

Finding Balance: Split Graphs and Related Classes

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Abstract

A graph is a split graph if its vertex set can be partitioned into a clique and a stable set. A split graph is unbalanced if there exist two such partitions that are distinct. Cheng, Collins and Trenk (2016), discovered the following interesting counting fact: unlabeled, unbalanced split graphs on n vertices can be placed into a bijection with all unlabeled split graphs on $n - 1$ or fewer vertices. In this paper we translate these concepts and the theorem to different combinatorial settings: minimal set covers, bipartite graphs with a distinguished block and posets of height one.

Keywords: split graph, set cover, bipartite graph, bipartite poset, bijection

1 Introduction

1.1 Background

In this paper, we consider unlabeled graphs as in [12], that is, two graphs are considered the same if there is an isomorphism between them. For any graph G , the number of vertices in a largest clique is denoted by $\omega(G)$ and the number of vertices in a largest stable (independent) set is denoted $\alpha(G)$. We denote by $G[X]$, the graph induced in G by $X \subseteq V(G)$. A finite graph G is a *split graph* if its vertex set can be partitioned into $K \cup S$ where $G[K]$ is a clique and $G[S]$ is a stable set. We refer to such a partition as a *split graph partition* or a KS -partition of G .

It is easy to see from the definition that the complement of a split graph is again a split graph, and that a split graph does not contain a chordless odd cycle on five or

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more vertices. Therefore by the Strong Perfect Graph Theorem, split graphs are perfect. Indeed, the closely related class of double split graphs are one of the five families of graphs that form the base case of the inductive proof of the Strong Perfect Graph Theorem, conjectured in [1] and proven in [4].

In [3], we categorized split graphs based on their KS -partitions and defined *balanced* and *unbalanced* split graphs. The terms *balanced* and *unbalanced* in Definition 1 refer to a split graph G while the terms K -max and S -max refer to a particular KS -partition of G .

Definition 1. A split graph G is *balanced* if it has a KS -partition satisfying $|K| = \omega(G)$ and $|S| = \alpha(G)$ and *unbalanced* otherwise. A KS -partition is S -max if $|S| = \alpha(G)$ and K -max if $|K| = \omega(G)$.

The first and last columns of Figure 1 show all nine split graphs on four vertices. The first column shows a K -max partition of the vertices and the last column shows an S -max partition. The graphs in the first eight rows are unbalanced and the graph in the ninth row is balanced. As this example illustrates, the KS -partitions of an unbalanced split graph are not unique. The next theorem follows from the work of Hammer and Simeone [9] and appears in [7].

Theorem 2. (Hammer and Simeone) *For any KS -partition of a split graph G , exactly one of the following holds:*

- (i) $|K| = \omega(G)$ and $|S| = \alpha(G)$. (balanced)
- (ii) $|K| = \omega(G) - 1$ and $|S| = \alpha(G)$. (unbalanced, S -max)
- (iii) $|K| = \omega(G)$ and $|S| = \alpha(G) - 1$. (unbalanced, K -max)

Moreover, in (ii) there exists $s \in S$ so that $K \cup \{s\}$ is complete and in (iii) there exists $k \in K$ so that $S \cup \{k\}$ is a stable set.

In cases (ii) and (iii) of Theorem 2, we call such vertices s or k *swing* vertices of G . Additional background on split graphs can be found in [3] and [7]. The next remarks follow directly from Theorem 2 and give alternative conditions for a split graph to be balanced or unbalanced.

Remark 3. A split graph is unbalanced if and only if it has a swing vertex.

Remark 4. Since we are considering unlabeled graphs, an unbalanced split graph will have a unique K -max partition and a unique S -max partition. A balanced split graph has a unique partition that is both K -max and S -max.

1.2 Overview

In this paper we study bijections and counting questions for split graphs and related classes. There is a complicated formula for the number of split graphs on n vertices resulting from the work of Clarke [5] and Royle [11]. In [3] we count the number of balanced and unbalanced split graphs on n vertices. That proof uses a sequence of bijections

	Split graph K-max S K	XY-graph X Y	Minimal set cover	Bipartite poset	Split graph S-max S K
1.					
2.					
3.					
4.					
5.					
6.					
7.					
8.					
9.					

