On almost-planar graphs

Guoli Ding*

Mathematics Department
Louisiana State University
Baton Rouge, Louisiana, U.S.A.

ding@math.lsu.edu

Joshua Fallon

Mathematics Department
Louisiana State University
Baton Rouge, Louisiana, U.S.A.

jfallo3@math.lsu.edu

Emily Marshall

Computer Science and Mathematics Department
Arcadia University
Glenside, Pennsylvania, U.S.A.

marshalle@arcadia.edu

Submitted: Nov 11, 2016; Accepted: Feb 13, 2018; Published: Mar 16, 2018
Mathematics Subject Classifications: 05C75, 05C10, 05C83

Abstract

A nonplanar graph G is called *almost-planar* if for every edge e of G, at least one of $G\backslash e$ and G/e is planar. In 1990, Gubser characterized 3-connected almost-planar graphs in his dissertation. However, his proof is so long that only a small portion of it was published. The main purpose of this paper is to provide a short proof of this result. We also discuss the structure of almost-planar graphs that are not 3-connected.

1 Introduction

A nonplanar graph G is called *almost-planar* if for every edge e of G, at least one of $G\backslash e$ and G/e is planar. The following is an equivalent definition. For any specified set $\mathcal{S}$ of graphs, let us call a graph $\mathcal{S}$*-free* if it does not contain any graph in $\mathcal{S}$ as a minor. Using this terminology, we can say that a graph G is almost-planar if and only if G is not $\{K_5, K_{3,3}\}$-free but for every edge e of G, at least one of $G\backslash e$ and G/e is $\{K_5, K_{3,3}\}$-free.

This notion of almost-planar is in fact a special case of a concept in matroid theory. Given a set $\mathcal{S}$ of matroids, we say a matroid M is $\mathcal{S}$*-free* if no matroid in $\mathcal{S}$ is a minor of M. If $\mathcal{M}$ denotes the class of all $\mathcal{S}$-free matroids, then a matroid M is called $\mathcal{S}$*-fragile* or *almost-*$\mathcal{M}$ if M is not $\mathcal{S}$-free but for every element e of M, at least one of $M\backslash e$ and

*Supported in part by NSF Grant DMS - 1500699

M/e is $\mathcal{S}$-free. Fragility has recently received significant attention since it plays a crucial role in proving Rota's Conjecture [5], and it is an effective stepping stone for determining excluded minors. Classes of $\mathcal{S}$-fragile matroids have been determined for several choices of $\mathcal{S}$, and one of the first of these results is a characterization of $\{K_5, K_{3,3}\}$-fragile 3-connected graphic matroids. In purely graph-theoretic terms, this is a characterization of 3-connected almost-planar graphs. This result was obtained by Gubser in his dissertation [2] and Kingan and Lemos used this result to determine all almost-graphic matroids [3].

Gubser's result is important beyond simply being one of the first results on fragility. In terms of explicit graph structures, very few excluded-minor characterizations of minor-closed classes of graphs are known beyond that of planar graphs. For instance, the complete list of forbidden minors is not known for the class of toroidal graphs, or for *apex graphs*, the class $\{G : G\backslash v$ is planar for some $v \in V(G)\}$, or for the class $\{G : G\backslash e$ is planar for some $e \in E(G)\}$ (these are special toroidal graphs). This disappointing situation motivates careful examination of graphs that are, in some meaningful sense, close to being planar because these graphs could bring new insight to our understanding of nonplanarity. Almost-planar graphs are a family of such graphs and examining this family is one of the main motivations of this paper. In this vein, Wagner proved in a recent paper [8] that almost-planar graphs, as well as several other classes of graphs, can be Delta-wye reduced to certain small graphs. Since there are other graphs that admit such a reduction, Wagner's result only provides a necessary condition for being almost-planar; it does not imply Gubser's result. However, it seems that Wagner's result does follow from Gubser's characterization.

We remark that loops and parallel edges can be ignored in the study of almost-planar graphs. Certainly no almost-planar graphs have loops. If e and f are parallel edges in a nonplanar graph G, then certainly neither $G\backslash e$ nor $G\backslash f$ is planar. If, however, G/e is planar, then $G' = G\backslash f$ is nonplanar and G'/e is planar. This implies that non-simple almost-planar graphs are precisely those obtained from simple almost-planar graphs G by adding edges parallel to edges e of G for which G/e is planar. For this reason we assume that all graphs in this paper are simple. We also assume that after any graph operation the resulting graph is automatically simplified so we only need to handle simple graphs.

To state the characterization of Gubser we need to define a few families of graphs. Let $n \geqslant 3$ be an integer. A *wheel* of size n, denoted by W_n, is the graph obtained from a cycle of length n (called the *rim* of the wheel) by adding a *hub* vertex and joining it to all vertices of the cycle with *spokes*. A *double wheel* of size n, denoted by DW_n, is the graph obtained from W_n by adding a second hub, a new vertex adjacent to all vertices of the wheel. The edge between the hubs is the *axle* of DW_n. A *Möbius ladder* of length n, denoted by M_n, is obtained from a cycle of length $2n$ (called the *rim* of the ladder) by joining opposite pairs of vertices on the cycle. Let $\mathcal{W}$ denote the set of all graphs constructed by identifying three triangles from three wheels. In other words, each graph $G \in \mathcal{W}$ admits a partition (V_0, V_1, V_2, V_3) of its vertex set such that $G[V_0]$ is a triangle, $G[V_0 \cup V_i]$ is a wheel for $i = 1, 2, 3$, and G has no edges other than those in these three wheels ($G[X]$ is the subgraph of G induced by the vertices of X). Finally, for each $G \in \{K_5, K_{3,3}\}$, let G^+ be obtained from G by adding a new vertex v and a new edge e

between v and a vertex of G, and let G^h be obtained from G by subdividing two adjacent edges and then joining the two new vertices with a new edge called a handle edge.

Theorem 1 (Gubser [1])**.** *Let G be a 3-connected nonplanar graph. Then the following are equivalent.*

(i) G is almost-planar;

(ii) G is a minor of a double wheel, a Möbius ladder, or a graph in $\mathcal{W}$;

(iii) G is $\mathcal{F}$-free, where $\mathcal{F} = \{EX_1, EX_2, EX_3, EX_6, EX_8\}$ shown in Figure 1.

