On the adjacency spectra of hypertrees

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Abstract

We show that λ is an eigenvalue of a k-uniform hypertree ($k \geqslant 3$) if and only if it is a root of a particular matching polynomial for a connected induced subtree. We then use this to provide a spectral characterization for power hypertrees. Notably, the situation is quite different from that of ordinary trees, i.e., 2-uniform trees. We conclude by presenting an example (an 11 vertex, 3-uniform non-power hypertree) illustrating these phenomena.

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1 Introduction

The following beautiful result was shown in [8]: the set of roots of a certain matching polynomial of a k-uniform hypertree (an acyclic k-uniform hypergraph) is a subset of its homogeneous adjacency spectrum.

Theorem 1. *([8]) λ is a nonzero eigenvalue of a hypertree H with the corresponding eigenvector $\mathbf{x}$ having all elements nonzero if and only if it is a root of the polynomial*

$$\varphi(H) = \sum_{i=0}^{m} (-1)^i |\mathcal{M}_i| x^{(m-i)r}$$

where $\mathcal{M}_i$ is the collection of all i-matchings of H.

In the present work, we show how to obtain *all* of the eigenvalues of a hypertree, and use this description to give a spectral characterization of "power" hypertrees (defined below). Here the notion of a hypergraph's eigenpairs is the homogeneous adjacency spectrum, q.v. Qi [7], Lim [4], and Cooper-Dutle [1]. We extend Theorem 1 as follows to describe the spectrum of a hypertree, answering the main open question in [8].

Theorem 2. *Let $\mathcal{H}$ be a k-uniform hypertree, for $k \geqslant 3$; λ is a nonzero eigenvalue of $\mathcal{H}$ if and only if there exists an induced subtree $H \subseteq \mathcal{H}$ such that λ is a root of the polynomial $\varphi(H)$.*

We note that Theorem 2 is not true for (2-uniform) trees (c.f. Cauchy's interlacing theorem), making it an unusual example of a result in spectral hypergraph theory that fails in the graph case. The necessity for $k \geqslant 3$ is established by Theorem 7 ([8], [9]).

Below, we reserve the use of the term "hypergraph" for the case of k-uniform hypergraphs with $k \geqslant 3$, and use the language of "graphs" exclusively for $k = 2$. We also make use of the nomenclature k-graph and k-tree to mean k-uniform hypergraph and k-uniform hypertree, respectively. We maintain much of the notation of [8] and refer the interested reader to their paper. In the next section, we provide necessary definitions and prove Theorem 2. We then use this result to show that power trees are characterized by their spectra being cyclotomic (Theorem 9) and provide an example demonstrating these phenomena.

2 Proof of Theorem 2

A vector is *totally nonzero* if each coordinate is nonzero and an eigenpair $(\lambda, \mathbf{x})$ is totally nonzero if $\lambda \neq 0$ and $\mathbf{x}$ is a totally nonzero vector. Given a vector $\mathbf{x} \in \mathbb{C}^n$ the *support* $\mathrm{supp}(\mathbf{x})$ is the set of all indices of non-zero coordinates of $\mathbf{x}$. Let $\mathbf{x}^\circ$ denote the totally nonzero projection (by restriction) of $\mathbf{x}$ onto $\mathbb{C}^{|\mathrm{supp}(\mathbf{x})|}$. For ease of notation, we assume that the coordinate indices of vectors agree with the vertex labeling of the hypergraph under consideration. We denote the induced subgraph of $\mathcal{H}$ on $U \subseteq V(\mathcal{H})$ by

$$\mathcal{H}[U] = (U, \{v_1 \ldots v_k \in E(\mathcal{H}) : v_i \in U\})$$

and write $H \sqsubseteq \mathcal{H}$ to mean $H = \mathcal{H}[U]$ for some $U \subseteq V(\mathcal{H})$.

The following establishes the forward direction of Theorem 2.