Figure 1: All split graphs, minimal set covers, and bipartite posets on four vertices. All XY -graphs on 3 vertices. Entries in rows 1–8 are unbalanced, entries in row 9 are balanced.

involving families of graphs called NG-graphs. The following surprising theorem about split graphs is also proven in [3], and it shows that the number of unbalanced split graphs on n vertices equals the total number of split graphs on $n - 1$ or fewer vertices. In Section 6 of this paper, we give a direct and natural proof of Theorem 5. Throughout this paper we prove similar results showing that there is a bijection between two sets, one of which is the unbalanced objects of size n and the other is all the objects of size up to and including $n - 1$. We call such results *Compilation Theorems*.

Theorem 5. (*Compilation Theorem for Split Graphs*) *There is a bijection between the class of unbalanced split graphs on n vertices and the class of split graphs on $n - 1$ or fewer vertices.*

Royle’s work provides a bijection between split graphs and minimal set covers. In searching the *Online Encyclopedia of Integer Sequences* (OEIS), we found on the split graph page (A048194) two additional classes of combinatorial objects with similar sizes. The first class is bipartite graphs with a distinguished block (XY -graphs). The number of XY -graphs on n vertices is equal to the number of principal (or fundamental) transversal matroids of size n , as enumerated by Brylawski [2]. In 2000, Vladeta Jovivic noted on the OEIS page that the number of unlabeled bipartite graphs with n vertices and no isolated vertices in the distinguished bipartite block equals the number of unlabeled split graphs on n vertices. The second class is unlabeled posets of height at most one. Detlef Pauly noted on the OEIS page that the number of such posets with n elements equals number of unlabeled split graphs on n vertices.

Class	Unbalanced subclass characterized by the existence of a ...
Split Graphs	swing vertex
Minimal Set Cover \mathcal{C} on set V	set of size $ V - \mathcal{C} + 1$
XY -graph with no isolate in Y	universal vertex in X
Bipartite Poset	full support point

Table 2: Unbalanced subclasses characterized by the existence of a structure.

In this paper we consider these four combinatorial settings. We define balanced and unbalanced subclasses for each of the other three classes. As detailed in Table 2, for each class, the unbalanced category is characterized by the existence of a structure. We provide bijections between each pair of classes, and show our bijections preserve balance, that is, our bijections map unbalanced objects of the first type to unbalanced objects of the second type, and balanced objects of the first type to balanced objects of the second

type. These bijections prove that a Compilation Theorem holds for each of the other three classes. We also prove each of these directly, and each new context contributes something new: either a shorter and more intuitive proof, and/or an intriguing new theorem. It is interesting to look at these proofs in their own settings without reverting to split graphs. Some of these proofs are subtle and would be more difficult to discover without their connection to one of the other classes.

Other authors have studied counting questions for related classes of graphs and posets. Hanlon [10] counted unlabeled bipartite graphs using generating functions. In 2014, Gainer-Dewar and Gessel counted both unlabeled bipartite graphs and blocks in a bipartite graph [6]. Also in 2014, Guay-Paquet et al. use similar techniques to count both labeled and unlabeled $(3 + 1)$ -free posets [8].

The rest of the paper is organized as follows: we focus on minimal set covers in Section 2, XY -graphs in Section 3, and bipartite posets in Section 4. In Section 5 we show the remaining bijections between different classes. Finally, in Section 6, we give a short proof of Theorem 5 and in Section 7 we present concluding remarks.

2 Minimal set covers

In 1990, Clarke [5] found counting formulas for labeled and unlabeled set covers and in 2000, Royle [11] recognized that the number of unlabeled minimal set covers equalled the number of unlabeled split graphs for small values of n . He demonstrated a bijection between unlabeled split graphs and unlabeled minimal set covers, thus confirming that Clarke's formula counts split graphs. In this section, we translate the concepts of balanced and unbalanced split graphs to the setting of minimal set covers. A minimal set cover is unlabeled when the elements in the ground set of a set cover are unlabeled and the sets in the cover are also unlabeled. Column 3 of Figure 1 shows the nine minimal set covers of a set with four elements; the set cover in row 1 contains four sets while that in row 2 contains 3 sets.

Definition 6. Given a set of unlabeled vertices V , a *set cover* \mathcal{C} is a collection of subsets of V whose union is V . A set cover \mathcal{C} is *minimal* if no set in \mathcal{C} is contained in the union of the remaining sets in \mathcal{C} . A vertex in V is *loyal* if it is in a unique set of \mathcal{C} .

In column three of Figure 1, the loyal vertices are colored black. The next remark follows directly from Definition 6.

Remark 7. A set cover \mathcal{C} of V is minimal if and only if each set in \mathcal{C} contains a loyal vertex.

