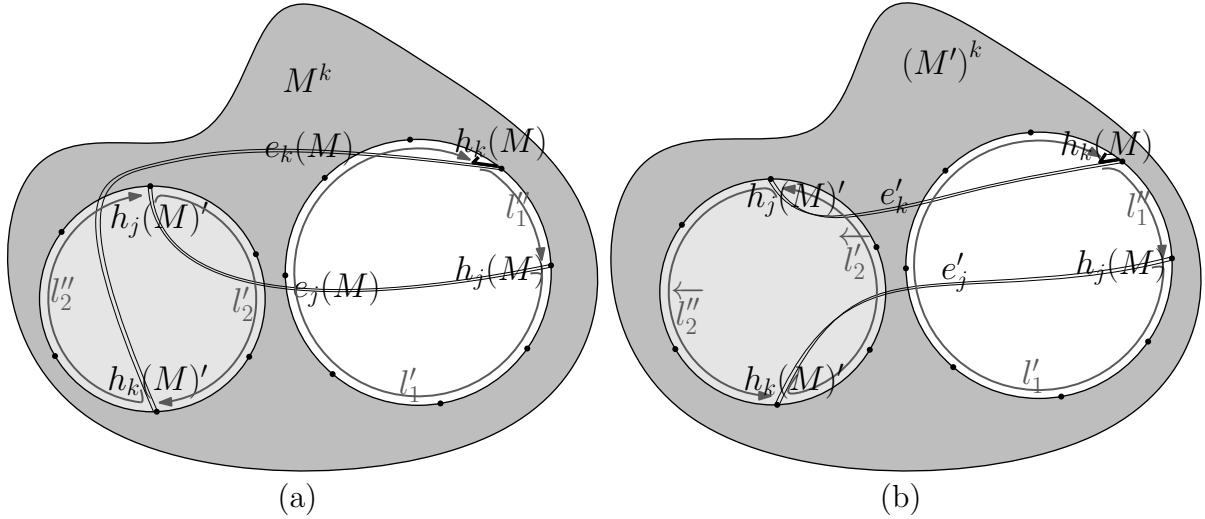


Figure 9: (a) represents diagrammatically a map M^k and (b) represents diagrammatically a map $(M')^k$. Indices of edges $e'_k = e_1((M')^k)$ and $e'_j = e_{j-k+1}((M')^k)$ distinguished in (b) correspond to their labels in M' .

face of the map M^{k+1} with respect to the root orientation is given by the concatenation $l_1 \cdot l_2$ of the words l_1 and l_2 . See Fig. 8. For a given word w , we denote by \overleftarrow{w} the word obtained from w by reading it backwards (from right to left). Then the word given by reading consecutive corners of the unique face of $(\tau_j M)^{k+1} = \tau_{j-k} M^{k+1}$ with respect to the root orientation is given by the concatenation $l_1 \cdot \overleftarrow{l_2}$.

