The complexity of computing the cylindrical and the t-circle crossing number of a graph

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Abstract

A plane drawing of a graph is *cylindrical* if there exist two concentric circles that contain all the vertices of the graph, and no edge intersects (other than at its endpoints) any of these circles. The *cylindrical crossing number* of a graph G is the minimum number of crossings in a cylindrical drawing of G. In his influential survey on the variants of the definition of the crossing number of a graph, Schaefer lists the complexity of computing the cylindrical crossing number of a graph as an open question. In this paper, we prove that the problem of deciding whether a given graph admits a cylindrical embedding is NP-complete, and as a consequence we show that the t-cylindrical crossing number problem is also NP-complete. Moreover, we show

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an analogous result for the natural generalization of the cylindrical crossing number, namely the *t-circle crossing number*.

Keywords: cylindrical crossing number; book crossing number; t-circle crossing number

Mathematics Subject Classifications: 05C10, 68R10, 05C85

1 Introduction

This work is motivated by a question posed by Marcus Schaefer in his survey on the variants of the definition of the crossing number of a graph. In [11], Schaefer listed as open the problem of the complexity of computing the cylindrical crossing number of a graph. We recall that a *cylindrical drawing* of a graph G is a plane drawing where all the vertices are in two concentric cycles, and no circle is intersected by the interior of an edge. The *cylindrical crossing number* $\mathrm{cr}_{\circledcirc}(G)$ of a graph G is the minimum number of crossings in a cylindrical drawing of G.

The concept of a cylindrical drawing is motivated by a family of graph drawings of the complete graph K_n, originally conceived by the British artist Anthony Hill. As narrated in the lively account given in [5], Hill's construction produces drawings of K_n that are cylindrical, according to the definition above, and have exactly $Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$ crossings. It is a long-standing conjecture that the crossing number of K_n is $Z(n)$, for every $n \geqslant 3$ [9]. In [1], Ábrego et al. proved that $\mathrm{cr}_{\circledcirc}(K_n) = Z(n)$, for every $n \geqslant 3$.

Let $\mathcal{D}$ be a plane drawing of a graph G. We say that a Jordan curve ρ (that is, a simple closed curve) is *clean* (with respect to $\mathcal{D}$) if the interior of no edge of G intersects ρ. Now suppose that there are two clean disjoint circles with respect to $\mathcal{D}$, say ρ_1 and ρ_2, such that every vertex of G is in $\rho_1 \cup \rho_2$. Note that not only concentricity is not assumed, but also it is not required that the disk bounded by one of these circles contains the other circle. It is a straightforward exercise in plane topology that there is a cylindrical drawing $\mathcal{D}'$ with the same cellular structure as $\mathcal{D}$; in particular, $\mathcal{D}'$ has the same number of crossings as $\mathcal{D}$. Thus, for crossing number purposes, it is totally valid to adopt the following definition of a cylindrical drawing.

Definition 1 (Equivalent definition of cylindrical drawing)**.** A plane drawing of a graph G is *cylindrical* if there exists two disjoint clean circles ρ_1, ρ_2 such that every vertex of G is in $\rho_1 \cup \rho_2$.

The advantage of adopting this definition of a cylindrical drawing is that it allows us to generalize this notion to an arbitrary number of circles, as follows. We should mention that the term "t-circle drawing" has been suggested by Éva Czabarka and Marcus Schaefer (private communication).

Definition 2 (***t***-circle drawing and ***t***-circle crossing number)**.** Let $t \geqslant 1$ be an integer. A plane drawing of a graph G is a *t-circle drawing* if there exist t pairwise disjoint clean circles $\rho_1, \dots, \rho_t$, such that every vertex of G is in $\rho_1 \cup \cdots \cup \rho_t$. The *$t$-circle crossing* *number* $\mathrm{cr}_{t\circ}(G)$ of a graph G is the minimum number of crossings in a t-circle drawing of G.