While the definition of loyal vertices in set covers comes from [11], we next define an analog in split graphs.

Definition 8. A vertex in a split graph G is *loyal* if it is contained in a unique maximal clique of G .

Remark 9. For any KS -partition of a split graph G , each vertex in S is loyal. For any S -max KS -partition of a split graph G , the only loyal vertices in G are those in S unless there is a unique swing vertex $s \in S$. If such an s exists, the additional loyal vertices, if any, are those vertices of K whose only adjacency in S is s .

In the last column of Figure 1, the graphs with loyal vertices in K are those in rows 2, 3, 4, and 7. The next theorem appears in [11]. We include a proof here because we refer to the bijection in the proof of Theorem 10 when proving Theorem 12. The construction in the proof of Theorem 10 is illustrated in Figure 1 by comparing the entries in columns 3 and 5.

Theorem 10 (Royle [11]). *There is a bijection between split graphs on n vertices and minimal set covers on n vertices.*

Proof. Let \mathcal{C} be a minimal set cover of an n element set V . Form a graph G on the same vertex set as follows. Choose one (representative) loyal vertex from each set of \mathcal{C} , let S be the collection of these loyal vertices, and let $K = V - S$. Form the edge set of G as follows: $uv \in E(G)$ if and only if either u and v are together in a set of \mathcal{C} or if u, v are both in K . The result is a split graph and $K \cup S$ forms a KS -partition. Indeed, by construction, $|S| = |\mathcal{C}|$. Furthermore, for each $k \in K$ there exists $s \in S$ for which k and s are in the same set of \mathcal{C} and thus $ks \in E(G)$. Consequently $K \cup S$ is an S -max KS -partition of G .

The process is reversible. Let $G = (V, E)$ be a split graph with an S -max KS -partition. For each $s \in S$, form a set consisting of s and its neighbors. The resulting collection of sets is a set cover \mathcal{C} of V since each $k \in K$ is adjacent to some $s \in S$. The vertices of S are loyal in \mathcal{C} by Remark 9, hence by Remark 7, \mathcal{C} is a minimal set cover. \square

We next define analogs of balanced and unbalanced for minimal set covers so that the bijection given in Theorem 10 preserves balance. The unbalanced class is defined by the existence of a set of a particular size. It is easy to check that the split graphs and minimal set covers in rows 1–8 of Figure 1 are unbalanced while the ones in row 9 are balanced.

Definition 11. Let \mathcal{C} be a minimal set cover of V . Then \mathcal{C} is *unbalanced* if it contains a set with cardinality $|V| - |\mathcal{C}| + 1$ and *balanced* otherwise.

Theorem 12. *The bijection given in the proof of Theorem 10 preserves balance.*

Proof. Let \mathcal{C} be an unbalanced minimal set cover of V . Choose a set $Y \in \mathcal{C}$ with $|Y| = |V| - |\mathcal{C}| + 1$. Let V' be the vertices in V that are not in Y and let \mathcal{C}' be the collection of sets in \mathcal{C} other than Y . Hence, $|V'| = |V| - (|V| - |\mathcal{C}| + 1) = |\mathcal{C}| - 1 = |\mathcal{C}'|$. By Remark 7, each set in \mathcal{C}' has a loyal vertex and by Definition 6, these loyal vertices cannot be in Y . Thus they all come from V' and therefore each vertex in V' belongs to a unique set in \mathcal{C}' and each set in \mathcal{C}' consists of one vertex from V' and a (perhaps empty) subset of Y .

Let u be a loyal vertex chosen from Y , and use the bijection from Theorem 10 to get a split graph G . The set of loyal vertices chosen from the sets in \mathcal{C}' will be V' . In the resulting KS -partition of G , the vertex set of clique K is $Y - \{u\}$. Adding vertex u to K gives a larger clique since the bijection transforms the set Y into a clique. Thus the split graph G is unbalanced as desired.

Conversely, let G be an unbalanced split graph with vertex set V . Fix an S -max KS -partition of G . The bijection in Theorem 10 produces a set cover \mathcal{C} of V where each set in \mathcal{C} corresponds to an element of S together with its neighborhood. Thus $|\mathcal{C}| = |S|$. Since G is unbalanced and our partition is S -max, there exists a vertex in S adjacent to every vertex in K . Let u be such a vertex, thus $K \cup \{u\}$ is a maximal clique in G and its vertices form a set Y in \mathcal{C} . Thus in the resulting set cover \mathcal{C} , we have $|V| - |\mathcal{C}| + 1 = (|K| + |S|) - |S| + 1 = |K| + 1$. The set Y in \mathcal{C} has cardinality $|Y| = |K \cup \{u\}| = |K| + 1$, so \mathcal{C} is unbalanced as desired. \square

The next theorem is the Compilation Theorem for Minimal Set Covers. Our proof focuses on a set Y whose existence is the defining property of unbalanced minimal set covers.