Lemma 3. *Let $(\lambda, \mathbf{x})$ be a nonzero eigenpair of the normalized adjacency matrix of a k-uniform hypergraph $\mathcal{H}$. Then $(\lambda, \mathbf{x}^\circ)$ is a totally nonzero eigenpair of $\mathcal{H}[supp(\mathbf{x})]$.*

Proof. As $(\lambda, \mathbf{x})$ is an eigenpair of $\mathcal{H}$,

$$\sum_{i_2.i_2,\ldots,i_k=1}^{n} a_{ji_2i_3\ldots i_k} x_{i_2} x_{i_3} \ldots x_{i_k} = \lambda x_j^{k-1}$$

for $j \in [n]$ by definition. Let $m = |\mathrm{supp}(\mathbf{x})|$ and suppose without loss of generality that supp($\mathbf{x}$) $= [m]$. For $j \in [m]$ we have

$$\begin{aligned}
\lambda(x^\circ)_j^{k-1} &= \lambda x_j^{k-1} \\
&= \sum_{i_2.i_2,\ldots,i_k=1}^{n} a_{ji_2i_3\ldots i_k} x_{i_2}x_{i_3}\ldots x_{i_k} \\
&= \sum_{i_2.i_2,\ldots,i_k=1}^{m} a_{ji_2i_3\ldots i_k} x_{i_2}x_{i_3}\ldots x_{i_k}.
\end{aligned}$$

Thus, $(\lambda, \mathbf{x}^\circ)$ is an eigenpair of $\mathcal{H}[m]$ by definition; moreover, $(\lambda, \mathbf{x}^\circ)$ is totally nonzero, as each coordinate of $\mathbf{x}^\circ$ is nonzero by construction. $\square$

The following result, from [1], shows how the eigenvalues of a disconnected hypergraph arise from the eigenvalues of its components.

Theorem 4. *([1]) Let H be a k-graph that is the disjoint union of hypergraphs H_1 and H_2. Then as sets, $spec(H) = spec(H_1) \cup spec(H_2)$.*

The *matching polynomial* $\varphi(H)$ of H is defined by

$$\varphi(H) = \sum_{i=0}^{m}(-1)^i|\mathcal{M}_i|x^{(m-i)r}$$

where $\mathcal{M}_i$ is the collection of all i-matchings of H. We now show that φ is multiplicative over connected components.

Claim 5. *If $\mathcal{H} = \bigsqcup_{i=1}^{t} H_i$ is a disjoint union of k-uniform hypertrees then $\varphi(\mathcal{H}) = \prod_{i=1}^{t}\varphi(H_i)$.*

Proof. Clearly, the result follows inductively if it is true for $t = 2$. Denote matching numbers $m(H_1) = m_1$ and $m(H_2) = m_2$. Indexing the sum of H_1 and H_2 by i and j, respectively, we compute

$$\begin{aligned}
\varphi(H_1)\varphi(H_2) &= \left(\sum_{i=0}^{m_1}(-1)^i|\mathcal{M}_i|x^{(m_1-i)k}\right)\left(\sum_{j=0}^{m_2}(-1)^i|\mathcal{M}_j|x^{(m_2-i)k}\right) \\
&= \sum_{\substack{0\leqslant i\leqslant m_1 \\ 0\leqslant j\leqslant m_2}} (-1)^{i+j}|\mathcal{M}_i||\mathcal{M}_j|x^{(m_1+m_2-(i+j))k}.
\end{aligned}$$

Let

$$\varphi(H_1 \sqcup H_2) := \sum_{\ell=0}^{m}(-1)^\ell|\mathcal{M}_\ell|x^{(m-\ell)k}.$$

Since matchings of H are unions of matchings of H_1 and H_2, $m = m_1 + m_2$ is the size of the the largest matching of H. Furthermore, for any $0 \leqslant \ell \leqslant m$,

$$|\mathcal{M}_\ell| = \sum_{\substack{0\leqslant i\leqslant m_1\\ 0\leqslant j\leqslant m_2\\ i+j=\ell}} |\mathcal{M}_i||\mathcal{M}_j|,$$

since a matching of ℓ edges in H consists of a matching of i edges in H_1 and j edges in H_2, where $i + j = \ell$. Substituting yields

$$\varphi(H_1)\varphi(H_2) = \sum_{\ell=0}^{m} (-1)^\ell |\mathcal{M}_\ell| x^{(m-\ell)k} = \varphi(H_1 \sqcup H_2)$$

as desired. □

Recall from [8] that a *pendant* edge is a k-uniform edge with exactly $k - 1$ vertices of degree 1.