Thus a cylindrical drawing is simply a 2-circle drawing. Moreover, for $t = 1$, there is an immediate connection with 2-page drawings. We recall that a 2-*page drawing* of a graph is a drawing in which the vertices lie on the x-axis, and each edge is contained (except for its endpoints) either in the upper halfplane, or in the lower halfplane. A straightforward argument shows that a 1-circle drawing can be transformed into a 2-page drawing with the same cellular structure.

Thus the 1-circle crossing number of a graph coincides with its 2-page crossing number, and the 2-circle crossing number of a graph is the same as its cylindrical crossing number. The 3-circle crossing number is related to the *pair of pants crossing number* [11], but these are different notions, since in the latter it is required that none of the disks bounded by the circles contains another circle, and that no edge intersects the interior of any of these disks.

For the arguments we will use in this paper, it will be useful to relax the condition that the clean Jordan curves in Definition 2 need to be circles:

Definition 3 (***t***-curve drawing and ***t***-curve crossing number)**.** Let $t \geqslant 1$ be an integer. A plane drawing of a graph G is a t-*curve drawing* if there exist t pairwise disjoint clean Jordan curves $\rho_1, \ldots, \rho_t$ such that every vertex of G is in $\rho_1 \cup \cdots \cup \rho_t$. The t-*curve crossing number* of a graph G is the minimum number of crossings in a t-curve drawing of G.

It follows from the Jordan-Schönflies theorem that if $\mathcal{D}$ is a t-curve drawing of a graph G, then there is a self-homeomorphism of the plane that takes $\mathcal{D}$ to a t-circle drawing. In particular, for any graph G, its t-circle crossing number and its t-curve crossing number are the same. Thus the difference between these notions is rather cosmetic. On the other hand, as we hinted above, the advantage of dealing with t-curve drawings instead of t-circle drawings is being able to work with arbitrary Jordan curves, instead of exclusively with circles, which makes our arguments simpler.

As we mentioned above, our motivation in this work is to settle the complexity of computing the cylindrical crossing number of a graph, that is, the complexity of the decision problem CylindricalCrossingNumber: "given a graph G and an integer k, is $\mathrm{cr}_{\circledcirc}(G) \leqslant k$?". As we shall see, this question will be settled as a consequence of Theorem 4, which establish the computational complexity of the decision problem CylindricalEmbedding: "given a graph G, is $\mathrm{cr}_{\circledcirc}(G) = 0$?". As it happens, with very little additional effort we can settle the complexity of the decision problem t-curveEmbedding, that considers a fixed integer t and asks "given a graph G, is there a t-curve drawing of G with no crossings?".

Chung, Leighton, and Rosenberg [8] proved that 2-pageEmbedding is NP-complete. This implies that 1-curveEmbedding is NP-complete, as testing if a graph has pagenumber 2 is equivalent to testing if it has a 1-curve embedding. As we shall see, the NP-hardness proof for $t \geqslant 2$ works by reducing it to the case $t = 1$.

Theorem 4. *For each fixed integer $t \geqslant 1$, t-*curveEmbedding *is NP-complete.*

Given that a t-circle embedding of a graph is homeomorphic to a t-curve embbeding, Theorem 4 settles the complexity of the decision problem t-CIRCLEEMBEDDING, that considers a fixed integer t and asks "given a graph G, is $\mathrm{cr}_{t\circ}(G) = 0$?". Since a cylindrical embedding of G is also a 2-circle embedding of G, Theorem 4 settles in particular the complexity of cylindrical embedding. For completeness, we state these observations formally:

Corollary 5. *For each fixed integer $t \geqslant 1$, t-CIRCLEEMBEDDING is NP-complete. In particular* CYLINDRICALEMBEDDING *is NP-complete.*

The following corollary is another consequence of Theorem 4, and settles the computational complexity of the decision problem t-CURVECROSSINGNUMBER for each fixed integer $t \geqslant 2$. We recall that such a problem takes a fixed integer t and asks "given a graph G and an integer k, is the t-curve crossing number of G at most k?". For $t = 1$, Bannister and Eppstein [4] proved that 2-page crossing number (equivalently, 1-curve crossing number) is fixed-parameter tractable.