(iv) G is $\{K_5^+, K_{3,3}^+, K_5^h, K_{3,3}^h\}$-free.

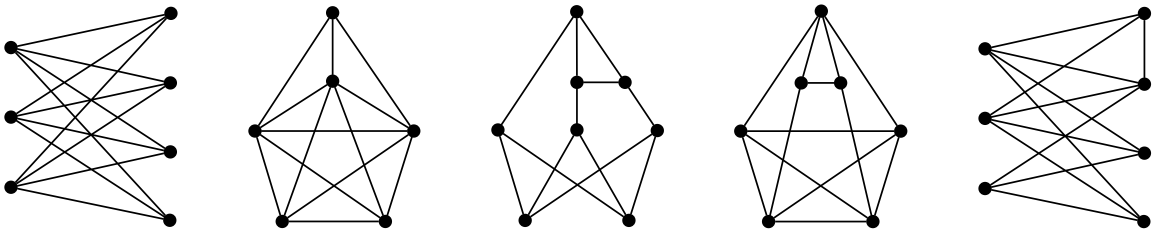

Figure 1: Forbidden graphs EX_1, EX_2, EX_3, EX_6, and EX_8

We remark that terminology $EX_1, EX_2, \ldots, EX_8$ is inherited from [1], and that our formulation of this theorem is slightly different from that given in [1]. First, we modified the definition of graphs in $\mathcal{W}$ to better illustrate the structure of these graphs. Second, set $\mathcal{F}$ given in [1] consists of eight graphs, instead of five graphs. These eight include the five listed in Figure 1 and another three graphs, which were denoted by EX_4, EX_5, and EX_7. Since each of these three extra graphs contains EX_8 as a minor, they are removed from our statement. Finally, the theorem as presented in [1] does not include statement (iv). We remark that graphs EX_3 and EX_6 are isomorphic to $K_{3,3}^h$ and K_5^h, respectively.

Seymour's Splitter Theorem [7] implies every almost-planar graph can be generated from K_5 or $K_{3,3}$ by repeated application of two *extensions*: addition of an edge or splitting a vertex, although not every graph constructed this way has to be almost-planar. Therefore, to prove that every almost-planar graph must be one of the three types listed in (ii), one only needs to show that any extension of any graph of those three types either results in another graph of one of the three types or contains a member of $\mathcal{F}$ as a minor. The proof in [2] took this approach and was divided into analysis of fifteen cases, depending on which graph is extended and by what sequence of extensions. Since the case checking is lengthy, only two of the fifteen cases are published in [1].

In addition to being quite long, Gubser's proof focuses only on local operations, losing sight of the global structure of almost-planar graphs. For instance, graphs in $\mathcal{W}$ are defined descriptively in [1] (by naming the vertices and edges) while our definition is constructive, revealing the structure of these graphs. In addition, in our constructive proof, each step of the construction (stated as lemmas) is interesting in its own right and is potentially useful elsewhere.

In Sections 2 and 3 we provide a short proof of Theorem 1. We conclude in Section 4 with a discussion of the structure of almost-planar graphs that are not 3-connected, correcting a few flaws appearing in [1].

We close this section by making two more remarks. Notice that graphs in $\mathcal{W}$ can be naturally divided into three groups, depending on how the three wheels' hubs are distributed on the common triangle. In the case that each hub is a distinct vertex of the triangle, the resulting graph is in fact a minor of a Möbius ladder, as illustrated in Figure 2. We could therefore define $\mathcal{W}$ differently so that these graphs are not included in $\mathcal{W}$ but we leave them in $\mathcal{W}$ since it makes the definition more concise. This is another difference between our formulation of Theorem 1 and the formulation given in [2].

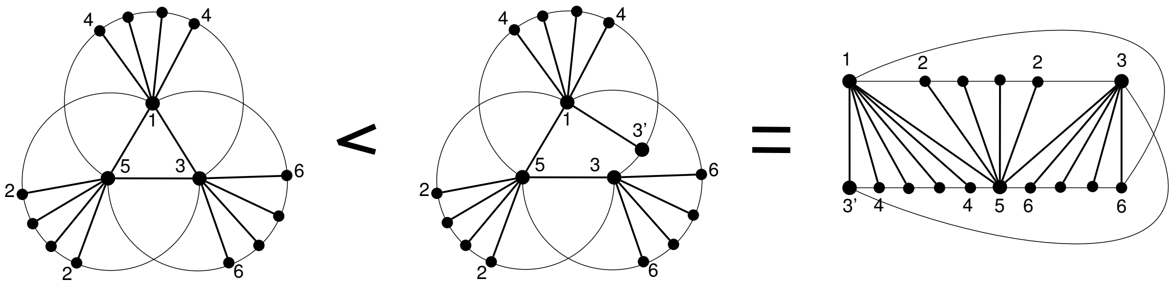

Figure 2: Some graphs in $\mathcal{W}$ are minors of a Möbius ladder

For $G \in \{K_5, K_{3,3}\}$, let e be the edge added to obtain G^+. Since both $G^+\backslash e$ and G^+/e are nonplanar, G^+ is not almost-planar. Notice that EX_2 contains K_5^+, while EX_1 and EX_8 contain $K_{3,3}^+$. Hence EX_1, EX_2, and EX_8 are not minimal in the set of non-almost-planar nonplanar graphs. However, by Theorem 1, within the universe of 3-connected graphs, these three graphs are minimal and in fact, the only five minimal graphs are precisely the members of $\mathcal{F}$. In Section 4, we determine all minimal graphs without imposing any connectivity.

2 A few lemmas

In this section we present a few lemmas that will be used in our main proof. As usual, let $|G| := |V(G)|$. Since we restrict our discussion to simple graphs, we use the following definition of *separation*. For any integer $k \geqslant 0$, a k-separation of a graph G is an unordered pair $\{G_1, G_2\}$ of proper induced subgraphs of G such that $G_1 \cup G_2 = G$ and $|G_1 \cap G_2| = k$. A graph on more than k vertices is *k-connected* if it has no k'-separation for any $k' < k$. Our first lemma is (2.4) of [6], which we state using our terminology. We remark that our G is obtained from G of (2.4) by adding a new vertex x and joining x to all vertices of $\bar{\Omega}$.