Theorem 13. *There is a bijection between unbalanced minimal set covers of an n -set and minimal set covers of a set with $n - 1$ or fewer elements.*

Proof. Let V be an n -set and \mathcal{C} an unbalanced minimal set cover of V . We show how to map \mathcal{C} to a minimal set cover \mathcal{C}' on a set with $n - 1$ or fewer elements. Let $k = |\mathcal{C}|$. Since set cover \mathcal{C} is unbalanced, there exists a set Y in \mathcal{C} with $|Y| = n - k + 1$. Let V' be the set of elements of V covered by Y and not covered by any other set in \mathcal{C} . We know $|V'| \geq 1$ since Y has a loyal vertex, so $|V - V'| \leq n - 1$. Define \mathcal{C}' to be the set cover of $V - V'$ consisting of all the sets in \mathcal{C} except Y . Then $|\mathcal{C}'| = |\mathcal{C}| - 1$. Since \mathcal{C} is a minimal set cover, each set S in \mathcal{C} contains a loyal vertex which is still loyal to S in \mathcal{C}' . Thus \mathcal{C}' is a minimal set cover of $V - V'$ and $|V - V'| \leq n - 1$.

Conversely, let \mathcal{C}' be a minimal set cover of a set V' with t elements, where $0 \leq t \leq n - 1$. Let $k' = |\mathcal{C}'|$. For each set in \mathcal{C}' , designate a loyal element. Create a new set V consisting of V' and $n - t$ additional elements, and a set cover \mathcal{C} consisting of the sets in \mathcal{C}' together with a set Y that contains the $n - t$ new elements together with the elements of V' other than the k' loyal elements designated earlier. It follows immediately that \mathcal{C} is a minimal set cover of n -set V and $|\mathcal{C}| = k' + 1$.

We next show that \mathcal{C} is unbalanced. Note that Y consists of the $n - t$ new elements together with $t - k'$ elements of V' , thus $|Y| = n - k'$. Furthermore, $|\mathcal{C}| = |\mathcal{C}'| + 1 = k' + 1$ and $|V| - |\mathcal{C}| + 1 = n - (k' + 1) + 1 = n - k' = |Y|$, so \mathcal{C} is unbalanced.

Finally, the second mapping reverses the first so we have a bijection. \square

3 XY-Graphs

In [3] we calculate the number of (unlabeled) unbalanced split graphs on n vertices for $n = 1, 2, 3, \dots$ and get the following sequence: 1, 2, 4, 8, 17, 38, 94, 258, ... According to the Online Encyclopedia of Integer Sequences (OEIS), this sequence also counts the number of bipartite graphs with a distinguished block. This motivated us to seek connections between these two classes, and indeed the proof of Theorem 15 gives an explicit bijection between these classes. We prefer to use the term *XY-graph* rather than *bipartite graph with a distinguished block* so that we may consistently refer to X as the distinguished block and Y as the other block.

Definition 14. An XY -graph is a bipartite graph together with a bipartition of the vertices as $X \cup Y$. The set X is the *distinguished part* of the bipartition.

While there are only three distinct bipartite graphs on three vertices, there are eight different XY -graphs on three vertices, as shown in Column 2 of Figure 1. For example, the bipartite graph P_3 is counted as two different XY -graphs, one with $|X| = 1$ and one with $|X| = 2$. These eight XY -graphs correspond to the unbalanced split graphs on four vertices, as described below in Theorem 15 and illustrated in the first two columns of Figure 1.

Theorem 15. *There is a bijection between the class of XY -graphs on n vertices and the class of unbalanced split graphs on $n + 1$ vertices.*

Proof. Let G be an XY -graph on n vertices and form a split graph H on $n + 1$ vertices by adding a new vertex v as follows. Let $K = Y \cup \{v\}$, let $S = X$, retain all edges from G , and add an edge between each pair of distinct vertices in K . Since vertex v is not adjacent to any vertex in S , we know S is not a maximum stable set in H and thus H is an unbalanced split graph and the given KS -partition is K -max.