Claim 6. *Let $\mathcal{H}$ be an r-uniform hypertree. If $H \subseteq \mathcal{H}$ is a sub-hypertree then there exists a sequence of edges $(e_1, e_2, \ldots, e_t)$ such that e_i is a pendant edge of H_{i-1} where $H_0 := \mathcal{H}$, $H_i := H_{i-1} - e_i$, and $H_t := H$.*

Proof. We prove our claim by induction on $||E(\mathcal{H})| - |E(H)||$. The base case of $||E(\mathcal{H})| - |E(H)|| = 0$ is immediate. Suppose that $||E(\mathcal{H})| - |E(H)|| = 1$. Since H is a hypertree, H is necessarily connected. In particular, H is formed by removing a pendant edge of $\mathcal{H}$, and the claim follows. □

In keeping with the notation of [8], let $s(\mathcal{H})$ be the set of all sub-hypertrees of $\mathcal{H}$.

Theorem 7. *([8]) For any k-uniform hypertree $\mathcal{H}$ where $k \geqslant 3$, the roots of*

$$\prod_{H_i \in s(H)} \varphi(H_i)$$

are eigenvalues of $\mathcal{H}$. Moreover, the largest root is the spectral radius of H.

We now present a proof of Theorem 2.

Proof. Theorem 1 establishes the case when λ is nonzero and has an eigenvector which is totally nonzero.

Let $(\lambda, \mathbf{x})$ be a nonzero eigenpair of $\mathcal{H}$. From Claim 3, it follows that $(\lambda, \mathbf{x}^\circ)$ is a totally nonzero eigenpair of $\mathcal{H}[\mathrm{supp}(\mathbf{x})]$. For the moment, let

$$H = \mathcal{H}[\mathrm{supp}(\mathbf{x})] \subseteq \mathcal{H}.$$

If H is connected we apply Theorem 1 to conclude that λ is a root of $\varphi(H)$ as desired. If instead H is disconnected, then we may write $H = \bigsqcup_{i=1}^{t} H_i$, where each H_i is connected. Appealing to Theorem 4, λ is a root of $\varphi(H)$, whence $(x - \lambda) \mid \varphi(H_i)$ for some i. Let $\mathbf{y}$

be the totally nonzero projection (by restriction) of $\mathbf{x}$ onto $\mathbb{C}^{|V(H_i)|}$. Indeed, $(\lambda, \mathbf{y})$ is a totally nonzero eigenpair of H_i. Therefore, as H_i is a connected subgraph of $\mathcal{H}$ we apply Theorem 1 to conclude that λ is a root of $\varphi(H_i)$, as desired.

Now suppose that λ is a root of $\varphi(H)$ for $H \subseteq \mathcal{H}$. The case of $H = \mathcal{H}$ was established by Theorem 1: suppose further that $H \subsetneq \mathcal{H}$. If H is connected then by Theorem 1 $(\lambda, \mathbf{x}^\circ)$ is a totally nonzero eigenpair of H. By Claim 6 and Theorem 7 we conclude that λ is an eigenvalue of $\mathcal{H}$. Suppose even further that $H = \bigsqcup_{i=1}^{t} H_i$ where H_i is a connected component. We have shown in Claim 5,

$$\varphi(H) = \prod_{i=1}^{t} \varphi(H_i).$$

As λ is a root of $\varphi(H)$, it is true that λ is a root of $\varphi(H_i)$ for some i. Appealing to Theorem 1, once more we have that $(\lambda, \mathbf{x}^\circ)$ is a totally nonzero eigenpair of H_i; therefore, $(\lambda, \mathbf{x})$ is an eigenpair of $\mathcal{H}$ by Claim 6 and Theorem 7 as desired. $\square$

3 The spectra of power trees

The following generalizes the definition of powers of a hypergraph from [2].

Definition 8. Let H be an r-graph for $r \geqslant 2$. For any $k \geqslant r$, the k^{th} power of G, denoted H^k, is a k-uniform hypergraph with edge set

$$E(H^k) = \{e \cup \{v_{e,1}, \ldots, v_{e,k-r}\} : e \in E(G)\},$$

and vertex set

$$V(H^k) = V(G) = V(G) \cup \{i_{e,j} : e \in E(G), j \in [k-r]\}.$$

In other words, one adds exactly enough new vertices (each of degree 1) to each edge of H so that H^k is k-uniform. Note that, if $k = r$, then $H^k = H$. Adhering to this nomenclature we refer to a power of a 2-tree simply as a *power tree*. In this section we prove the following characterization of power trees.

Theorem 9. *Let $\mathcal{H}$ be a k-tree and let $\sigma(\mathcal{H})$ denote its (multiset) spectrum. Then $\sigma(\mathcal{H}) \subseteq \mathbb{R}[\zeta_k]$ if and only if $\mathcal{H}$ is a power tree, where ζ_k is a principal k^{th} root of unity.*

We recall the following Theorem from Cooper-Dutle.