Lemma 2. *Let x be a vertex of a graph G and let σ be a circular permutation of E_x, the set of edges incident with x. Suppose G has no k-separation $\{G_1, G_2\}$ with $k \leqslant 2$ and $E_x \subseteq E(G_i)$ for some i. Then either G admits a planar drawing such that σ is the local rotation at x or G has a subgraph G' such that either G' is a subdivision of $K_{3,3}$ with $\deg_{G'}(x) = 3$ or G' is a union of two cycles C, D with $V(C \cap D) = \{x\}$ and under σ the four edges of $E_x \cap E(C \cup D)$ alternate between C and D.*

The next result we need follows immediately from Lemma 2.

Lemma 3. *Let x be a cubic vertex of a nonplanar graph G. Suppose G has no k-separation $\{G_1, G_2\}$ with $k \leqslant 2$ such that some G_i contains x and all its three neighbors. Then G has a subgraph G' such that G' is a subdivision of $K_{3,3}$ and x is cubic in G'.*

A 3-connected graph G with $|G| \geqslant 5$ is called *internally* 4*-connected* if for every 3-separation $\{G_1, G_2\}$ of G, exactly one of G_1, G_2 is $K_{1,3}$. We point out that no cubic vertex belongs to a triangle in an internally 4-connected graph, as we will use this property repeatedly. Let C_n^2 be the graph obtained from a cycle of length n by joining all pairs of vertices of distance two on the cycle.

Lemma 4. *Let G be an internally 4-connected nonplanar minor of a Möbius ladder. Then $G = M_n$ or C_{2n-1}^2 for some integer $n \geqslant 3$.*

Proof. Let $k \geqslant 3$ be the smallest integer such that G is a minor of M_k. Let M_k consist of a Hamilton cycle C and pairwise crossing chords. The minimality of k implies that no chord is deleted in obtaining G. Moreover, since G is nonplanar, no chord of C is contracted and no edge of C is deleted. So G is obtained by contracting some edges of C. Since no cubic vertex of G belongs to a triangle and contraction of an edge of C creates two cubic vertices in a triangle, either the triangle edge in C must be contracted, implying G is a minor of M_{k-1} (and contradicting the minimality of k), or the two edges incident to these cubic vertices and outside the triangle must be contracted. In this case, new cubic vertices in a triangle will be created and the process must be iterated. If k is even, then this process leads to the contraction of all of C so k is not even. If k is odd, then the process terminates with C_k^2. □

We remark that the lemma holds even if we drop the nonplanarity assumption.

The next is a theorem of Maharry and Robertson [4] on M_4-free graphs. An *alternating double wheel* of length $2n$ ($n \geqslant 2$), denoted by AW_{2n}, is obtained from a cycle $v_1 v_2 \dots v_{2n} v_1$ by adding two new adjacent vertices u_1, u_2 such that u_i is adjacent to v_{2j+i} for all $i = 1, 2$ and $j = 0, 1, \dots, n-1$.

Lemma 5 (Maharry and Robertson [4])**.** *If an internally* 4*-connected graph G is M_4-free then at least one of the following holds.*

(i) G is planar;

(ii) $|G| \geqslant 8$ and $G \backslash \{v_1, v_2, v_3, v_4\}$ is edgeless, for some distinct $v_1, v_2, v_3, v_4 \in V(G)$;

(iii) $G = DW_{n+3}$ or AW_{2n} for some $n \geqslant 3$;

(iv) G is $L(K_{3,3})$, the line graph of $K_{3,3}$;

(v) G has fewer than eight vertices.

To concisely state the next result and its proof, we introduce some structural terminology and notation. For any two vertices x, y of a path P, let $P[x, y]$ denote the subpath of P between x and y. If G is a subdivision of a graph H, then *branch vertices* of G are vertices of G that correspond to vertices of H, and *arcs* of G are paths of G that correspond to edges of H. A *triad addition* of H is obtained by adding a new vertex v to H and joining v to three distinct vertices of H.

Lemma 6. *Let H be a graph with $|H| \geqslant 3$. If a 3-connected graph G has a subgraph H' such that H' is a subdivision of H and $|H'| < |G|$, then G contains a triad addition of H as a minor.*

Proof. Let G' be a subgraph of G such that G' is a subdivision of H and $|G'| < |G|$. Let $v \in V(G) \setminus V(G')$ and let P_1, P_2, P_3 be three paths of G from v to G' that are disjoint except for v. For $i = 1, 2, 3$ let v_i be the endvertex of P_i in G'. We first prove that G', P_1, P_2, P_3 can be chosen so that v_1, v_2, v_3 are not contained in a single arc of G'.

Suppose v_1, v_2, v_3 are contained in an arc A of G'. Let x, y be the two ends of A and let x, v_1, v_2, v_3, y be the order in which these vertices appear along A. Choose P_1, P_2, P_3 and A such that $|A[x, v_1] \cup A[y, v_3]|$ is minimized. Since G is 3-connected, $G \backslash \{v_1, v_3\}$ has a path Q with one endvertex s in $(P_1 \cup P_2 \cup P_3 \cup A[v_1, v_3]) \backslash \{v_1, v_3\}$ and the other endvertex t in $G' \backslash V(A[v_1, v_3])$. By the minimality of $|A[x, v_1] \cup A[y, v_3]|$, t must belong to $G' \backslash V(A)$. It follows that $G' \cup P_1 \cup P_2 \cup P_3 \cup Q$ contains a subdivision of a triad addition of H.

Now we assume that v_1, v_2, v_3 are not contained in a single arc. Since this property can be preserved when we contract G' to H, we obtain in this case a triad addition of H as a minor. □

3 A proof of Theorem 1

Before proving Theorem 1 we establish a few lemmas, which are the main parts of our proof. We begin with some definitions. A *3-sum* of two disjoint graphs G_1, G_2 is obtained by identifying a triangle T_1 of G_1 with a triangle T_2 of G_2, and then deleting 0, 1, 2, or 3 of the three identified edges. We call T_i the *summing triangle* of G_i ($i = 1, 2$). Observe that 3-sums are not uniquely determined by G_1, G_2; rather, they depend on both which vertices are identified and which edges of the summing triangles remain. For disjoint graphs $G_0, \ldots, G_k$ ($k \geqslant 0$), let $S(G_0) = G_0$ and let $S(G_0, \ldots, G_k)$ ($k \geqslant 1$) denote a graph obtained by iteratively 3-summing $G_1, \ldots, G_k$ to G_0 over distinct summing triangles of G_0. In other words, we define $S(G_0, \ldots, G_k)$ inductively as a 3-sum of $S(G_0, \ldots, G_{k-1})$ and G_k, such that the summing triangle of $S(G_0, \ldots, G_{k-1})$ is contained in G_0 and is different from all previous summing triangles.