Now, we recall our assumption that $C(M^k) \neq C((\tau_j M)^k)$. This is equivalent to saying that the set of corners belonging to the root face of M^k is different than the set of corners belonging to the root face of $(\tau_j M)^k$ (since both M^k and $(\tau_j M)^k$ have exactly two faces). Thus, the half-edge $h_k(M)$ is lying in some corner belonging to l_1 and incident to $h_{k+1}(M)$ and $h_k(M)$ divides this corner into two new corners. Similarly $h_k(M)'$ is lying in some corner belonging to l_2 and divides it into two new corners (indeed, if both half-edges $h_k(M)$, and $h_k(M)'$ are lying in l_1 , then, clearly, the sets of corners belonging to the root face of M^k and to the root face of $(\tau_j M)^k$ coincide, which gives a contradiction with our assumption). Let l'_1, l''_1 and l'_2, l''_2 , respectively be two new words obtained by reading consecutive corners between $h_k(M)$ and $h_j(M)$ and between $h_k(M)'$ and $h_j(M)'$, respectively, as depicted in Fig. 9a. In other words, l_1 is a concatenation of $(l'_1)_-$, a letter c_1 which corresponds to the corner of M^{k+1} containing $h_k(M)$, and $-(l'_1)$, where w_- ($_-w$, respectively) is a word obtained from w by removing its last (first, respectively) letter. Similarly, l_2 is a concatenation of $(l'_2)_-$, a letter c_2 which correspond to the corner of M^{k+1} containing $h_k(M)'$, and $-(l'_2)$. Then the word given by reading consecutive corners of the root face of M^k , starting from the first corner visited after the root corner, is given by the concatenation $l''_2 \cdot l'_1$, while the word obtained by reading the corners visited consecutively in the root face of $(\tau_j M)^k$, starting from the first corner visited after the root corner, is given by the concatenation $l''_2 \cdot \overleftarrow{l'_1} \cdot l'_2 \cdot l'_1$. In particular, the map $(\tau_j M)^k$ is unicellular,

and its root is twisted, which shows that $C(M^k) \neq C((\tau_j M)^k)$, as we assumed.

Finally, in order to construct a map M' as in (P1) such that $C((M')^k) = C(M^k)$ we need to merge $h_k(M)$ with $h_j(M)'$ and $h_j(M)$ with $h_k(M)'$ in a way such that the word given by reading consecutive corners of the root face of $(M')^k$ with respect to the root orientation and starting from the first visited corner after the root corner is given by the concatenation of $\overleftarrow{l_2''}$ and l_1' . See Fig. 9b. We recall that for $k+1 \leq m \leq j$, if the unicellular map M^m ($(M')^m$, respectively) contains $h_j(M)$, then the edge containing $h_j(M)$ is a handle. Moreover, if we equip two faces f_1, f_2 of $M^m \setminus \{e_j(M^m)\}$ (f_1', f_2' of $(M')^m \setminus \{e_j(M^m)\}$, respectively), where f_1 (f_1' , respectively) is the root face, into the orientation inherited from the the root face of M^m ($(M')^m$, respectively), then the orientations of f_1 , and f_1' are the same, while the orientations of f_2 and f_2' are opposite to each other (compare Fig. 9a to Fig. 9b). By Definition 5.1

$$\{\eta_{\mathcal{O}}(M^j), \eta_{\mathcal{O}}((M')^j)\} = \{0, 1\}.$$

Thus, $e_j(M)$ and $e_j(M')$ are handles of different kinds. In particular, for all integers $m \geq k$ the types of the root edges of M^m and $(M')^m$ are the same. Since $C((M')^k) = C(M^k)$ and since for all $m \in [k-1]$ the root edges of M^m and $(M')^m$ contain the same pairs of half-edges and are not twisted nor bridges, they have the same types, which finishes the proof. \square

Now we are going to show that the assumption of Lemma 5.3 that the root edge of M^k is a border is in fact implied by the assumption that $C(M^k) \neq C((\tau_j M)^k)$.

Lemma 5.4. *Let $M \in \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}$, let $i < j$ be the labels of the edges that are handles in the root-deletion process of M , and let $k < j$ be the largest positive integer such that $C(M^k) \neq C((\tau_j M)^k)$ (we assume that it exists). Then necessarily $i < k < j$ and the root edge of M^k is a border.*

Proof. We recall from the beginning of Section 5.2.3 that there are two possible cases:

- for all positive integers $m \in [j] \setminus \{i, j\}$ the root edge of M^m is a bridge;
- there exists the unique positive integer $i < k < j$ such that the root edge of M^k is a border and for all positive integers $m \in [j] \setminus \{i, k, j\}$ the root edge of M^m is a bridge.