Corollary 6. *For each fixed integer $t \geqslant 2$, t-CURVECROSSINGNUMBER is NP-complete.*

As we mentioned above, the t-circle crossing number of a graph and the t-curve crossing number are the same. This fact and Corollary 6 imply that both decision problems t-CIRCLECROSSINGNUMBER and t-CYLINDRICALCROSSINGNUMBER are NP-complete whenever $t \geqslant 2$.

Before proceeding to the proof of Theorem 4 (Section 3), we establish in the next section a result on plane triangulations that are minimal with respect to having a t-curve embedding.

2 Minimal t-curve embeddings

An essential ingredient in the proof that t-CURVEEMBEDDING is NP-hard is the existence of plane triangulations that are minimal with respect to having a t-curve embedding. Our aim in this section is to establish this result (Lemma 8 below). We will need the following statement.

Proposition 7. *Let G be a maximal planar graph, and let t be a positive integer. Suppose that G has a t-curve embedding. Then there is a collection $\{H_1, \ldots, H_t\}$ of pairwise disjoint subgraphs of G with the following properties: (i) if H_i has at least 3 vertices for some $i \in \{1, \ldots, t\}$, then H_i is a cycle; and (ii) $\bigcup_{i=1}^{t} H_i$ contains all the vertices of G.*

Proof. Let $\mathcal{E}$ be a t-curve embedding of G, and let $\rho_1, \ldots, \rho_t$ be the underlying t clean Jordan curves of $\mathcal{E}$. Let $i \in \{1, \ldots, t\}$. If ρ_i does not contain any vertex, then we let H_i be the null graph. If ρ_i contains at least one vertex, let $v_1, \ldots, v_{m_i}$ be the vertices on ρ_i, in the (cyclic) order in which they appear in ρ_i. If $m_i = 1$, then we let H_i be the subgraph of G that consists only of the vertex v_1. If $m_i \geqslant 2$, we proceed as follows.

For $j = 1, \ldots, m_i$, there is a subarc of ρ_i whose endpoints are v_j and v_{j+1} (indices are taken modulo m_i), and that is otherwise disjoint from G. This implies that for

$j = 1, \ldots, m_i$, there is a face incident with v_j and v_{j+1}. Since G is maximal planar, $\mathcal{E}$ is a plane triangulation and it is the unique plane embedding of G (up to homeomorphism). Therefore the existence of a face incident with v_j and v_{j+1} implies that v_j and v_{j+1} are adjacent.

If $m_i = 2$, then we let H_i be the subgraph of G that consists of the vertices v_1 and v_2, and the edge joining them. If $m_i \geqslant 3$, then $v_1 v_2 \ldots v_{m_i} v_1$ is a cycle C_i of G, and we let $H_i = C_i$.

Since each vertex of G is contained in a curve in $\{\rho_1, \ldots, \rho_t\}$, and these curves are pairwise disjoint, it follows that the collection $\{H_1, \ldots, H_t\}$ satisfies the required conditions. □

Lemma 8. *For every $t \geqslant 2$ there is a 3-connected simple graph G_t such that (i) G_t triangulates the plane; (ii) G_t has a t-curve embedding; and (iii) G_t has no $(t-1)$-curve embedding.*

Proof. The heart of the proof is the existence of plane triangulations whose longest cycles are relatively small. Since all graphs under consideration in this proof are plane graphs, we often make no distinction between a graph and its drawing.

Following Chen and Yu [7], let $T_1, T_2, \ldots$ be the family of plane triangulations constructed as follows. First, T_1 is the plane triangulation induced by K_4. Now, T_{i+1} is constructed from T_i, for $i = 1, 2, \ldots$, as follows: in each inner face of T_i, add one new vertex and join it to the vertices of T_i incident with the face containing it. We refer the reader to Figure 1.