A 3-connected graph G is called 3^+*-connected* if $G \backslash X$ has at most two components for every set X of three vertices. Given a 3-separation $\{G_1, G_2\}$ of G, for $i = 1, 2$, let G_i^Δ be obtained from G_i by adding all missing edges among vertices of $G_1 \cap G_2$; let G_i^Y be obtained from G_i by adding a new vertex v_i adjacent to all three vertices of $G_1 \cap G_2$.

Lemma 7. *Every 3^+-connected nonplanar graph G can be expressed as $S(G_0, \ldots, G_k)$ ($k \geqslant 0$), where G_0 is an internally 4-connected nonplanar minor of G. Moreover, for any $1 \leqslant i_1 < \ldots < i_t \leqslant k$, there is a minor of G that can be expressed as $S(G_0, G_{i_1}, \ldots, G_{i_t})$.*

Proof. We begin with the first assertion of the lemma. Suppose the assertion is false, and consider a counterexample G with $|G|$ as small as possible. Clearly, G is not internally 4-connected since $G_0 = G$ would trivially satisfy the lemma. Thus G must have a 3-separation $\{H_1, H_2\}$ such that neither part is $K_{1,3}$. Let $V(H_1 \cap H_2) = \{x, y, z\}$. Since G

is nonplanar, at least one of H_1^Y and H_2^Y is nonplanar. Without loss of generality, suppose H_1^Y is nonplanar. Let us choose such a 3-separation with $|H_1|$ as small as possible. Since $H_2 \neq K_{1,3}$, H_1^Δ is a 3-connected minor of G. Notice that H_1^Δ is also 3^+-connected, because any 3-cut of H_1^Δ leaving more than two components would do the same to G. Moreover, H_1^Δ is nonplanar, because otherwise, since H_1^Y is nonplanar, triangle xyz would not be a facial triangle of H_1^Δ, which implies $G\backslash\{x, y, z\}$ would have more than two components, contradicting the 3^+-connectivity of G. Therefore, H_1^Δ satisfies the assumptions of the lemma. Since $|H_1^\Delta| < |G|$, H_1^Δ can be expressed as $S(G_0, \ldots, G_k)$ for an internally 4-connected minor G_0 of H_1^Δ. By the minimality of H_1, triangle xyz must be contained in G_0 and G can be expressed as $S(G_0, \ldots, G_k, H_2^\Delta)$. This contradiction proves the first assertion of the lemma.

To prove the second assertion, let $G = S(G_0, \ldots, G_k)$ and for $i = 1, \ldots, k$ let $T_i \subseteq G_0$ be the triangle over which G_0 and G_i are 3-summed. To simplify our notation in the rest of this proof, let us slightly abuse our terminology by assuming each $V_i := V(G_i)$ is a subset of $V(G)$. Then for each $i = 1, \ldots, k$, the summing triangle of G_i is also T_i and the only difference between G_i and $G[V_i]$ is that T_i is a triangle of G_i but not all edges of T_i are necessarily in $G[V_i]$. The difference between G_0 and $G[V_0]$ is similar. For $i = 0, \ldots, k$, let $S_i = S(G_0, \ldots, G_i)$. From the last paragraph we deduce that $S_i[V_i]$ can be contracted to T_i ($i = 1, \ldots, k$). Equivalently, $S_i[V_i]$ contains a cycle. Note that even when $S_i[V_i]$ contains a cycle, $G[V_i]$ may be a tree, and in such a case $G[V_i] = K_{1,3}$. The discrepancy occurs because 3-summing $G_{i+1}, \ldots, G_k$ to S_i removed some edges of $S_i[V_i]$. Therefore, if $G[V_i] = K_{1,3}$ then T_i has a common edge with T_j for some $j > i$. We use this property in the remaining proof and refer to it as property (*).

Let G be expressed as $S(G_0, \ldots, G_k)$ with property (*). To prove the second assertion we only need to show: for each $i = 1, \ldots, k$, G has a minor that can be expressed as $S(G_0, \ldots, G_{i-1}, G_{i+1}, \ldots, G_k)$ with property (*). If $G[V_i] \neq K_{1,3}$, let G' be obtained from G by contracting $G[V_i]$ to T_i. Then G' is a required minor. On the other hand, if $G[V_i] = K_{1,3}$, by property (*), there exists $j > i$ such that $|T_i \cap T_j| = 2$. Let $V_i = \{w, x, y, z\}$, where y, z are in T_j. Then G/wx is a required minor. □

Lemma 8. *Let G be 3-connected and $\{K_{3,3}^+, K_{3,3}^h\}$-free. If $\{G_1, G_2\}$ is a 3-separation of G such that G_1^Y is nonplanar, then G_2^Δ is a wheel.*

Proof. Let $V(G_1 \cap G_2) = \{v_1, v_2, v_3\}$ and let x be the only vertex of $G_1^Y \backslash V(G_1)$. Since G_1^Y is nonplanar and G is 3-connected, by Lemma 3, G_1^Y has a subgraph G' such that G' is a subdivision of $K_{3,3}$ and $\deg_{G'}(x) = 3$. Let $u \in V(G_2)\backslash\{v_1, v_2, v_3\}$. Since G is 3-connected, G_2 contains three induced paths P_1, P_2, P_3 from u to each of v_1, v_2, v_3, respectively, such that the three paths are disjoint except for u. Then $(G'\backslash x) \cup P_1 \cup P_2 \cup P_3 \subseteq G$ is a subdivision of $K_{3,3}$. Since G is $K_{3,3}^+$-free, G_2 contains no vertex outside $P_1 \cup P_2 \cup P_3$; since G is $K_{3,3}^h$-free, $G_2\backslash\{v_1, v_2, v_3\}$ contains no edge outside $P_1 \cup P_2 \cup P_3$.