Theorem 10. *[1] The (multiset) spectrum of a k-cylinder is invariant under multiplication by any k^{th} root of unity.*

One can show by straightforward induction that a k-tree is a k-cylinder, so its spectrum is symmetric in the above sense. The following result, from [9], shows that power trees have spectra which satisfy a much more stringent condition: they are cyclotomic, in the sense that they belong to $\mathbb{R}[\zeta_k]$.

Theorem 11. *[9] If $\lambda \neq 0$ is an eigenvalue of any subgraph of G, then $\lambda^{2/k}$ is an eigenvalue of G^k for $k \geqslant 4$.*

We restate Theorem 11 with the additional assumption that the underlying graph is a tree; the proof is easily obtained by applying Claim 6 to the proof of Theorem 11 appearing in [9].

Corollary 12. *If $\lambda \neq 0$ is an eigenvalue of any subgraph of a tree T, then $\lambda^{2/k}$ is an eigenvalue of T^k for $k \geqslant 3$.*

Note that Theorem 9 provides a converse to Corollary 12 in the case of power trees. In particular, appealing to Theorem 2, Corollary 12, and the fact that the spectrum of a graph is real-valued, we have that the spectrum of a power tree is a subset of $\mathbb{R}[\zeta_k]$. All that remains to be shown is that if a k-tree is not a power tree then it has a root in $\mathbb{C} \setminus \mathbb{R}[\zeta_k]$. To that end, we introduce the k-comb.

Let $\textsc{comb}_k$ be the k-graph where

$$\textsc{comb}_k = ([k^2], \{[k] \cup \{\{i + tk : 0 \leqslant t \leqslant k-1\} : i \in [k]\}\}.$$

We refer to $\textsc{comb}_k$ as the k-comb. By the definition of power tree, a non-power tree H must contain an edge e incident to a family $\mathcal{F}$ consisting of at least three other edges which are mutually disjoint. This edge e, together with $\mathcal{F}$, form a connected induced subgraph H' of H which is the k^{th} power of a t-comb for $t = |\mathcal{F}| \geqslant 3$. It is straightforward to see that $\varphi(H')(x) = \varphi(\textsc{comb}_t^k)(x) = \varphi(\textsc{comb}_t)(x^{k/t})$, since matchings in H' are simply k^{th} powers of matchings in $\textsc{comb}_t$; therefore, roots of $\varphi(H')$ are k^{th} roots of reals if and only if the roots of $\varphi(\textsc{comb}_t)$ are t^{th} roots of reals. We presently show that the spectrum of the k-comb is not contained within the k^{th} cyclotomic extension of $\mathbb{R}$, completing the proof of Theorem 9.

Lemma 13. *There exists a root λ of $\varphi(\textsc{comb}_k)$ for $k \geqslant 3$ such that $\lambda \in \mathbb{C} \setminus \mathbb{R}[\zeta_i]$.*

Proof. Let $\mathcal{H}$ be a k-comb where $k \geqslant 3$. By a simple counting argument,

$$\varphi(\mathcal{H}) = \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} \alpha^{k-i} \right) - \alpha^{k-1}$$

where $\alpha = x^k$. Appealing to the binomial theorem we have

$$\varphi(\mathcal{H}) = (1-\alpha)^k - \alpha^{k-1}.$$

Let $\beta = \alpha^{-1}$. Setting $\varphi(\mathcal{H}) = 0$ yields

$$(\beta - 1)^k = \beta. \tag{1}$$

It is easy to see that (1) has precisely one solution when $k \geqslant 3$ is odd and precisely two solutions when it is even. In either case, as the number of solutions is strictly less than k it follows that there must be a non-real solution and the claim follows. □

4 Concluding Remarks

We conclude our note by presenting an example demonstrating Theorems 2 and 9.