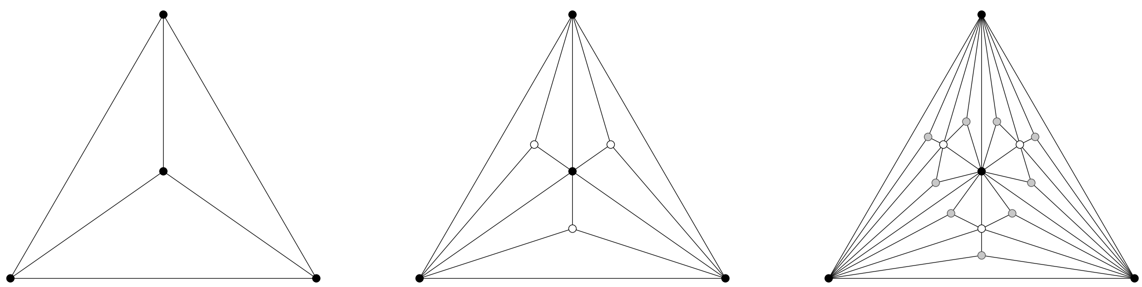

Figure 1: On the left hand side we have the triangulation T_1. For each inner face T of T_1, we add a (white) vertex inside T and join it with edges to the three vertices incident to T; the result is the middle triangulation T_2. We obtain T_3 (right hand side) similarly: for each inner face T of T_2, we add a (grey) vertex inside T, and join it with edges to the three vertices incident to T.

In [7] it is proved that, for $i \geqslant 1$, the length of the longest cycle of T_i is less than $\frac{7}{2}|V(T_i)|^{\log_3 2}$. Now let j be an integer large enough such that $\frac{7}{2}|V(T_j)|^{\log_3 2} \cdot (t-1) < |V(T_j)|$. Toward a contradiction, suppose that T_j has a $(t-1)$-curve embedding, and let $\{H_1, \ldots, H_{t-1}\}$ be the subgraphs of T_j guaranteed by Proposition 7. Since $V(T_j) = \bigcup_{i=1}^{t-1} V(H_i)$, it follows that there is some H_i such that $|V(H_i)| > \frac{7}{2}|V(T_j)|^{\log_3 2}$. Since $\frac{7}{2}s^{\log_3 2} \geqslant 3$ for every $s \geqslant 1$, it follows from Proposition 7 that H_i must be a cycle, contradicting that the length of the longest cycle of T_j is less than $\frac{7}{2}|V(T_j)|^{\log_3 2}$. Thus T_j has no $(t-1)$-curve embedding.

In the previous paragraph, we have shown that the family of graphs that are not $(t-1)$-curve embeddable is not empty. Now we will choose the required graph from such a family. Let m be the least integer such that T_m has no $(t-1)$-curve embedding. Note that $m \geqslant 3$, since T_2 has a 1-curve embedding, and thus a $(t-1)$-curve embedding for every $t \geqslant 2$. By the minimality of m, T_{m-1} has a $(t-1)$-curve embedding. Let $Q_1 := T_{m-1}, Q_2, \ldots, Q_k := T_m$ be a sequence of triangulations (subtriangulations of T_m) such that Q_{i+1} is obtained from Q_i by adding a new vertex and its three incident edges, for $i \in \{1, \ldots, k-1\}$. Let ℓ be the largest integer such that Q_ℓ has a $(t-1)$-curve embedding. Let v be the vertex that gets added (together with its three incident edges) to Q_ℓ, in order to get $Q_{\ell+1}$.

The maximality of ℓ implies that $Q_{\ell+1}$ does not have a $(t-1)$-curve embedding, and we claim that $Q_{\ell+1}$ has a t-curve embedding. To see this, let x, y, z be the three vertices adjacent to v in $Q_{\ell+1}$. Thus x, y, z form a 3-cycle, which bounds the face f in Q_ℓ in which v is placed. Let $\rho_1, \ldots, \rho_{t-1}$ be clean Jordan curves that witness the $(t-1)$-curve embeddability of Q_ℓ. It is easy to see that if one of these Jordan curves intersects f, then we can slightly perturb it so that it also intersects v (see Figure 2). But this is impossible, since then $Q_{\ell+1}$ would be a $(t-1)$-curve embedding. Thus none of $\rho_1, \ldots, \rho_{t-1}$ intersects v or its incident edges, and so they are also clean Jordan curves in $Q_{\ell+1}$. We now draw in a small neighborhood of v a clean Jordan curve ρ_t that only contains v, so that $\rho_1, \ldots, \rho_t$ is a collection of pairwise disjoint clean Jordan curves that contain all the vertices of $Q_{\ell+1}$. Therefore $Q_{\ell+1}$ is a t-curve embedding, as claimed.

