Topological infinite gammoids, and a new Menger-type theorem for infinite graphs

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Abstract

Answering a question of Diestel, we develop a topological notion of gammoids in infinite graphs which, unlike traditional infinite gammoids, always define a matroid.

As our main tool, we prove for any infinite graph G with vertex-subsets A and B, if every finite subset of A is linked to B by disjoint paths, then the whole of A can be linked to the closure of B by disjoint paths or rays in a natural topology on G and its ends.

This latter theorem implies the topological Menger theorem of Diestel for locally finite graphs. It also implies a special case of the infinite Menger theorem of Aharoni and Berger.

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1 Introduction

Unlike finite gammoids, traditional infinite gammoids do not necessarily define a matroid. Diestel [9] asked whether a suitable topological notion of infinite gammoid might mend this, so that gammoids always give rise to a matroid. We answer this in the positive by developing such a topological notion of infinite gammoid. Our main tool is a new topological variant of Menger's theorem for infinite graphs, which is also interesting in its own right.

Given a directed graph[1] G with a set $B \subseteq V(G)$ of vertices, the set $\mathcal{L}(G, B)$ contains all vertex-subsets I that can be linked by vertex-disjoint directed paths[2] to B. If G is finite, $\mathcal{L}(G, B)$ is the set of independent sets of a matroid, called the *gammoid of G with respect to B*. If G is infinite, $\mathcal{L}(G, B)$ does not always define a matroid [1].

[1] We allow loops and parallel edges.

[2] In this paper, *paths* are always finite.

In 1968, Perfect [11] looked at the question of when $\mathcal{L}(G, B)$ is a matroid. As usual at that time, she restricted her attention to matroids with every circuit finite, now called *finitary matroids*. In [6], Bruhn et al found a more general notion of infinite matroids, which are closed under duality and need not be finitary. Afzali, Law and Müller [1] studied infinite gammoids in this more general setting and found conditions under which $\mathcal{L}(G, B)$ is a matroid. In this paper, we introduce a topological notion of gammoids in infinite graphs that always define a matroid.

These gammoids can be defined formally without any reference to topology, as follows. A ray R in G *dominates* B if G contains infinitely many vertex-disjoint directed paths from R to B, see Figure 1. A vertex v *dominates* B if there are infinitely many directed paths from v to B that are vertex-disjoint except in v. A path *dominates* B if its last vertex dominates B. A *domination linkage* from A to B is a family of vertex-disjoint directed paths or rays $(Q_a \mid a \in A)$ where Q_a starts in a and either ends in some vertex of B or else dominates B. A vertex-subset I is in $\mathcal{L}_T(G, B)$ if there is a domination linkage from I to B.

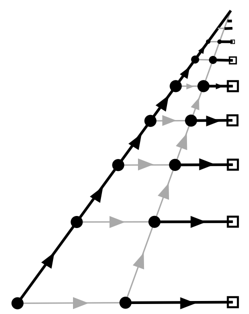

Figure 1: The vertices of B are squares, all other vertices are circles. In black, we see a domination-linkage into B which contains one ray dominating B.

We offer the following solution to Diestel's question:

Theorem 1. *$\mathcal{L}_T(G, B)$ is a finitary matroid.*

When G is undirected[3], Theorem 1 has the following topological interpretation. On G and its ends consider the topology whose basic open sets are the components C of $G \setminus X$ where X is a finite set of inner points of edges, together with the ends that have rays in C. The closure of $B \subseteq V(G)$ consists of B, the vertices dominating B, and the ends ω whose rays $R \in \omega$ dominate B. Thus $I \in \mathcal{L}_T(G, B)$ if and only if the whole of I can be linked to the closure of B by vertex-disjoint paths or rays.[4] We will not need this topological interpretation.

[3]Formally, we consider those directed graphs G obtained from an undirected graph by replacing each edge by two parallel edges directed both ways.

[4]Instead of just taking paths and rays, one might want to take all 'topological arcs'. However, this would result in a weaker theorem than the one proved here.

Theorem 1 can be used to prove that under certain conditions the naive, non-topological, gammoid $\mathcal{L}(G, B)$ is a matroid, too:

Corollary 2. *Let G be a digraph with a set B of vertices such that there are neither infinitely many vertex-disjoint rays dominating B nor infinitely many vertices dominating B. Then $\mathcal{L}(G, B)$ is a matroid.*

Theorem 2 does not follow from the existence criterion of Afzali, Law and Müller for non-topological gammoids. Also its converse is not true, see Section 5 for details.

Theorem 1 is a natural example of a phenomeon that happens often in topological infinite graph theory: the naive extension of the finite theorem that $\mathcal{L}(G, B)$ is a matroid for every digraph G and every vertex-subset B is false. However, there is a natural topological extension that is true for all graphs and all vertex-subsets (for a nontrivial reason).