Consider the 3-uniform hypergraphs

$$\begin{aligned}\mathcal{H}_1 &= ([9], \{\{1,2,3\},\{1,4,7\},\{2,5,8\},\{3,6,9\}\}) = \textsc{comb}_3\\ \mathcal{H}_2 &= ([9], \{\{1,2,3\},\{1,4,7\},\{3,6,9\},\{1,10,11\}\})\\ \mathcal{H}_3 &= ([11], E(\mathcal{H}_1) \cup E(\mathcal{H}_2)).\end{aligned}$$

Let $\phi(\mathcal{H})$ denote the characteristic polynomial of the normalized adjacency hypermatrix of $\mathcal{H}$ as described in [1]. We have computed via [5],

$$\begin{aligned}\phi(\mathcal{H}_1) &= x^{567}(x^9-4x^6+3x^3-1)^{81}(x^6-3x^3+1)^{81}(x^3-2)^{27}(x^3-1)^{147}\\ \phi(\mathcal{H}_2) &= x^{999}(x^6-4x^3+2)^{81}(x^6-3x^3+1)^{54}(x^3-3)^{27}(x^3-2)^{63}(x^3-1)^{75}\\ \phi(\mathcal{H}_3) &= x^{3767}(x^9-5x^6+5x^3-2)^{243}(x^9-4x^6+3x^3-1)^{162}(x^6-4x^3+2)^{162}\\ &\quad \cdot (x^6-3x^3+1)^{135}(x^3-3)^{27}(x^3-2)^{180}(x^3-1)^{483}.\end{aligned}$$

Let P_n and S_n denote the 3-uniform loose path and star with n edges, respectively. We list the non-trivial induced subgraphs of $\mathcal{H}_3$ and their matching polynomials below.

$H \sqsubseteq \mathcal{H}_3$	$\varphi(H)$
$P_1 = S_1$	x^3-1
$P_2 = S_2$	x^3-2
P_3	x^6-3x^3+1
S_3	x^3-3
$\mathcal{H}_1$	$x^9-4x^6+3x^3-1$
$\mathcal{H}_2$	x^6-4x^3+2
$\mathcal{H}_3$	$x^9-5x^6+5x^3-2$

Figure 1 gives a drawing of $\mathcal{H}_3$ (the striped subgraph is $\mathcal{H}_1$) and a plot of the roots of $\phi(\mathcal{H}_3)$, with a circle centered at each root in the complex plane whose area is proportional to the multiplicity of the root. Notice that, despite the rotational symmetry (turning by a third), the cubes of the roots are not all real, i.e., some of the roots do not lie on the the rays with argument 0, $2\pi/3$, or $4\pi/3$.

Observe that each matching polynomial divides wholly into the characteristic polynomials. A priori, this is a symptom of the matching polynomials having distinct roots. A preliminary question: what can one say about the roots of $\varphi(H_1)$ and $\varphi(H_2)$ for k-trees $H_1 \sqsubseteq H_2$? With this question in mind we conjecture the following.

Conjecture 14. If $H \sqsubseteq \mathcal{H}$ are k-trees for $k \geqslant 3$ then $\varphi(H) \mid \phi(\mathcal{H})$. In particular, if $H \sqsubseteq \mathcal{H}$ then $\phi(H) \mid \phi(\mathcal{H})$.

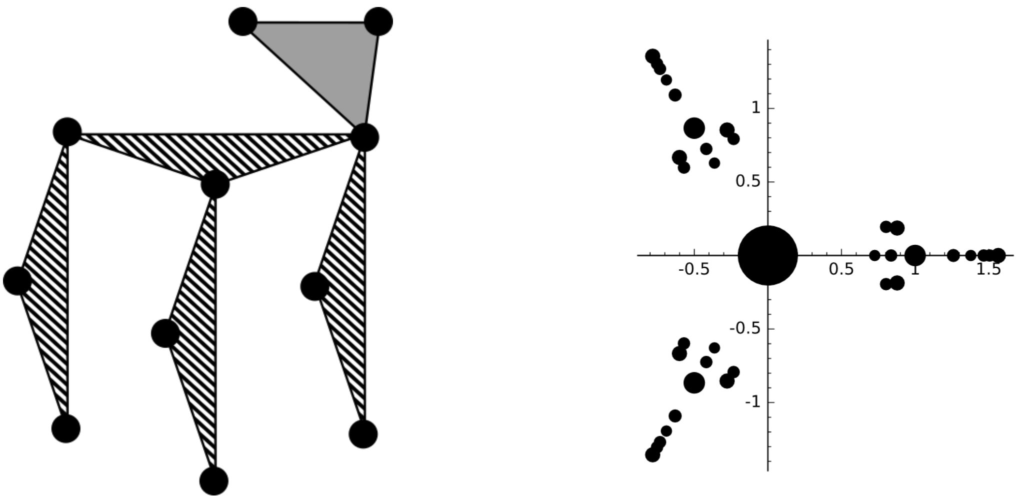

Figure 1: $\mathcal{H}_3$ and its spectrum.

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