We are going to compare the sets $C(M^{m+1})$ and $C(M^m)$ in the following cases:

- the root edge of M^m is a bridge: there exists $l > m+1$ such that $M^m \setminus \{e_m(M)\} = M^{m+1} \cup M^l$ and there exist two corners $c_1 \in f_1 \in C(M^{m+1})$, and $c_2 \in f_2 \in C(M^l)$ which correspond to the root corners of M^{m+1} and M^l , respectively, and which are divided by $h_m(M)$ and $h_m(M)'$, respectively, into two pairs of new corners c_1', c_1'' ,

and c'_2, c''_2 such that

$$\begin{aligned}
C(M^m) = & \underbrace{C(M^{m+1}) \setminus \{f_1\}}_{\text{the set of corners belonging to the faces of } M^{m+1} \text{ different than the root face}} \\
& \cup \underbrace{C(M^l) \setminus \{f_2\}}_{\text{the set of corners belonging to the faces of } M^l \text{ different than the root face}} \\
& \cup \underbrace{\{f_1 \setminus \{c_1\} \cup f_2 \setminus \{c_2\} \cup \{c'_1, c''_1, c'_2, c''_2\}}}_{\text{the set of corners belonging to the root face of } M^m \text{ obtained by merging root faces of } M^{m+1} \text{ and } M^l}.
\end{aligned}$$

- the root edge of M^m is a handle: there exist two corners $c_1 \in f_1 \in C(M^{m+1})$, and $c_2 \in f_2 \in C(M^{m+1})$ containing $h_m(M)$ and $h_m(M)'$, respectively, such that $f_1 \neq f_2$, and $h_m(M)$, $h_m(M)'$, respectively, divides c_1 and c_2 into two pairs of new corners c'_1, c''_1 , and c'_2, c''_2 , respectively. Thus,

$$\begin{aligned}
C(M^m) = & \underbrace{C(M^{m+1}) \setminus \{f_1, f_2\}}_{\text{the set of corners belonging to the faces of } M^{m+1} \text{ different from the faces merged by a handle } e_m(M)} \\
& \cup \underbrace{\{f_1 \setminus \{c_1\} \cup f_2 \setminus \{c_2\} \cup \{c'_1, c''_1, c'_2, c''_2\}}}_{\text{the set of corners belonging to the root face of } M^m \text{ obtained by merging faces } f_1 \text{ and } f_2}.
\end{aligned}$$

If for all $m \in [j] \setminus \{i, j\}$ the root edge of M^m is a bridge then by Proposition 3.5 also the root edge of $(\tau_j M)^m$ is a bridge thus for all $m \leq j$ types of the root edges of M^m and $(\tau_j M)^m$ coincide and they are either bridges or handles. Since for all $m > j$ the maps M^m and $(\tau_j M)^m$ are the same, above analysis gives immediately that $C(M^m) = C((\tau_j M)^m)$ holds true for all $m \in [n]$. This proves that if there exists $k \in [n]$ such that $C(M^k) \neq C((\tau_j M)^k)$ and such that k is the greatest possible, then necessarily $i < k < j$ and the root edge of M^k is a border, which proves Lemma 5.4. □

We are finally ready to construct the promised involution.

Lemma 5.5. *Let \mathcal{O} be a set of orientations and let us fix a positive integer n and partitions $\mu, \nu \vdash n$ such that $\ell(\mu) + \ell(\nu) = n - 3$. Then, there exists an involution*

$$\sigma_{\mathcal{O}} : \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)} \rightarrow \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}$$

such that

$$\eta_{\mathcal{O}}(\sigma_{\mathcal{O}}(M)) = 2 - \eta_{\mathcal{O}}(M).$$

Proof. Let us fix $M \in \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}$ and let $i < j$ be the labels of the edges that are handles in the root-deletion process of M .