The main tool in our proof of Theorem 1 is a purely graph-theoretic Menger-type theorem, which seems to be interesting in its own right. It is not difficult to show that if there is a domination linkage from A to B, then there is a linkage from every finite subset of A to B. Our theorem says that the converse is also true:

Theorem 3. (i) *In any infinite digraph with vertex-subsets A and B, there is a domination linkage from A to B if and only if every finite subset of A can be linked to B by vertex-disjoint directed paths.*

(ii) *In any infinite undirected graph G, a set A of vertices can be linked by disjoint paths and rays to the closure of another vertex-subset B if and only if every finite subset can be linked to B by vertex-disjoint paths.*

We remark that the proof of Theorem 3 is non-trivial and not merely a compactness result. Applying compactness, one would get a topological linkage from A to the closure of B by arbitrary topological arcs, not necessarily paths and rays. Our graph-theoretical version of Theorem 3 is considerably stronger than this purely topological variant.

In Section 4 we study the relationship between Theorem 3 and existing Menger-type theorems for infinite graphs: the Aharoni-Berger theorem [3] and the topological Menger theorem for arbitrary infinite graphs. The latter was proved by Bruhn, Diestel and Stein [5], extending an earlier result of Diestel [8] for countable graphs. In the special case of infinite graphs with 'well-separated' sets A and B (defined in Section 4), Theorem 3 implies and strengthens the Aharoni-Berger theorem. This in turn allows us to give a proof of the topological Menger theorem for locally finite graphs which, unlike the earlier proofs, does not rely on the (countable) Aharoni-Berger theorem (which was proved earlier by Aharoni [2]).

The paper is organised as follows. After a short preliminary section we prove in Section 3 the directed edge version of Theorem 3. In Section 4 we sketch how this variant implies Theorem 3, and how Theorem 3 implies the Aharoni-Berger theorem for 'well-separated' sets A and B, and the topological Menger theorem for locally finite graphs. In Section 5 we summarise some basics about infinite matroids, and prove Theorem 1 and Theorem 2.

2 Preliminaries

Throughout, notation and terminology for graphs are that of [7]. In Sections 2 and 3, we will mainly be concerned with sets of edge-disjoint directed paths. Thus there, we abbreviate 'edge-disjoint' by 'disjoint', 'edge-separator' by 'separator' and 'directed path' by 'path'. Given a digraph G and $A, B \subseteq V(G)$, a *linkage from A to B* is a set of disjoint paths from the whole of A to B. We update the definitions of 'a ray dominates B', 'a vertex dominates B', 'a path dominates B', and of 'domination linkage': these are the definitions made in the Introduction with 'vertex-disjoint' replaced by 'edge-disjoint'. In a slight abuse of notion, we will suppress the set brackets of $\{b\}$ and just talk about '(domination) linkages from I to b'. The proof of the following theorem takes the whole of Section 3.

Theorem 4. *Let G be a digraph and $b \in V(G)$, and $I \subseteq V(G)-b$. There is a domination linkage from I to b if and only if every finite subset of I has a linkage into b.*

We delay the proof that Theorem 4 implies Theorem 3 until Section 4.

One direction of Theorem 4 is indeed easy:

Lemma 5. *If there is a domination linkage from I to b, then every finite subset S of I has a linkage into b.*

Proof. For $s \in S$, let P_s be the path or ray from the domination linkage starting in s. Suppose for a contradiction, there is no linkage from S into b. Then by Menger's theorem, there is a set F of at most $|S|-1$ edges such that after its removal there is no (directed) path from S to b.

Suppose for a contradiction that there is some P_s not containing an edge of F. Then P_s cannot end at b. So P_s dominates b, and thus there is some P_s-b-path avoiding F, contradicting the fact that F was a separator. Thus each P_s contains an edge of F. As the P_s are disjoint, $|F| \geqslant |S|$, which is the desired contradiction. $\square$

3 Proof of Theorem 4

The proof of Theorem 4 takes the whole of this section.

3.1 Exact graphs

The core of the proof of Theorem 4 is the special case where G is exact (defined below). In this subsection, we show that the special case of Theorem 4 where G is exact implies the general theorem. More precisely, we prove that the Theorem 6 below implies Theorem 4.

Given a vertex-subset D, an edge is *D-crossing (or crossing for D)* if its starting vertex is in D and the endvertex is outside. We abbreviate $V(G) \setminus D$ by $D^{\complement}$. The *order of D* is the number of D-crossing edges. The vertex-subset D is *exact* (for some set $I \subseteq V(G)$ and $b \in V(G)$) if $b \notin D$ and the order of D is finite and equal to $|D \cap I|$. A graph is *exact* (for b and I) if for every $v \in V(G) - b$, there is an exact set D containing v.