If P_1, P_2, P_3 are all single edges, then G_2^Δ is a wheel W_3. Otherwise, suppose without loss of generality that P_1 is not a single edge and let w be any internal vertex of P_1. Since P_1 is an induced path and G is 3-connected, w must be incident with an edge outside P_1. As observed above, the only possible neighbors of w outside P_1 are v_2 and v_3. If w is adjacent to both v_2 and v_3 then G has a $K_{3,3}^+$ minor. So w is adjacent to exactly one of v_2, v_3. If w, w' are distinct internal vertices of P_1 such that both wv_2 and $w'v_3$ are edges of G, then G again has a $K_{3,3}^+$ minor. Therefore, all internal vertices of P_1 are cubic and they have a common neighbor outside P_1, which we may assume to be v_2.

Since G is $K_{3,3}^+$-free, we deduce that $P_2 = uv_2$ and no internal vertex of P_3 is adjacent to v_1. All vertices of $P_3 \backslash v_3$ are therefore cubic and adjacent to v_2, so G_2 consists of path $P_1 \cup P_3$, all edges from v_2 to $(P_1 \backslash v_1) \cup (P_3 \backslash v_3)$, and possibly edges between v_1, v_2, v_3, implying G_2^Δ is a wheel. □

Lemma 9. *Let G be connected and $\{K_{3,3}^+, K_5^h\}$-free. If G contains an M_4 minor then G is a minor of M_n for some $n \geqslant 4$.*

Proof. Since M_4 is cubic (and a minor of G), G necessarily has a subgraph H isomorphic to a subdivision of M_4. Since G is $K_{3,3}^+$-free, H must be a spanning subgraph and no chord of M_4 is subdivided. It follows that the rim of H is a Hamilton cycle C of G. Let $v_0, \ldots, v_7$ be the branch vertices of H, listed in the order they appear on C. We denote the path of C from v_i to v_{i+1} by A_i, where $i = 0, \ldots, 7$. (In this proof the indices are taken modulo 8.)

Let $e \in E(G) \backslash E(H)$. We prove the endvertices of e are in A_i and A_{i+4} for some i. First, observe that no A_i contains both ends of e because then $H + e$ contains a $K_{3,3}^+$ minor. Next, assume one end of e is an internal vertex of some A_i. If the other end of e does not belong to A_{i+4}, it is straightforward to verify that $H + e$ can be contracted to $M_4 + v_j v_{j+2}$, for some j, and thus G contains a $K_{3,3}^+$ minor. So both ends of e are cubic vertices of H, implying $e = v_i v_{i+t}$ $(t = 3, 5)$, which satisfies the requirement.

To finish the proof, we only need to show that any two non-adjacent chords e, f of C must cross. The previous discussion makes this clear if the four ends of e, f are not contained in $A_i \cup A_{i+4}$ for any i. If all ends of e, f are contained in $A_i \cup A_{i+4}$ for some i and if e, f do not cross each other, then $H + e + f$ can be contracted to $M_4 + v_i v_{i+5} + v_{i+1} v_{i+4}$, which contains a K_5^h minor. Therefore, e, f must cross and that completes our proof. □

Lemma 10. *Let G be internally 4-connected, nonplanar, and $\{K_5^+, K_{3,3}^+, K_5^h, K_{3,3}^h\}$-free. Then G is M_n or C_{2n+1}^2 or DW_n or AW_{2n} for some $n \geqslant 3$.*

Proof. If G has an M_4 minor then the result follows immediately from Lemmas 9 and 4. So we assume G is M_4-free and observe nonplanarity implies one of (ii)-(v) of Lemma 5. Case (iv) does not hold since $L(K_{3,3})$ contains a $K_{3,3}^h$ minor, as shown in Figure 3, and case (iii) is one of the results of the current lemma.

Suppose (ii) holds. Let $V(G) \backslash \{v_1, v_2, v_3, v_4\} = \{u_1, \ldots, u_k\}$. Note that each u_i has degree either 3 or 4. If two or more of them are of degree 4 then $K_{3,3}^+$ is a minor of G. If every u_i is cubic, since no two of them have the same set of neighbors, it follows that $k = 4$ and G is the cube, contradicting the nonplanarity of G. We may therefore assume u_1 has degree 4 and all other u_i are cubic. As a result, $k = 4$ or 5. If $k = 4$ then G is AW_6. If $k = 5$ then $G \backslash u_1$ is the cube and G has a $K_{3,3}^+$ minor, as shown in Figure 3.

It remains to consider case (v). If $|G| \leqslant 5$, then since G is nonplanar we have $G = K_5 = DW_3$. If $|G| = 6$, then G contains a $K_{3,3}$ subgraph, as every 3-connected nonplanar

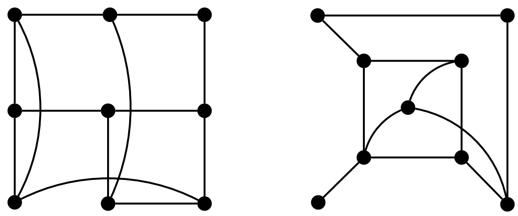

Figure 3: $K^h_{3,3}$ is a minor of $L(K_{3,3})$, and $K^+_{3,3}$ is a minor of Cube$+v$

graph with more than five vertices contains a $K_{3,3}$ minor. Since no cubic vertex of G belongs to a triangle, it follows that either $G = K_{3,3} = M_3$, or for each color class of $K_{3,3}$, G contains at least two edges with both endvertices in that color class. If G contains no other edges then $G = DW_4$; if G contains at least one other edge then G contains K^+_5.

If $|G| = 7$, we may assume G contains a subgraph H obtained from $K_{3,3}$ by subdividing an edge exactly once. Let $B = \{b_1, b_2, b_3\}$ and $R = \{r_1, r_2, r_3\}$ be a partition of the cubic vertices of H corresponding to the two color classes of $K_{3,3}$ and let v be the vertex subdividing edge b_1r_1. Since v belongs to a triangle of G, we deduce that v has degree at least 4 in G. If v is adjacent to both b_2 and b_3 or both r_2 and r_3, then G contains $K^+_{3,3}$. Thus we see the degree of v is 4 and v has one additional neighbor in each of B and R, say b_2 and r_2. Since b_1 and r_1 are in triangles, they must have degree at least 4. If G contains the edge b_1r_1, then G contains $K^+_{3,3}$, so b_1 must be adjacent to b_2 or b_3, and r_1 must be adjacent to r_2 or r_3. It follows that b_3 and r_3 are in triangles and thus they must also have degree at least 4. Let H' be the graph obtained from H by adding edges vb_2 and vr_2, and observe that $H' + b_1b_3 + r_1r_2 \cong H + b_1b_3 + r_2r_3$, which contains K^h_5. This implies G is obtained from H' either by adding edges b_1b_3 and r_1r_3, so $G = C^2_7$, or by adding paths $b_1b_2b_3$ and $r_1r_2r_3$, in which case $G = DW_5$. □

Lemma 11. *Suppose that a $\{K^+_5, K^+_{3,3}, K^h_5, K^h_{3,3}\}$-free graph G can be expressed as*

$$S(G_0, \ldots, G_k),$$

where $G_0 = K_5$ and each G_i $(i > 0)$ is a wheel. Then G is a minor of some M_n or DW_n.