There are two possible cases: for all positive integers m sets $C(M^m)$ and $C((\tau_j M)^m)$ are the same, or not. If they coincide, we consider two maps: $M_1 := \tau_i \tau_j M$, and $M_2 := \tau_j M$. Proposition 3.5 ensures that labels in M_1 , and M_2 are the same as in M . Thus,

types of the root edges of all three maps M^m, M_1^m , and M_2^m coincide, too. In particular, the root edges of M^j , and $M_1^j = M_2^j = \tau_1(M^j)$ are handles so we have

$$\{\eta_{\mathcal{O}}(M^j), \eta_{\mathcal{O}}(\tau_1 M^j)\} = \{0, 1\}. \quad (20)$$

Moreover, the root edge of both M_1^i , and M_2^i is a handle and

$$\{\eta_{\mathcal{O}}(M_1^i), \eta_{\mathcal{O}}(M_2^i)\} = \{\eta_{\mathcal{O}}(\tau_1(\tau_j M)^i), \eta_{\mathcal{O}}((\tau_j M)^i)\} = \{\eta_{\mathcal{O}}((\tau_j M)^{i+1}), \eta_{\mathcal{O}}((\tau_j M)^{i+1}) + 1\}.$$

Combining this with Eq. (20), we obtain that there exists an integer $l \in \{1, 2\}$ such that

$$\eta_{\mathcal{O}}(M_l) = 2 - \eta_{\mathcal{O}}(M).$$

In other words, there exists $\epsilon \in \{0, 1\}$ such that $M_l = \tau_i^\epsilon \tau_j M$. We define:

$$\sigma_{\mathcal{O}}(M) := \tau_i^\epsilon \tau_j M.$$

Now, it is straightforward from the construction that $\sigma_{\mathcal{O}}(M) \in \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}$ and $\sigma_{\mathcal{O}}(\sigma_{\mathcal{O}}(M))$ is again of the same form. That is, $\sigma_{\mathcal{O}}(\sigma_{\mathcal{O}}(M)) := \tau_i^\epsilon \tau_j \sigma_{\mathcal{O}}(M)$. Thus, by Proposition 3.5,

$$\sigma_{\mathcal{O}}(\sigma_{\mathcal{O}}(M)) = \tau_i^{2\epsilon} \tau_j^2 M = M,$$

which finishes the proof in the first case.

Now, we are going to treat the second case. That is, we assume that there exists $k \in [n]$ such that $C(M^k) \neq C((\tau_j M)^k)$, and we choose the greatest possible k with this property. Let M' be the unique map as in Lemma 5.3 (P2) associated with M and we define two maps $M_1 := M', M_2 := \tau_i M'$. First of all, Proposition 3.5, Lemma 5.3, and Lemma 5.4 state that for all positive integers m types of the root edges of all three maps M, M_1 and M_2 coincide. Thus, $M, M_1, M_2 \in \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}$. Moreover, Proposition 3.5 and Lemma 5.3 say that for all positive integers m the roots of all three maps M^m, M_1^m , and M_2^m are the same. Thus, for all positive integers $m > i$ maps M_1^m and M_2^m coincide and

$$\{\eta_{\mathcal{O}}(M^j), \eta_{\mathcal{O}}(M_1^j) = \eta_{\mathcal{O}}(M_2^j) = \eta_{\mathcal{O}}((M')^j)\} = \{0, 1\}, \quad (21)$$

by Lemma 5.3 (P2). Also

$$\begin{aligned} \{\eta_{\mathcal{O}}(M_1^i), \eta_{\mathcal{O}}(M_2^i)\} &= \{\eta_{\mathcal{O}}(\tau_1(M')^i), \eta_{\mathcal{O}}((M')^i)\} \\ &= \{\eta_{\mathcal{O}}((M')^{i+1}), \eta_{\mathcal{O}}((M')^{i+1}) + 1\} = \{\eta_{\mathcal{O}}((M')^j), \eta_{\mathcal{O}}((M')^j) + 1\}, \end{aligned}$$

and combining it with Eq. (21), we obtain that there exists an integer $l \in \{1, 2\}$ such that