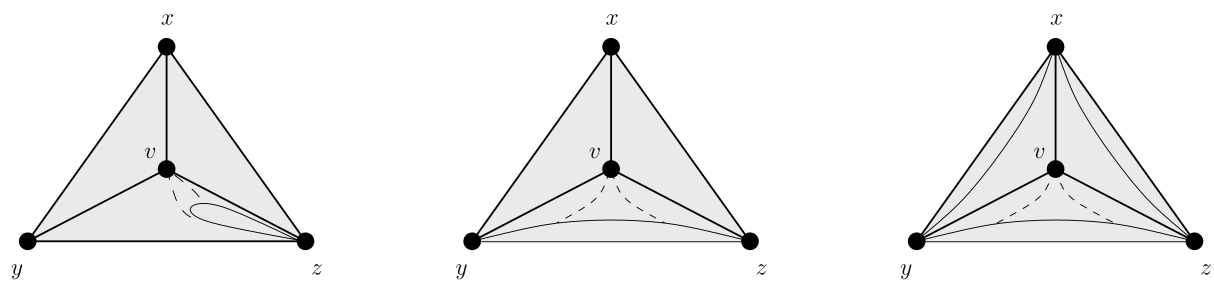

Figure 2: A slightly perturbation of a Jordan curve (dashed line) passing through the vertex v.

Let G_t be the underlying graph of the triangulation $Q_{\ell+1}$. It is readily checked that G_t is 3-connected and simple, and $Q_{\ell+1}$ witnesses that G_t triangulates the plane, and that G_t has a t-curve embedding. Since G_t is 3-connected, it follows that $Q_{\ell+1}$ is its unique embedding (up to isomorphism) in the plane. Since $Q_{\ell+1}$ is not a $(t-1)$-curve embedding, it follows that G_t does not have a $(t-1)$-curve embedding. $\square$

3 Proof of Theorem 4

First we prove membership in NP, and then we prove NP-hardness.

(A) t-CURVEEMBEDDING is in NP.

Proof. Let $\mathcal{D}$ be an embedding of a graph G, and let R be a collection of t clean Jordan curves with respect to $\mathcal{D}$. Now we regard each of these curves as the edge set of a cycle that gets added to $\mathcal{D}$. (We remark that, for this purpose, we regard a graph that consists of a pair of vertices joined by two parallel edges, or of a vertex with a loop-edge, as a cycle.) We let $\mathcal{D}'$ denote the drawing that is obtained from $\mathcal{D}$ by adding the edges these t cycles, which we color blue to help comprehension.

The fact that G has a t-curve embedding can be attested in polynomial time by verifying the existence of such an embedding $\mathcal{D}'$, with the properties that the blue cycles are pairwise disjoint, and each vertex of G is contained in a blue cycle. □

(B) t-CURVEEMBEDDING is NP-hard.

Proof. Let $t \geqslant 2$ be fixed and consider G and G' two graphs such that G' is the disjoint union of G and a graph G_t that satisfies the conditions in Lemma 8. It was proved in [8] that testing if a graph has a 2-page embedding is NP-complete. Since the size of G' is bounded by a polynomial function of $|V(G)|+|E(G)|$ (the size of G_t is a constant, for each fixed t), it follows that to prove (B), it suffices to show that G has a 2-page embedding if and only if G' has a t-curve embedding.