Proof. Let $V(G_0) = \{1, 2, 3, 4, 5\}$ and let $\mathcal{T}$ be the set of summing triangles of G_0 used to obtain G. We first determine $\mathcal{T}$ and then determine how wheels are 3-summed to each member of $\mathcal{T}$. We assume G is M_4-free because otherwise the result holds by Lemma 9.

Let $\mathcal{T}_1 = \{123, 124, 134\}$ and $\mathcal{T}_2 = \{123, 234, 145\}$. We claim that, up to permutation of vertex labels, either $\mathcal{T} = \mathcal{T}_1$ or $\mathcal{T} \subseteq \mathcal{T}_2$. For any triangle xyz, we can *turn it into a triad* by deleting all three edges between x, y, and z and adding a degree 3 vertex adjacent to x, y and z. First we observe that turning triangles 123, 124, 125 into triads in G_0 results in $K^h_{3,3}$, and turning 123, 134, 145 into triads results in M_4. Thus, by the second half of Lemma 7 neither $\{123, 124, 125\}$ nor $\{123, 134, 145\}$ is a subset of $\mathcal{T}$. Similarly, $\{123, 124, 134, 234\} \nsubseteq \mathcal{T}$ because otherwise G contains a $K^+_{3,3}$ minor (deleting any of the triads still leaves a nonplanar graph). Now it is straightforward to verify that either some vertex i belongs to at least three summing triangles, in which case $\mathcal{T} = \mathcal{T}_1$, or every vertex i belongs to at most two summing triangles, in which case $\mathcal{T} \subseteq \mathcal{T}_2$.

Suppose a wheel W_t is 3-summed to G_0 over $xyz \in \mathcal{T}$. If $t = 3$ then at least one of the three edges of xyz is not in G, because otherwise G contains a K_5^+ minor. If $t > 3$ and the hub of W_t is identified with x then $yz \notin E(G)$ because otherwise G contains a K_5^h minor (Figure 4(i)).

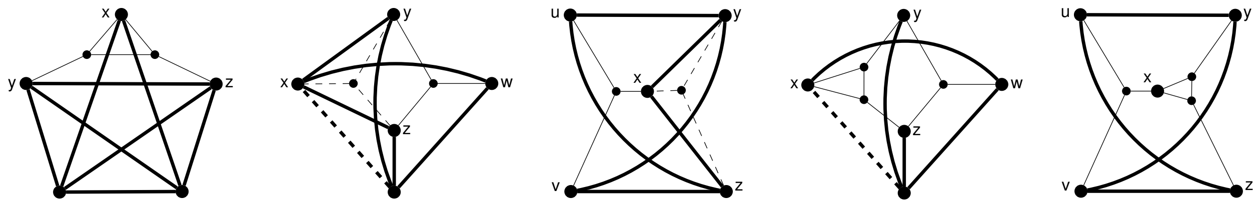

Figure 4: G contains K_5^h, $K_{3,3}^+$, or $K_{3,3}^h$

We call x a *hub* of xyz if either $t > 3$ and the hub of W_t is identified with x, or $t = 3$ and $yz \notin E(G)$. The above observations imply that every summing triangle has a hub, and if x is a hub then $yz \notin E(G)$. To complete the proof we need three observations about $xyz \in \mathcal{T}$.

(a) *If $yz \notin E(G)$ then either xyz has only one hub or x is a hub of xyz.*

(b) *If $yzw \in \mathcal{T}$ then either y or z is a hub of xyz, where we assume $w \neq x$.*

(c) *If $xuv \in \mathcal{T}$ then either y or z is a hub of xyz, where we assume $\{y, z\} \cap \{u, v\} = \emptyset$.*

Statement (a) is clear since if $t > 3$ then xyz has only one hub and if $t = 3$ then x is a hub. To prove (b) and (c) we assume that neither y nor z is a hub of xyz. Then either $t > 3$ and the hub of W_t is identified with x, or $t = 3$ and both xy and xz are edges of G. In both cases (b) and (c), if $t = 3$ then G contains a $K_{3,3}^+$ minor (Figure 4(ii-iii)), and if $t > 3$ then G contains a $K_{3,3}^h$ minor (Figure 4(iv-v)). These contradictions prove both (b) and (c).

Suppose $\mathcal{T} = \mathcal{T}_1$. Then none of the edges 23, 24, 34 are in G, because otherwise G contains a $K_{3,3}^+$ minor. This observation, together with (a) and (b), implies that 1 is a hub for all three summing triangles. As a result, $G \backslash \{1, 5\}$ is a cycle and G is a minor of some DW_n.

Suppose $\mathcal{T} = \mathcal{T}_2$. We first note that $14 \notin E(G)$ because otherwise G contains a $K_{3,3}^h$ minor. This and observations (a)-(c) imply that 5 is a hub of 145 and that $\{2, 3\}$ contains a hub of 123 and a hub of 234. If 2 (or 3) is a hub for both 123 and 234, then G is a minor of a double wheel (Figure 5(i)); if 2 is a hub of 123 and 3 is a hub of 234, then G is a minor of a Möbius ladder (Figure 5(ii)).

Finally, suppose $\mathcal{T} \subset \mathcal{T}_2$. Up to symmetry there are four possible choices for $\mathcal{T}$: $\emptyset$, $\{123\}$, $\{123, 234\}$, and $\{123, 145\}$. All these are degenerate forms of the last case. It is routine to verify using (b) and (c) that G is a minor of a double wheel, except when 2 is a hub of 123 and 3 is a hub of 234, in which case G is a minor of a Möbius ladder. $\square$

Now we are ready to prove Theorem 1.