Conversely, let H be an unbalanced split graph on $n + 1$ vertices and fix a KS -partition of G that is K -max. Chose a swing vertex v in K to remove and let G be the XY -graph with $X = S$, $Y = K - \{v\}$ and $E(G) = \{xy : x \in X, y \in Y, xy \in E(H)\}$. It is easy to see that these functions are inverses, so they provide a bijection. \square

In an XY -graph, we say a vertex of Y is an *isolate* if it has no neighbors (in X) and that a vertex of X is *universal* if it is adjacent to every vertex of Y .

While Theorem 15 provides a connection between XY -graphs and split graphs, the number of vertices changes from n to $n + 1$ in that result. In the next theorem, the number of vertices is n for both classes and we see that XY -graphs with no isolates in Y serve as an analog of split graphs. Column 5 in Figure 1 shows split graphs on four vertices with an S -max KS -partition; the XY -graphs on four vertices with no isolates in Y are identical where $X = S$, $Y = K$, and we remove all edges between vertices in K .

Theorem 16. *There is a bijection between split graphs on n vertices and XY -graphs on n vertices with no isolates in Y .*

Proof. Let G be a split graph on n vertices and fix an S -max KS -partition of G . Let $X = S$ and $Y = K$ and let H be the XY -graph with $E(H) = \{xy \in E(G) : x \in X, y \in Y\}$. Then $|V(H)| = n$ and there are no isolates in Y since the KS -partition of G was S -max.

Conversely, let H be an XY -graph on n vertices with no isolates in Y . Form a split graph G by letting $S = X$, $K = Y$ and $E(G) = E(H) \cup \{yz : y, z \in Y, y \neq z\}$. Then $|V(G)| = n$ and the resulting KS -partition is S -max since in graph H there were no isolates in Y . These functions are inverses, so provide a bijection. \square

As we saw in Theorem 16, XY -graphs with no isolates in Y provide an analog of split graphs. We next define balanced and unbalanced for XY -graphs with no isolates in Y and then prove that the bijection given in Theorem 16 preserves balance. As in our previous

definitions of unbalanced, the class is defined by the existence of some structure, in this case, a universal vertex in X (see Table 2).

Definition 17. An XY -graph with no isolates in Y is *unbalanced* if there exists a universal vertex in X , and *balanced* otherwise.

Note that we could have instead chosen XY -graphs with a universal vertex in X for our analogue of split graphs. However, the definition of unbalanced would remain the same, namely, XY -graphs with a universal vertex in X and no isolates in Y .

Theorem 18. *The bijection given in the proof of Theorem 16 preserves balance.*

Proof. By Remark 3, a split graph is unbalanced if and only if its S -max KS -partition has a swing vertex in S . An XY -graph with no isolate in Y is unbalanced if and only if it has a universal vertex in X . The bijection given in the proof of Theorem 16 maps swing vertices in S to universal vertices in X and vice versa, so it preserves balance. \square

The next result is our Compilation Theorem for XY -graphs with no isolates in Y . Note that by definition, an XY -graph that is balanced or unbalanced has no isolates in Y .

Theorem 19. *There is a bijection between the set of unbalanced XY -graphs on n vertices and the union of the sets of unbalanced and balanced XY -graphs on $n-1$ or fewer vertices.*

Proof. Let G be an unbalanced XY -graph on n vertices and let u be a universal vertex in X . Form the $X'Y'$ -graph H as follows: let $X' = X - \{u\}$, let Y' consist of the elements of Y that are adjacent in G to a vertex of X' , and $E(H) = E(G) \cap \{xy : x \in X', y \in Y'\}$. By construction, there are no isolates in Y' and $|V(H)| \leq n-1$ as desired.

Conversely, let H be an $X'Y'$ -graph on at most $n-1$ vertices that is balanced or unbalanced, and hence has no isolates in Y' . Let Y consist of the vertices of Y' plus $(n-1) - |V(H)|$ additional isolates and let X consist of the vertices of X' plus a universal vertex u so that $E(G) = E(H) \cup \{uy : y \in Y\}$. The result is an XY -graph on n vertices with no isolates in Y (because each vertex of Y is adjacent to u). These functions are inverses and thus provide a bijection between the two classes. \square

4 Bipartite Posets

A *poset* P consists of a non-empty set V together with a relation \prec on V that is irreflexive, transitive, and therefore antisymmetric. Two elements (or *points*) $u, v \in V$ are *comparable* if $u \prec v$ or $v \prec u$, and *incomparable* otherwise. A poset has *height at most one* if there do not exist three elements x, y, z with $x \prec y \prec z$. Such posets are also known as *bipartite posets*. In a bipartite poset, an element v has *height 1* if there is a u with $u \prec v$ and height 0 otherwise. In this paper, we consider *unlabeled* posets, that is, two posets are considered the same if they are isomorphic. Figure 1 shows all bipartite posets on four points, where the height 1 points are shown in black. The examples in the last two columns of this figure can be used to illustrate the proof of the next theorem.