$$\eta_{\mathcal{O}}(M_l) = 2 - \eta_{\mathcal{O}}(M).$$

In other words, there exists $\epsilon \in \{0, 1\}$ such that $M_l = \tau_i^\epsilon M'$, and we define:

$$\sigma_{\mathcal{O}}(M) := \tau_i^\epsilon M'.$$

Now, it is straightforward from the construction that $\sigma_{\mathcal{O}}(\sigma_{\mathcal{O}}(M))$ is again of the same form. That is $\sigma_{\mathcal{O}}(\sigma_{\mathcal{O}}(M)) := \tau_i^\epsilon \sigma_{\mathcal{O}}(M)'$, where $\sigma_{\mathcal{O}}(M)'$ is a map given by Lemma 5.3 (P2). Thus,

$$\sigma_{\mathcal{O}}(\sigma_{\mathcal{O}}(M)) = \tau_i^\epsilon (\tau_i^\epsilon M')' = \tau_i^{2\epsilon} (M')' = M,$$

where the last two equalities are clear from the construction of M' given in the proof of Lemma 5.3 (P2) – see Fig. 9.

Since there are no other cases, we have constructed the required involution, which finishes the proof. \square

Now, we have all necessary ingredients to prove Theorem 5.2.

Proof of Theorem 5.2. Let \mathcal{O} be an orientation of all rooted maps and let us fix a positive integer n and partitions $\mu, \nu \vdash n$ such that $\ell(\mu) + \ell(\nu) = n - 3$. Thanks to Proposition 3.7 we know that $(a_{\eta_{\mathcal{O}}})_{\mu, \nu; 2}^{(n)}$ is the polynomial in β of the following form:

$$(a_{\eta_{\mathcal{O}}})_{\mu, \nu; 2}^{(n)}(\beta) := \sum_{M \in \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}} \beta^{\eta_{\mathcal{O}}(M)} = a + b\beta + c\beta^2,$$

where a, b, c are nonnegative integers. Moreover, Lemma 5.5 gives the following equality

$$\sum_{M \in \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}} \beta^{\eta_{\mathcal{O}}(M)} = \sum_{M \in \widetilde{\mathcal{M}}_{\mu, \nu; 2}^{(n)}} \beta^{2 - \eta_{\mathcal{O}}(M)}.$$

Hence

$$(a_{\eta_{\mathcal{O}}})_{\mu, \nu; 2}^{(n)}(\beta) = a + b\beta + a\beta^2.$$

Finally, Proposition 3.9 says that $(a_{\eta_{\mathcal{O}}})_{\mu, \nu; 2}^{(n)}(-1) = 0$. Thus, there exists a positive integer $(\widetilde{a_{\eta_{\mathcal{O}}}})_{\mu, \nu; 2}^{(n)}$ such that

$$(a_{\eta_{\mathcal{O}}})_{\mu, \nu; 2}^{(n)}(\beta) = (\widetilde{a_{\eta_{\mathcal{O}}}})_{\mu, \nu; 2}^{(n)} \cdot (1 + \beta)^2.$$

Corollary 3.3 asserts that for all positive integers n and partitions $\mu, \nu \vdash n$ such that $\ell(\mu) + \ell(\nu) > n - 3$ the set $\widetilde{\mathcal{M}}_{\mu, \nu; i}^{(n)}$ is empty for $i \geq 2$. Thus, Proposition 3.7 and Proposition 3.9 give that for any positive integer n , partitions $\mu, \nu \vdash n$ such that $\ell(\mu) + \ell(\nu) > n - 3$, and a nonnegative integer i , there exists a positive integer $(\widetilde{a_{\eta_{\mathcal{O}}}})_{\mu, \nu; i}^{(n)}$ such that

$$(a_{\eta_{\mathcal{O}}})_{\mu, \nu; i}^{(n)}(\beta) = (\widetilde{a_{\eta_{\mathcal{O}}}})_{\mu, \nu; i}^{(n)} (1 + \beta)^i.$$