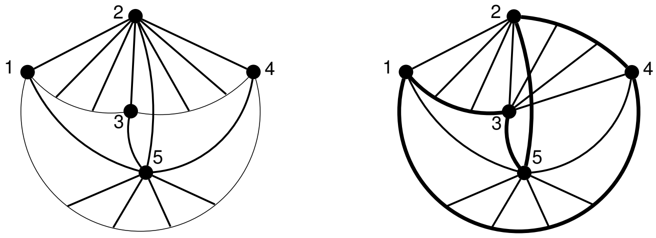

Figure 5: G is a minor of a double wheel or a Möbius ladder

Proof of Theorem 1. We prove implications (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): Note that, if $\mathcal{P}$ is the class of all planar and almost-planar graphs, then $\mathcal{P}$ is closed under taking minors. Therefore, we only need to verify that each Möbius ladder, each double wheel, and each graph in $\mathcal{W}$ is almost-planar. In M_n, deleting a rim edge or contracting a chord results in a planar graph, so M_n is almost-planar. In DW_n, deleting the axle or a rim edge or contracting a spoke results in a planar graph, so DW_n is almost-planar. In any $G \in \mathcal{W}$, contracting a spoke or an edge in the common triangle or deleting a rim edge results in a planar graph, so G is almost-planar.

(i) $\Rightarrow$ (iii): Since $\mathcal{P}$ is closed under taking minors, we only need to show that each $H \in \mathcal{F}$ is not in $\mathcal{P}$. To do this, we only need to find an edge e such that both $H\backslash e$ and H/e are nonplanar. In EX_1 every edge satisfies the requirement. In EX_2 and EX_8, any edge incident with the top cubic vertex (of the drawing shown in Figure 1) has the required property. In $K_{3,3}^h$ and K_5^h (EX_3 and EX_6, respectively), the handle edge meets the requirement.

(iii) $\Rightarrow$ (iv): Since $\{K_5^h, K_{3,3}^h\} \subseteq \mathcal{F}$, we only need to show that $\mathcal{F}$-free G is $\{K_5^+, K_{3,3}^+\}$-free. Suppose G has a minor in $\{K_5^+, K_{3,3}^+\}$. Then G has a vertex v such that $G\backslash v$ is nonplanar, which implies that G has a non-spanning subgraph G' that is a subdivision of K_5 or $K_{3,3}$. By Lemma 6, G contains a triad addition of K_5 or $K_{3,3}$ as a minor. Since EX_2 is the only triad addition of K_5, and EX_1, EX_8 are the only triad additions of $K_{3,3}$, by contrapositive (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (ii): Assume G is 3-connected, nonplanar, and $\{K_5^+, K_{3,3}^+, K_5^h, K_{3,3}^h\}$-free. We need to show G is a minor of a double wheel, a Möbius ladder, or a graph in $\mathcal{W}$. If G is not 3^+-connected, then there exists $V_0 \subseteq V(G)$ such that $|V_0| = 3$ and $G\backslash V_0$ has more than two components. Since G is $K_{3,3}^+$-free, $G\backslash V_0$ must have exactly three components. Let V_1, V_2, V_3 be the vertex sets of these three components and let G_i ($i = 1, 2, 3$) be obtained from $G[V_0 \cup V_i]$ by adding all missing edges between vertices of V_0. By applying Lemma 8 to each 3-separation $\{G\backslash V_i, G[V_0 \cup V_i]\}$ of G ($i = 1, 2, 3$), we conclude that G_i is a wheel, which implies G is a subgraph of a member of $\mathcal{W}$. We may therefore assume G is 3^+-connected.

By Lemma 7, G can be expressed as $S(G_0, \ldots, G_k)$ for an internally 4-connected nonplanar minor G_0 of G. We deduce from Lemma 10 that $G_0 = M_n$ or C_{2n+1}^2 or DW_n or AW_{2n} for some $n \geqslant 3$. We also deduce from Lemma 8 that each G_i ($i > 0$) is a wheel. Note that (ii) holds if $k = 0$ because C_{2n+1}^2 is a minor of M_{2n+1} and AW_{2n} is a minor of DW_{2n}. So we assume $k \geqslant 1$. Since M_n and AW_{2n} do not have any triangles, it follows that $G_0 = C^2_{2n+1}$ or DW_n for some $n \geqslant 3$.

Suppose $G_0 = C^2_{2n+1}$ $(n \geqslant 3)$. Observe that a 3-sum of G_0 and a wheel is a graph with an M_4 minor, so by Lemma 9 G is a minor of a Möbius ladder.

The case $G_0 = DW_3 = K_5$ is covered by Lemma 11. So we assume $G_0 = DW_n$ $(n \geqslant 4)$. Let G_0 have hubs u_1, u_2 and rim cycle $C = v_0v_1 \dots v_{n-1}$ (indices are taken modulo n in this case). Let H be a 3-sum of G_0 and a wheel W_t over a triangle T. If T is $u_1u_2v_i$ then H contains a $K^+_{3,3}$ minor since $H \backslash v_{i+2}$ contains a $K_{3,3}$ minor. There is only one other type of triangle in G_0, so suppose T is $u_1v_1v_2$. If v_1v_2 is an edge of H then H contains a K^h_5 minor, which can be seen by contracting paths $v_3 \dots v_{n-1}$ and $W_t \backslash V(T)$. An almost identical configuration also shows if $t \geqslant 4$ then the hub of W_t is not identified with v_1 or v_2. The effect of 3-summing W_t to G_0 is therefore to subdivide an edge v_iv_{i+1} and join all the subdividing vertices with some u_j, creating a subgraph of a larger double wheel obtained by deleting some spokes. Many wheels can be added to DW_n in the same fashion. Thus 3-summing wheels to DW_n either creates a $K^+_{3,3}$ or K^h_5 minor or results in a minor of a double wheel. The proof of Theorem 1 is complete. □

4 Graphs of low connectivity

In the published version [1] of [2], graphs of low connectivity are also considered. Some of the statements in [1] are not accurate. Its second main theorem (Theorem 2.2) states: *A graph is neither planar nor almost-planar if and only if it has a* $\{EX_i : 1 \leqslant i \leqslant 8\}$*-minor.* As we pointed out in the introduction, $K^+_{3,3}$ is a counterexample to this statement. In this section we prove a corrected version of this theorem.

For any graph G, let $G \oplus e$ be obtained from G by adding two adjacent new vertices, and let G^* be obtained from G by deleting all its isolated vertices (assuming that $E(G) \neq \emptyset$). Let $D(G)$ be the set of edges e of G such that $G \backslash e$ is planar.