Suppose that G has pagenumber 2. Let $\mathcal{E}$ be a t-curve embedding of G_t. Let ρ be one of the t clean Jordan curves that witness that $\mathcal{E}$ is a t-curve embedding, and let p be a point on ρ that is not a vertex of G. Let δ be a disk with center p, small enough so that δ does not intersect any vertex or edge of G_t. Then we can embed G in the interior of δ, with the vertices lying on $\rho \cap \delta$. This yields a t-curve embedding of G'.

For the other direction, suppose that $\mathcal{D}'$ is a t-curve embedding of G'. Let $R := \{\rho_1, \ldots, \rho_t\}$ be a set of clean Jordan curves that witness that $\mathcal{D}'$ is a t-curve embedding. We let $\mathcal{E}_t$ denote the restriction of $\mathcal{D}'$ to G_t. Then obviously the collection R witnesses that $\mathcal{E}_t$ is a t-curve embedding.

CLAIM. *Let f be any face of $\mathcal{E}_t$. Then there is at most one curve in R that intersects f.*

Proof. Let f be any face of $\mathcal{E}_t$. By Lemma 8, every face in an embedding of G_t is a triangle, and so f is bounded by a 3-cycle C. Let u, v, w be the vertices of C.

To prove the claim, first note that at most three curves in R can intersect f; this follows simply because C has exactly three vertices, and the curves in R are pairwise disjoint and clean with respect to $\mathcal{E}_t$. Suppose that exactly two curves ρ_i, ρ_j in R intersect f. Since the curves in R are pairwise disjoint, it is not possible that each of ρ_i and ρ_j intersects two vertices of C. Thus at least one of these curves, say ρ_i, must be a loop based on a vertex of C, say u. In fact, the loop ρ_i is the whole clean Jordan curve, otherwise ρ_i would have a self-intersecction at u. The other curve ρ_j either contains both v and w, or

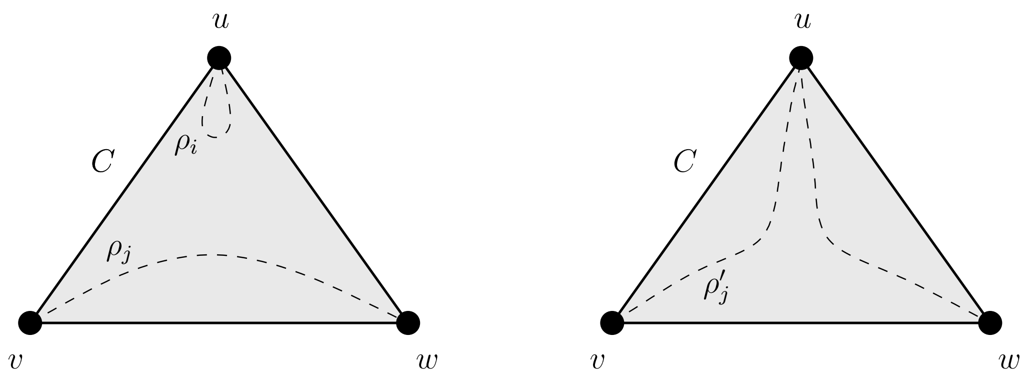

Figure 3: The curve ρ_j intersects the (shaded) face f, and contains v and w. Since the clean curve ρ_i contains u, then $\rho_i \setminus \{u\}$ must be contained in f. In this case, we can replace these two curves by a single curve ρ'_j that contains u, v, and w, as shown on the right-hand side.

exactly one of them. Suppose first that ρ_j contains both v and w. Thus the scenario is as depicted on the left side of Figure 3. We can then remove ρ_i, and reroute the part of ρ_j inside f, so that the resulting curve ρ'_j contains v, u, and w, as illustrated on the right side of Figure 3. Hence $(R \setminus \{\rho_i, \rho_j\}) \cup \rho'_j$ is a set of $t-1$ pairwise disjoint clean Jordan curves whose union contains all the vertices of G_t. Therefore G_t has a $(t-1)$-curve embedding, contradicting (iii) in Lemma 8.