Plugging it into Eq. (15), one has the following expression

$$(H_{\eta_{\mathcal{O}}})_{\mu, \nu}^{(n)}(\beta) = \sum_{0 \leq i \leq \lfloor g/2 \rfloor} (\widetilde{a_{\eta_{\mathcal{O}}}})_{\mu, \nu; i}^{(n)} \beta^{g-2i} (\beta + 1)^i,$$

where $g = n + 1 - (\ell(\mu) + \ell(\nu)) \leq 4$, and the above equation involves at most three coefficients $(\widetilde{a_{\eta\mathcal{O}}})_{\mu,\nu;i}^{(n)}$, where $i \in \{0, 1, 2\}$. Notice that Lemma 4.3 gives a similar expression for quantities $h_{\mu,\nu}^{(n)}(\beta)$:

$$h_{\mu,\nu}^{(n)}(\beta) = \sum_{0 \leq i \leq \lfloor g/2 \rfloor} a_{\mu,\nu;i}^{(n)} \beta^{g-2i} (\beta + 1)^i,$$

and, again, for $g \leq 4$, it involves at most three coefficients $a_{\mu,\nu;i}^{(n)}$, where $i \in \{0, 1, 2\}$. By Theorem 1.4 we know that

$$h_{\mu,\nu}^{(n)}(\beta) = (H_{\eta\mathcal{O}})_{\mu,\nu}^{(n)}(\beta)$$

for $\beta \in \{-1, 0, 1\}$. So for a fixed positive integer n and partitions $\mu, \nu \vdash n$ such that $\ell(\mu) + \ell(\nu) \geq n - 3$ it gives rise to a system of three equations with at most three indeterminates. It is easy to check that this system is non-degenerate. Thus, it has a unique solution. In other words $a_{\mu,\nu;i}^{(n)} = (\widetilde{a_{\eta\mathcal{O}}})_{\mu,\nu;i}^{(n)}$ for all nonnegative integers i , which finishes the proof. \square

6 Concluding remarks

We are going to finish this paper by posing some natural questions and remarks related to the combinatorial side of b -conjecture.

6.1 Removing edges in a different order

Note that the function η given by Definition 3.1 is built recursively by the root-deletion procedure, which gives a natural order on the set of edges of a given map M . One can wonder if there are some other, natural ways to define an order on $E(M)$ which have chances to give an affirmative answer to Conjecture 1.2 using the statistic η as in Definition 3.1, but with respect to the considered order. For instance, the author of this paper together with Féray and Śniady studied [6] a problem of understanding a combinatorial structure of Jack characters, already mentioned in Section 1.4. We proved that a similarly defined “measure of non-orientability”, but considered with respect to the uniform random order on $E(M)$, has many desired properties. Chapuy and the author of this paper constructed in [2] a certain directed graph associated with a bipartite quadrangulation \mathbf{q} (that is a map with all faces of degree 4), called the Dual Exploration Graph (DEG, for short), which is visiting all faces of \mathbf{q} in some particular order. Since maps (not necessarily bipartite) with n edges are in a natural bijection with bipartite quadrangulations with n faces and edges of a given map correspond to faces of an associated quadrangulation, DEG defines also an order on the set of edges of a given map and La Croix suggested [17] to use this order to define a measure of non-orientability η with respect to it. We did not study combinatorial properties of the statistic η defined in this way and we leave open the problem whether it gives the correct answer for b -conjecture.