Theorem 12. *The following are equivalent for any nonplanar graph G.*

(i) G is almost-planar;

(ii) G^* is obtained from a 3-connected almost-planar H by subdividing edges in $D(H)$;

(iii) G is $\mathcal{F}'$-free, where $\mathcal{F}' = \{K^+_5, K^+_{3,3}, K^h_5, K^h_{3,3}, K_5 \oplus e, K_{3,3} \oplus e\}$.

One possible sharpening of (ii) is to describe $D(H)$ explicitly. If H is a Möbius ladder or a double wheel or a graph in $\mathcal{W}$, then we have determined $D(H)$ in the proof of Theorem 1. For each nonplanar minor H' of H, one could also describe $D(H')$ because the structure of H is simple enough. However, we choose not to include such a description here since its derivation, although straightforward, is tedious. Statement (ii) is also touched on in [1], but the treatment is not rigorous. We include here the treatment of statement (ii) as presented in [1].

> Two corollaries of Theorem 2.1 characterize those almost-planar graphs that are not 3-connected. The elementary proofs are omitted.
>
> **Corollary 2.4.** *If G is a connected, almost-planar graph, then G is a series-parallel extension of a simple, 3-connected, almost-planar graph.*
>
> **Corollary 2.5.** *If G is a disconnected, almost-planar graph, then G is the union of a connected, almost-planar graph and a set of isolated vertices.*

In particular, both corollaries are not "if and only if" type of statements, and (2.4) cannot be turned into such a statement. Our (ii) corrects both problems.

Proof of Theorem 12. We prove implications (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i): Since adding isolated vertices to an almost-planar graph results in an almost-planar graph, we only need to show that G^* is almost-planar. This is clear because for each edge e of G^*, $G^*\backslash e$ is planar if e is an edge obtained by subdividing an edge of $D(H)$, while G^*/e is planar if e belongs to $E(H)\backslash D(H)$.

(i) $\Rightarrow$ (iii): As in the proof of (i) $\Rightarrow$ (iii) of Theorem 1, we only need to find, for each $H \in \mathcal{F}'$, an edge e of H such that both $H\backslash e$ and H/e are nonplanar. The case $H = K_5^h$ or $K_{3,3}^h$ was settled in the proof of Theorem 1. If $H = K_5^+$, $K_{3,3}^+$, $K_5 \oplus e$, or $K_{3,3} \oplus e$, then the edge outside K_5 or $K_{3,3}$ satisfies the requirement.

(iii) $\Rightarrow$ (ii): Suppose the implication does not hold. We choose a counterexample G with $|G|$ as small as possible. We first prove G is connected. If G is disconnected, then since G is nonplanar one component of G contains a K_5 or $K_{3,3}$ minor. If another component of G contains an edge, then G contains $K_5 \oplus e$ or $K_{3,3} \oplus e$ as a minor. So G must have an isolated vertex v. By the minimality of G, $G\backslash v$ satisfies (ii), implying G satisfies (ii), which is a contradiction. Thus G must be connected.

If G has a 1-separation $\{G_1, G_2\}$, then since G is nonplanar at least one G_i contains K_5 or $K_{3,3}$ as a minor. Since each G_i has at least one edge, this implies G contains K_5^+ or $K_{3,3}^+$ as a minor. This contradicts (iii), so G must be 2-connected.

Since 3-connected $\{K_5^+, K_{3,3}^+, K_5^h, K_{3,3}^h\}$-free nonplanar graphs are almost-planar, as shown in the proof of Theorem 1 (implications (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i)), G cannot be 3-connected and thus G must have a 2-separation $\{G_1, G_2\}$. Let $V(G_1 \cap G_2) = \{x, y\}$ and, for $i = 1, 2$, let $H_i = G_i$ (or $G_i + xy$, if $xy \notin E(G)$. Since G is nonplanar, we may assume without loss of generality that H_1 is nonplanar. If G_1 is nonplanar, then G contains a K_5^+ or $K_{3,3}^+$ minor. So $xy \notin E(G)$ and G_1 is planar.

Let P be an induced xy-path of G_2. Since $G_1 \cup P$ is a subdivision of nonplanar H_1, G_2 cannot have a vertex outside P, for then G contains a K_5^+ or $K_{3,3}^+$ minor. Therefore, $G_2 = P$ and G is obtained from H_1 by subdividing edge xy. By the minimality of G, H_1 must satisfy (ii). Let H_1 be obtained from a 3-connected almost-planar graph H by subdividing edges in $D(H)$. Each edge of $E(H)\backslash D(H)$ is not subdivided in the formation of H_1 and deleting such an edge in H_1 leaves a nonplanar graph. Therefore, xy is not such an edge since $H_1\backslash xy = G_1$ is planar. It follows that there is an edge $e \in D(H)$ such that xy belongs to a path (of H_1) obtained by subdividing e. Now it is clear that G is obtained by repeatedly subdividing e, and thus G satisfies (ii), which contradicts the assumption that G is a counterexample. This contradiction completes our proof. $\square$

References

[1] Bradley Gubser, A Characterization of Almost-Planar Graphs, *Combinatorics, Probability & Computing*, 5(3):227-245, 1996.

[2] Bradley Gubser, *Some problems for graph minors*, PhD thesis, Louisiana State University, 1990.

[3] S.R. Kingan and Manoel Lemos, Almost-graphic matroids, *Advances in Applied Mathematics*, 28(3-4):438-477, 2002.

[4] John Maharry and Neil Robertson, The structure of graphs not topologically containing the Wagner graph, *Journal of Combinatorial Theory Series* B, 121:398-420, 2016.

[5] Dillon Mayhew, Geoff Whittle, and Stefan H.M. van Zwam, Stability, fragility, and Rota's Conjecture, *Journal of Combinatorial Theory Series* B, 102:760-783, 2012.

[6] Neil Robertson and Paul Seymour, Graph minors IX: disjoint crossed paths, *Journal of Combinatorial Theory Series* B, 49:40-77, 1990.

[7] Paul Seymour, Decomposition of regular matroids, *Journal of Combinatorial Theory Series* B, 28:305-359, 1980.

[8] Donald K. Wagner, Delta-wye reduction of almost-planar graphs, *Discrete Applied Mathematics*, 180:158-167, 2015.