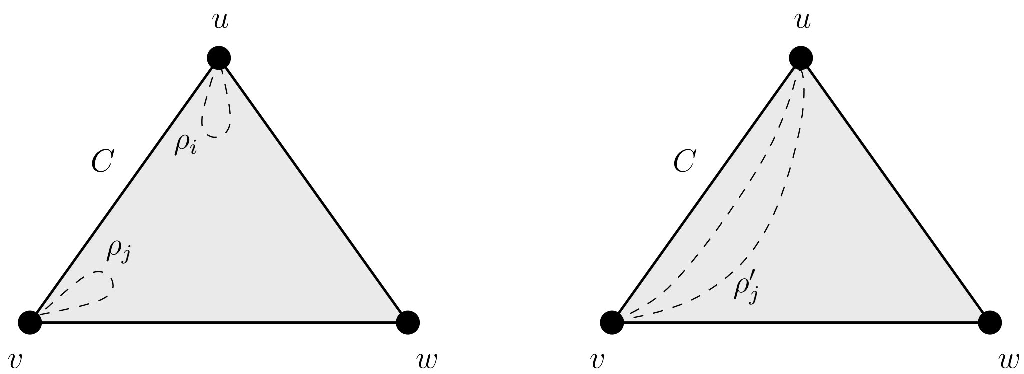

Figure 4: The curve ρ_i contains u, and is otherwise contained in f. The curve ρ_j contains v, and is otherwise contained in f. In this case, ρ_j can be re-routed inside f, as illustrated on the right-hand side, so that the result is a clean Jordan curve ρ'_j that contains both u and v.

Now, if ρ_j contains exactly one of v and w (say v, without loss of generality), then the scenario is as shown on the left side of Figure 4. In this case we can replace ρ_i and ρ_j by a curve ρ'_j that contains both u and v (as in the right side of Figure 4). Thus $(R \setminus \{\rho_i, \rho_j\}) \cup \rho'_j$ is a set of $t-1$ pairwise disjoint clean Jordan curves whose union contains all the vertices of G_t, again contradicting (iii) in Lemma 8.

□

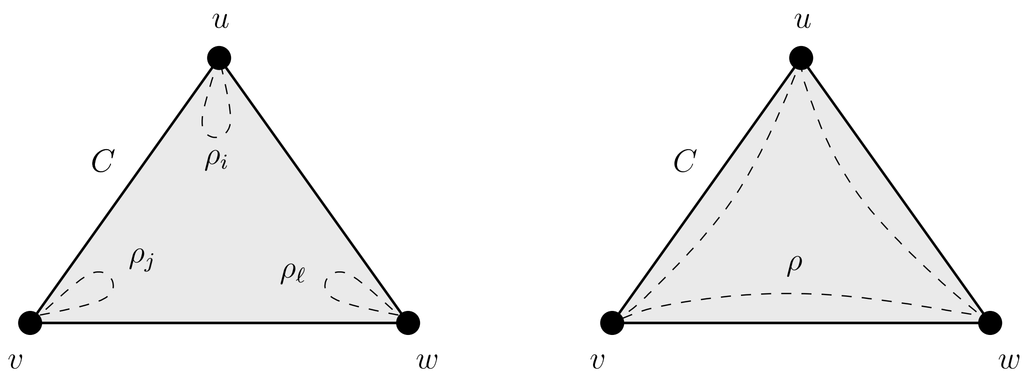

Figure 5: If the clean Jordan curves $\rho_i, \rho_j, \rho_\ell$ contain u, v, and w, respectively, and each of these curves intersects f, then $\rho_i \setminus \{u\}, \rho_j \setminus \{v\}$, and $\rho_\ell \setminus \{w\}$ are contained in f, as shown in the left-hand side figure. These three curves can then be replaced by a single curve ρ that contains u, v, and w.