Theorem 20. *There is a bijection between split graphs on n vertices and bipartite posets on n elements.*

Proof. Let P be a bipartite poset. Let K be the set of height one elements of P and S be the set of height 0 elements of P . Form a split graph G with KS -partition as follows: $V(G) = K \cup S$ and $E(G) = \{uv : u \prec v \text{ in } P\} \cup \{k\ell : k, \ell \in K, k \neq \ell\}$. Each vertex of K corresponds to a height 1 element of P which by definition must be comparable to a height 0 element of P . Thus each vertex of K is adjacent to a vertex of S and consequently, the KS -partition of G is S -max.

Conversely, let G be a split graph on n vertices and fix an S -max KS -partition of it. Form a poset $P = (V, \prec)$ as follows: $V = K \cup S$ and $x \prec y$ if and only if $x \in S, y \in K$, and $xy \in E(G)$. Since the KS -partition of G is S -max, each element of K is adjacent to an element of S , and thus the elements of K become height one elements of P .

These functions are inverses and thus provide bijections. \square

We next introduce the notions of balanced and unbalanced for bipartite posets. Once again, unbalanced bipartite posets are defined by the existence of some structure, in this case, a full support point (see Table 2). In column 4 of Figure 1, the full support points are shown as circles with a white interior.

Definition 21. Let P be a bipartite poset. A point at height 0 is called a *full support point* if it is comparable to every height one point. Points at height 0 that are not full support points are called *partial support points*. Poset P is *unbalanced* if it contains a full support point and *balanced* otherwise.

Theorem 22. *The bijection given in the proof of Theorem 20 preserves balance.*

Proof. Let P be an unbalanced bipartite poset and let v be a full support point in P . We form a split graph G as in the proof of Theorem 20. Since v is a full support point in P , it is a vertex in S that in G is adjacent to all vertices in K . Thus our KS -partition of G is not K -max and consequently, G is an unbalanced split graph.

Now suppose P is a balanced bipartite poset and consider the resulting KS -partition of G . The partition is K -max since there is no point in P which is a full support point. It is S -max since each height one point in P is comparable to some height 0 point in P . Thus G is a balanced split graph. \square

The next theorem is our Compilation Theorem for Bipartite Posets. While it follows from previous results, the proof we give here is a direct proof about posets.

Theorem 23. *There is a bijection between the set of unbalanced, bipartite posets on n points and the set of bipartite posets on at most $n - 1$ or fewer points.*

Proof. Let P be an n -element unbalanced, bipartite poset. We define $f(P)$ as follows. If each height one point in P is comparable to a partial support point, then simply remove all full support points to arrive at $f(P)$. Note that $f(P)$ has no full support points. Otherwise, there is a height one point in P that is comparable to precisely the set of full

support points. In this case, choose one such point, say u . Form $f(P)$ by removing all full support points and making $u \prec v$ for all $v \neq u$ that are height 1 points of P . By construction, the full support points of $f(P)$ include u .

Next we define the inverse function g . Let Q be a bipartite poset with t points where $0 \leq t \leq n - 1$. If Q has no full support point, add $n - t$ points at height 0 to $g(Q)$ and make them comparable to all height one points. The resulting n -point bipartite poset is $g(Q)$ and by construction, the new points are exactly the full support points of $g(Q)$. Otherwise, there is at least one full support point in Q . In this case, choose one such point v to become a height 1 element, and add $n - t$ points at height 0 and make them comparable to all height one points. The resulting n -point bipartite poset is $g(Q)$ and again, the new points are exactly the full support points of $g(Q)$. Since $n - t \geq 1$, in either case the resulting poset must have a full support point. Hence the result is an unbalanced, bipartite poset on n -points.