6.2 Unhandled maps and evaluation at $b = -1$

Corollary 3.3 suggests that unhandled maps are of special interest: indeed, for any measure of non-orientability η and for any partitions μ, ν, τ of a positive integer n , the top-degree coefficient in the polynomial $(H_\eta)_{\mu, \nu}^\tau(\beta)$ is given by unhandled maps of type $(\mu, \nu; \tau)$. In particular this top-degree part does not depend on the choice of η , but it does depend on the order of edges we are removing from a given map. Moreover, Eq. (16) ensures that in the case of one-part partition $\tau = (n)$, the top-degree coefficient in the polynomial $(H_\eta)_{\mu, \nu}^\tau(\beta)$ coincides, up to a sign, with the evaluation of this polynomial in $\beta = -1$. We can also prove that for any partitions μ, ν of a positive integer $n \geq 2$ and for any $l \in [n-1]$ the top-degree coefficient in the polynomial $(H_{\eta_O})_{\mu, \nu}^{(n-l, l)}(\beta)$ coincides with the evaluation of this polynomial in $\beta = -1$ for any set O of orientations (in fact, the main idea of the proof in the case when $\ell(\mu) + \ell(\nu) \geq n-3$ was given in Lemma 5.3, and the general case is almost the same). Thus, there are natural questions:

Question 6.1. *Is it true that for any measure of non-orientability η , for any positive integer n , and for any partitions $\mu, \nu, \tau \vdash n$ the following equality holds true:*

$$(H_\eta)_{\mu, \nu}^\tau(-1) = \# \left(\widetilde{M}_\eta \right)_{\mu, \nu; 0}^\tau ?$$

Question 6.2. *Is it true that for any measure of non-orientability η , for any positive integer n , and for any partitions $\mu, \nu, \tau \vdash n$ the following equality holds true:*

$$(H_\eta)_{\mu, \nu}^\tau(-1) = h_{\mu, \nu}^\tau(-1)?$$

Another interesting direction of the research initiated in this paper is an understanding of the combinatorial structure of an unhandled map of type $(\mu, \nu; \tau)$ for arbitrary partitions μ, ν, τ of a positive integer n . Note that the set of unhandled maps of a given type is not rooted invariant. However, Proposition 4.1 together with Theorem 4.2 imply that unicellular unhandled maps with the black vertex distribution μ and the white vertex distribution ν are in a bijection with *orientable maps* with the black vertex distribution μ , the white vertex distribution ν , and the *arbitrary* face distribution, which clearly *are rooted invariant* (one can even use a proof of Theorem 4.2 to construct such a bijection recursively). Therefore one can ask the following question:

Question 6.3. *Is it true that for any given type $(\mu, \nu; \tau)$ of unhandled maps, there exists some class of maps (orientable?), with the black vertex distribution μ , the white vertex distribution ν , and possibly some additional data (labeling faces?), which is rooted invariant, and which is in some natural bijection with the corresponding set of unhandled maps?*

Finally, one can refine Question 6.1 by asking:

Question 6.4. *Is it true that for any measure of non-orientability η , for any positive integer n , and for any partitions $\mu, \nu, \tau \vdash n$ the following equality holds true:*

$$(H_\eta)_{\mu, \nu}^\tau(\beta) = \begin{cases} \sum_{0 \leq i \leq [g/2]} \widetilde{a_{\mu, \nu; i}^\tau} \beta^{g-2i} (\beta+1)^i & \text{for } \ell(\mu) + \ell(\nu) + \ell(\tau) \leq 2+n, \\ 0 & \text{otherwise;} \end{cases}$$

where $g := 2 + n - (\ell(\mu) + \ell(\nu) + \ell(\tau))$ and $2^i \widetilde{a_{\mu,\nu;i}^\tau} = \# \left(\widetilde{M_\eta} \right)_{\mu,\nu;i}^\tau$?

We leave all these questions wide open for future research.

Acknowledgments

I thank Valentin Féray and Piotr Śniady for several years of collaboration on topics related to the current paper. I also thank Michael La Croix for a very interesting discussion concerning the b -Conjecture. English proofreading service was financed by *Narodowe Centrum Nauki*, grant number 2014/15/B/ST1/00064.