Lemma 6. *Let G be an exact digraph and $b \in V(G)$. Let $I \subseteq V(G) - b$ such that every finite subset of I has a linkage into b. Then there is a domination linkage from I to b.*

First we need some preparation. Let G be a graph and let $b \in V(G)$. Let $\mathcal{I}$ be the set of all sets $K \subseteq V(G) - b$ such that every finite subset of K has a linkage into b. The following is an easy consequence of Zorn's lemma.

Proposition 7. *Let $K \in \mathcal{I}$, and $X \subseteq V(G) - b$ containing K, then there is $J \in \mathcal{I}$ maximal with $K \subseteq J \subseteq X$.* □

Lemma 8. *Let G be a directed graph, and let $I \subseteq V(G) - b$ be maximal with the property that every finite subset of I has a linkage into b. Let $v \in (V(G) - b) \setminus I$. Then there is an exact D containing v.* □

Proof. By the maximality of I, there is a finite subset I' of I such that $I' + v$ cannot be linked to b. By Menger's theorem, there is a vertex-subset D of order at most $|I'|$ not containing b but containing $I' + v$. The order must be precisely $|I'|$ since I' can be linked to b. Thus D is exact, which completes the proof. □

Proof that Theorem 6 implies Theorem 4. By Theorem 5, it suffices to prove the 'if'-implication. Let G, b, I be as in Theorem 4. We obtain the graph H_1 from G by identifying b with all vertices v such that there are infinitely many disjoint v-b-paths. Note that in H_1 every vertex $v \neq b$ can be separated from b by a finite separator.

It suffices to prove the theorem for H_1 since then the set of dominating paths and rays we get for H_1 extends to a set of dominating paths and rays for G by adding a singleton path for every vertex in I that is identified with b in H_1.

We build an exact graph H_2 that has H_1 as a subgraph. Let $v \in V(H_1) - b$. Let k_v be the smallest order of some vertex-subset D containing v and not containing b. By construction of H_1, the number k_v is finite.

We obtain H_2 from H_1 by for each $v \in V(H_1) - b$ adding k_v-many vertices whose forward neighbourhood is that of v and that do not have any incoming edges. We will refer to these newly added vertices for the vertex v as the clones of v.

Now we extend I to a maximal set $I_2 \subseteq V(H_2) - b$ such that every finite subset of I_2 has a linkage into b. This is possible by Theorem 7.

Next we show that H_2 is exact with respect to I_2. Suppose for a contradiction that there is some $v \in I_2$ such that there is no exact D containing v. First we consider the case that $v \in V(H_1)$. Since v together with all its clones cannot be linked to b, there is a clone w of v that is not in I_2. Since $w \notin I_2$, there must be some exact D' containing w by Theorem 8. If $v \in D'$, we are done, otherwise we get a contradiction since there is no linkage from $((I \cap D') + v)$ to b. The case that $v \notin V(H_1)$ is similar.

Having shown that H_2 is exact, we now use the assumption that Theorem 6 is true for H_2 and I_2: We get for each $v \in I$ some path or ray that dominates b in H_2. This path or ray also dominates in H_1 because a clone-vertex cannot be an interior vertex of any (directed) path or ray. And it also dominates in G, which completes the proof that Theorem 6 implies Theorem 4. □

3.2 Exact vertex-subsets

In this subsection, we prove some lemmas needed in the proof of Theorem 6.

Until the end of the proof of Theorem 4, we will fix a graph G that is exact with respect to a fixed vertex b and some set $I \subseteq V(G) - b$. We further assume that every finite subset of I has a linkage into b. First we will prove some lemmas about exact vertex-subsets.

Lemma 9. *Let D be exact and let $P_1, \dots P_n$ be a linkage from $I \cap D$ to b. Then each P_i contains precisely one D-crossing edge, and each D-crossing edge is contained in one P_i.*

Proof. Clearly, each P_i contains a D-crossing edge. Since the P_i are disjoint no two of them contain the same crossing edge. Since D is exact, there are precisely n D-crossing edges, and thus there is precisely one on each P_i. □

Lemma 10. *Let $D, D' \subseteq V(G)$ such that $D' \subseteq D$, and D' is exact. Let $\mathcal{L}$ be a linkage from $(I \cap D)$ to b. If some $P \in \mathcal{L}$ starts at a vertex in $D \setminus D'$, then no vertex of P lies in D'.*

Proof. We recall that the set D' cannot contain the vertex b by the definition of exact. Since D' is exact, each D'-crossing edge lies on some path of $\mathcal{L}$. On the other hand $|I \cap D'|$ of the paths start in D', and thus contain an D'-crossing edge. So P cannot contain any D'-crossing edge. If P meets D', then it would meet D' in a last vertex, and the edge pointing away from this vertex would be a D'-crossing edge. Hence P does not meet D', which completes the proof. □

Lemma 11. *Let D and D' be exact.*

(i) Then $D \cup D'$ is exact.