We must show that for all Q , $f(g(Q)) = Q$ and likewise for all P , $g(f(P)) = P$. Start with a bipartite poset Q with t points where $0 \leq t \leq n - 1$. First consider the case in which there are no full support points in Q . In this case, function g adds a set of $n - t$ vertices at height 0 and makes them each a full support point to result in $g(Q)$. These $n - t$ elements are the only full support points in $g(Q)$. Now in $g(Q)$, each height 1 point is comparable to a partial support point, for otherwise it would not be at height 1 in Q . When f is applied to $g(Q)$, it removes the $n - t$ full support points and no other changes are made, so $f(g(Q)) = Q$.

Next consider the case in which there is a full support point in Q . Such vertices all have the same comparabilities and are thus interchangeable. The function g chooses one such vertex v , moves it to height 1 and adds a set B of $n - t$ full support vertices to arrive at $g(Q)$. In $g(Q)$, the set of full support vertices is exactly B . Vertex v is interchangeable with any other point at height 1 in $g(Q)$ whose only comparabilities are to points in B . When f is applied to $g(Q)$, the elements of B are removed and v becomes a height 0 point which is comparable to all remaining points at height 1. Thus, $f(g(Q)) = Q$.

Finally, we show $g(f(P)) = P$. Start with an unbalanced, bipartite poset P on n points. First consider the case in which all height 1 points are comparable to a partial support point. In this case, function f removes all full support points to arrive at $f(P)$. Since P is unbalanced, f removes at least one point. Note that there are no full support points in $f(P)$ and all points at height 1 in $f(P)$ are also at height 1 in P . Now apply g to $f(P)$. This adds the same number of full support points as were removed by f , resulting in the original poset P .

Now consider the case in which there are points at height 1 whose only comparabilities in P are to the full support points. These points are interchangeable. When f is applied to P , one such point v is chosen and moved to height 0 in $f(P)$ where it becomes a full support point in $f(P)$ and the original full support points in P are removed. In this case, $f(P)$ has at least one full support point, namely v , and if it has others, they are interchangeable with v . When g is applied to $f(P)$, the same number of full support points are added as were removed by f , and point v is moved to height 1 where it is above the new points, resulting in the original poset. Thus $g(f(P)) = P$. \square

5 Remaining Bijections

In each of the previous three sections, we gave a bijection between split graphs and another combinatorial class. Here we compare these combinatorial classes to each other, giving a direct bijection between each pair of the three new classes and proving these bijections preserve balance. The bijection in Theorem 24 is illustrated in Figure 1 by comparing entries in the same row of columns 3 and 4.

Theorem 24. *There is a balance-preserving bijection between minimal set covers on a set with n vertices and bipartite posets with n elements.*

Proof. Let \mathcal{C} be a minimal set cover on an n -set V and form a bipartite poset P as follows. Choose one (representative) loyal element from each set in \mathcal{C} and let S be the collection of these loyal elements. Form $P = (V, \prec)$ by placing the elements of S at height 0 and making $x \prec y$ precisely when $x \in S$, $y \in V - S$, and x and y are together in some set of \mathcal{C} . Since each $y \in V - S$ is in some set of \mathcal{C} and that set has a loyal element in S , each element of $V - S$ is at height 1.

The process is reversible. Let P be a bipartite poset with n elements and form a minimal set cover \mathcal{C} as follows. For each height 0 element in P , form a set in \mathcal{C} containing it and the points it is comparable to in P . These sets form a cover because each height 1 element is comparable to at least one height 0 element. By set-cov-poset, the points coming from height 0 elements are loyal, so by Remark 7, \mathcal{C} is minimal.

Furthermore, note that the number of height 0 elements in P is $|\mathcal{C}|$, and therefore the number of height 1 elements is $|V| - |\mathcal{C}|$. An element at height 0 will be a full support point of P if and only if it is the representative loyal vertex chosen from a set of \mathcal{C} with $|V| - |\mathcal{C}| + 1$ elements. Thus \mathcal{C} is unbalanced precisely when P is unbalanced. \square

Theorem 25. *There is a balance-preserving bijection between XY -graphs on n vertices with no isolates in Y and minimal set covers on a set with n vertices.*

Proof. Let G be an XY -graph on n vertices with no isolate in Y . For each vertex x in X , form a set consisting of x and its neighbors. The collection \mathcal{C} of these sets is a set cover of $X \cup Y$ because Y has no isolates, so each vertex in Y has a neighbor in X . Each set in \mathcal{C} contains precisely one element of X , and that element is loyal, so \mathcal{C} is minimal.