References

- [1] D. R. L. Brown and D. M. Jackson. A rooted map invariant, non-orientability and Jack symmetric functions. *J. Combin. Theory Ser. B*, 97(3):430–452, 2007.
- [2] G. Chapuy and M. Dołęga. A bijection for rooted maps on general surfaces. *J. Combin. Theory Ser. A*, 145:252–307, 2017.
- [3] A. Czyżewska-Jankowska and P. Śniady. Bijection between oriented maps and weighted non-oriented maps. *Electron. J. Combin.*, 24(3): #P3.7, 2017.
- [4] M. Dołęga and V. Féray. Gaussian fluctuations of Young diagrams and structure constants of Jack characters. *Duke Math. J.*, 165(7):1193–1282, 2016.
- [5] M. Dołęga and V. Féray. Cumulants of Jack symmetric functions and b -conjecture. *Trans. Amer. Math. Soc.*, 2017. [doi:10.1090/tran/7191](https://doi.org/10.1090/tran/7191).
- [6] M. Dołęga, V. Féray, and P. Śniady. Jack polynomials and orientability generating series of maps. *Sém. Lothar. Combin.*, 70:Art. B70j, 50 pp. (electronic), 2014.
- [7] B. Eynard. *Counting Surfaces*, volume 70. Birkhäuser Basel, 2016.
- [8] I. P. Goulden, J. L. Harer, and D. M. Jackson. A geometric parametrization for the virtual Euler characteristics of the moduli spaces of real and complex algebraic curves. *Trans. Amer. Math. Soc.*, 353(11):4405–4427 (electronic), 2001.
- [9] I. P. Goulden and D. M. Jackson. Connection coefficients, matchings, maps and combinatorial conjectures for Jack symmetric functions. *Trans. Amer. Math. Soc.*, 348(3):873–892, 1996.
- [10] I. P. Goulden and D. M. Jackson. Maps in locally orientable surfaces, the double coset algebra, and zonal polynomials. *Canad. J. Math.*, 48(3):569–584, 1996.
- [11] D. M. Jackson and T. I. Visentin. A character-theoretic approach to embeddings of rooted maps in an orientable surface of given genus. *Trans. Amer. Math. Soc.*, 322(1):343–363, 1990.
- [12] F. Knop and S. Sahi. A recursion and a combinatorial formula for Jack polynomials. *Inv. Math.*, 128(1):9–22, 1997.

- [13] A. I. Kanunnikov and E. A. Vassilieva. On the matchings-Jack conjecture for Jack connection coefficients indexed by two single part partitions. *Electron. J. Combin.*, 23(1):Paper #P1.53, 2016.
- [14] M. A. La Croix. *The combinatorics of the Jack parameter and the genus series for topological maps*. PhD thesis, University of Waterloo, 2009.
- [15] M. Lassalle. A positivity conjecture for Jack polynomials. *Math. Res. Lett.*, 15(4):661–681, 2008.
- [16] M. Lassalle. Jack polynomials and free cumulants. *Adv. Math.*, 222(6):2227–2269, 2009.
- [17] M. La Croix. Private communication, 2015.
- [18] S. K. Lando and A. K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. With an appendix by Don B. Zagier.
- [19] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [20] Gilles Schaeffer. Planar maps. In *Handbook of enumerative combinatorics*, Discrete Math. Appl. (Boca Raton), pages 335–395. CRC Press, Boca Raton, FL, 2015.
- [21] P. Śniady. Private communication, 2015.
- [22] P. Śniady. Top degree of Jack characters and enumeration of maps. [arXiv:1506.06361](https://arxiv.org/abs/1506.06361).
- [23] R. P. Stanley. Some combinatorial properties of Jack symmetric functions. *Adv. Math.*, 77(1):76–115, 1989.
- [24] E. A. Vassilieva. Polynomial properties of Jack connection coefficients and generalization of a result by Dénes. *Journal of Algebraic Combinatorics*, pages 1–21, 2014.