In the remaining case, exactly three curves $\rho_i, \rho_j, \rho_\ell$ intersect f. In this case each of these curves must contain exactly one of u, v, and w, as illustrated on the left side of Figure 5. We can then replace these three curves by a curve ρ contained in f, as shown on the right side of Figure 5. Thus $(R \setminus \{\rho_i, \rho_j, \rho_\ell\}) \cup \rho$ is a set of $t-2$ pairwise disjoint clean Jordan curves whose union contains all the vertices of G_t. Hence G_t has a $(t-2)$-curve embedding, (and therefore, a $(t-1)$-curve embedding), contradicting (iii) in Lemma 8.

Since G and G_t are disjoint, it follows that there is a face f of $\mathcal{E}_t$ such that, in $\mathcal{D}'$, G is drawn inside f. Thus it follows that some curve in R must intersect f.

From the Claim, there is exactly one curve ρ_m in R that intersects f. Since G is contained in f, it follows that all the vertices of G are contained in ρ_m. Since ρ_m is clean in $\mathcal{D}'$, it follows that ρ_m does not intersect any edge of G. Thus ρ_m witnesses that the restriction of $\mathcal{D}'$ to G is a 1-curve embedding. Hence we are done, since G has a 1-curve embedding if and only if it has a 2-page embedding. $\square$

Finally we show that t-curveCrossingNumber is NP-complete, as claimed in Corollary 6.

Proof of Corollary 6. Let $t \geqslant 2$ be a fixed integer. Given a graph G, let G' be the disjoint union of G and k disjoint copies of $K_{3,3}$. Since the (2-page) crossing number of $K_{3,3}$ is 1, then G has a t-curve embedding if and only if G' has a t-curve drawing with at most k crossings. Therefore t-curveCrossingNumber is at least as hard as t-curveEmbedding. The membership of t-curveCrossingNumber in NP follows from the fact that the time required to test whether a graph has a plane drawing with at most k crossing is polynomial. $\square$

4 Concluding remarks

It follows from the proof of Theorem 4 that, for each fixed $t \geqslant 2$, even the problem of deciding whether a given graph admits a t-curve embedding, is already NP-complete. As we have observed, this is also true for $t = 1$, as testing if a graph has pagenumber 2 (which is equivalent to testing if it has a 1-curve embedding) is NP-complete.

We recall that a *p-page book* consists of p halfplanes (the *pages*) whose boundaries lie on a common line (*the spine*). In a *p-page drawing*, all the vertices lie on the spine, and each edge (except for its endpoints) lies on a single page [6]. The *p-page crossing number* $\mathrm{bkcr}_p(G)$ of a graph G is the minimum number of crossings in a p-page drawing of G [12].

In the Book Crossing Number entry in [11], Schaefer mentions that testing if a graph G satisfies $\mathrm{bkcr}_p(G) = 0$ is NP-complete, for every integer $p \geqslant 2, p \neq 3$ (the case $p = 3$ remains open). We note that analogous arguments to those we used in part (B) of the proof of Theorem 4 can be used to prove the following.

Observation 9. *The decision problem "given a graph G, is $\mathrm{bkcr}_p(G) \leqslant k$?" is NP-complete for fixed $p \geqslant 2, p \neq 3$, and fixed $k \geqslant 0$.*

It is reasonable to argue that, alternatively to the definition of a t-circle drawing, we could obtain a generalization of the definition of a cylindrical drawing by asking that the vertices are contained in $t > 2$ clean concentric circles. To illustrate an issue with such a definition, let us consider drawings of the complete graph in which the vertices are placed on three clean concentric circles. Then there cannot be a vertex in the inner circle and a vertex in the outer circle, as then an edge joining these two vertices would necessarily cross the middle circle. Thus either all the vertices must lie in the union of the middle circle and the outer circle, or in the union of the middle circle and the inner circle. That is, any such drawing of the complete graph is necessarily cylindrical. Thus, for the complete graph, such an alternative definition of a t-circle drawing is not really more general than the definition of a cylindrical drawing. On the other hand, if we allow the interior of an edge to intersect each circle at most once, then we arrive at the radial crossing number [2, 10, 11] (see also the related notion of the cyclic level crossing number [3]).

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