Conversely, let \mathcal{C} be a minimal set cover of a set V with n elements. Choose a (representative) loyal vertex from each set in \mathcal{C} , let X be the collection of this set of loyal vertices, and let $Y = V - X$. Form XY -graph G by making $xy \in E(G)$ if and only if $x \in X$, $y \in Y$, and x and y are together in a set of \mathcal{C} . There are no isolates in Y because every set in \mathcal{C} has a loyal element in X . It is easy to see that these functions are inverses, so they provide a bijection.

By construction, $|\mathcal{C}| = |X|$. There exists a universal vertex in X if and only if the corresponding set of \mathcal{C} has $|Y| + 1$ elements. Since $|Y| + 1 = |V| - |X| + 1 = |V| - |\mathcal{C}| + 1$, unbalanced XY -graphs map to unbalanced minimal set covers. \square

Theorem 26. *There is a balance-preserving bijection between XY -graphs on n vertices with no isolates in Y and bipartite posets on n elements.*

Proof. Let G be an XY -graph on n vertices with no isolate in Y . Form a bipartite poset P by placing each element of X at height 0 and making $x \prec y$ if and only if $x \in X$, $y \in Y$ and $xy \in E(G)$. Since in graph G there are no isolates in Y , we know that the elements of Y are at height 1 in bipartite poset P .

Conversely, let P be a height 1 poset with n elements. Let X be the set of height 0 points and Y be the set of height 1 points. Form XY -graph G by making $xy \in E(G)$ if and only if $x \in X$, $y \in Y$, and $x \prec y$ in P . Since the elements of Y are at height 1 in P , there are no isolates in Y in graph G . It is easy to see that these functions are inverses, so they provide a bijection.

Moreover, a vertex in graph G is a universal vertex of X if and only if it is a full support point in the corresponding bipartite poset P . Thus the bijection given preserves balance. \square

6 Compilation Theorem for Split Graphs

In this section we give a direct proof of Theorem 5. The inspiration for this proof came from the bijections between split graphs and the other classes considered in this paper.

Proof of Theorem 5. Let G be an unbalanced split graph with n vertices and fix an S -max KS -partition of G . By Remark 3 there exists a swing vertex $s \in S$. Let $S' = S - \{s\}$ and K' be the set of vertices in K that are adjacent to a vertex in S' . Then $K' \cup S'$ is a split graph partition of a graph H with $|V(H)| \leq n - 1$. Since each vertex in K' is adjacent to a vertex in S' , there are no swing vertices in K' and thus this partition of H is S -max.

Conversely, let H be a split graph with $|V(H)| \leq n - 1$. Let $K'S'$ be an S -max partition of H . Form graph G with vertex set $K \cup S$ as follows: let $S = S' \cup \{s\}$, where s is a new vertex, and let K consist of the vertices in K' plus enough additional vertices to make $|K| + |S| = n$. The edge set of G consists of the edges of H , edges between each pair of distinct vertices in K , and an edge between s and each vertex in K . Since s is a swing vertex in this partition, G is unbalanced and the partition is S -max. The second mapping reverses the first, and thus we have a bijection. \square

7 Concluding Remarks

In this paper we have presented compilation theorems for split graphs, minimal set covers, XY -graphs, and bipartite posets. We can define compilation theorems in more general settings as follows. Let $(\mathcal{G})_n$ be a class of combinatorial objects on a ground set of size n , let $(\mathcal{G})_{\leq n-1} = (\mathcal{G})_0 \cup (\mathcal{G})_1 \cup (\mathcal{G})_2 \cup \dots \cup (\mathcal{G})_{n-1}$, and let $(\mathcal{H})_n$ be a subclass of $(\mathcal{G})_n$ whose members have an additional natural property. We say the pair $(\mathcal{G}, \mathcal{H})$ has a compilation theorem if $|(\mathcal{H})_n| = |(\mathcal{G})_{\leq n-1}|$ for each $n \geq 1$. For example, Theorem 5 is the compilation theorem for the pair (split graphs, unbalanced split graphs). The compilation theorems discussed in this paper all generate the sequence 1, 2, 4, 8, 17, 38, 94, ..., but other sequences are possible. For example, consider the sequence 1, 2, 4, 8, 16, 32, 64, ... Let $(\mathcal{G})_n$ be the set of subsets of an n -set and $(\mathcal{H})_n$ be the set of non-empty subsets of

an n -set. Then $|(\mathcal{H})_n| = 2^n - 1$ and $|(\mathcal{G})_{\leq n}| = 1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$, so there is a compilation theorem for the pair $(\mathcal{G}, \mathcal{H})$. In future work we hope to find compilation theorems in other contexts.

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