(ii) Then $D \cap D'$ is exact.

(iii) Then there does not exist an edge from $D \setminus D'$ to $D' \setminus D$.

Proof. Let $\mathcal{L}$ be a linkage from $I \cap (D \cup D')$ to b. For $X \subseteq D \cup D'$, let $\mathcal{L}(X)$ denote the set of those paths in $\mathcal{L}$ that have their starting vertex in X. For $X \subseteq V(G)$, let $\mathcal{C}(X)$ denote the set of X-crossing edges. It is immediate that.

$$|\mathcal{L}(D \cap D')| + |\mathcal{L}(D \cup D')| = |\mathcal{L}(D)| + |\mathcal{L}(D')| \tag{1}$$

Since D and D' are exact, (1) gives the following:

$$|\mathcal{L}(D \cap D')| + |\mathcal{L}(D \cup D')| = |\mathcal{C}(D)| + |\mathcal{C}(D')| \tag{2}$$

Next, we prove the following.

$$|\mathcal{C}(D \cap D')| + |\mathcal{C}(D \cup D')| \leqslant |\mathcal{C}(D)| + |\mathcal{C}(D')| \tag{3}$$

Each edge in both $\mathcal{C}(D \cap D')$ and $\mathcal{C}(D \cup D')$ points from $D \cap D'$ to $D^{\complement} \cap D'^{\complement}$, and hence is in both $\mathcal{C}(D)$ and $\mathcal{C}(D')$. Each edge in $\mathcal{C}(D \cap D')$ is in either $\mathcal{C}(D)$ or $\mathcal{C}(D')$. Similarly, each edge in $\mathcal{C}(D \cup D')$ is in either $\mathcal{C}(D)$ or $\mathcal{C}(D')$. This proves inequation (3). Note that if we have equality, we cannot have an edge from $D \setminus D'$ to $D' \setminus D$.

In order to prove (i) and (ii), it suffices to show that $|\mathcal{L}(D \cup D')| = |\mathcal{C}(D \cup D')|$ and that $|\mathcal{L}(D \cap D')| = |\mathcal{C}(D \cap D')|$. Since $\mathcal{L}(D \cup D')$ and $\mathcal{L}(D \cap D')$ are sets of disjoint paths, that each contain at least one crossing edge, it must be that $|\mathcal{L}(D \cup D')| \leqslant |\mathcal{C}(D \cup D')|$ and that $|\mathcal{L}(D \cap D')| \leqslant |\mathcal{C}(D \cap D')|$.

By equations (3) and (2), we get that

$$|\mathcal{C}(D \cap D')| + |\mathcal{C}(D \cup D')| \leqslant |\mathcal{L}(D \cap D')| + |\mathcal{L}(D \cup D')| \tag{4}$$

Combining this with the two inequalities before, we must have that $|\mathcal{L}(D \cup D')| = |\mathcal{C}(D \cup D')|$ and $|\mathcal{L}(D \cap D')| = |\mathcal{C}(D \cap D')|$, which proves (i) and (ii).

Now we must have equality in (3). So there cannot be an edge from $D \setminus D'$ to $D' \setminus D$, which proves (iii). This completes the proof. □

Lemma 12. *Let F be a finite set of vertices. Then there is an exact D with $F - b \subseteq D$.*

Proof. For each $v \in F - b$, there is an exact D_v containing v by exactness of G. Then $\bigcup_{v \in F-b} D_v$ is exact, which can easily be proved by induction over $|F - b|$, using (i) of Theorem 11 in the induction step. □

Let D be exact and let $\mathcal{L}$ be a linkage from $D \cap I$ to b. Then D' is called a *forwarder of D with respect to $\mathcal{L}$* if D' is exact and $\bigcup \mathcal{L} - b \subseteq D'$ and $D \subseteq D'$.

Lemma 13. *Each exact D has a forwarder with respect to each linkage $\mathcal{L}$ from some subset of $D \cap I$ to b.*

Proof. Apply Theorem 12 to the set of all vertices in $\bigcup \mathcal{L}$ to get a D' with all those vertices in $D' + b$. The desired forwarder is then $D \cup D'$. □

The *hull* $\hat{D}$ of a vertex-subset D consists of those vertices that are separated by the D-crossing edges from b. Note that $D \subseteq \hat{D}$ and that $\hat{D}^{\complement}$ consists of those vertices v such that there is a v-b-path all of whose internal vertices are outside D. Since every vertex on such a path is in $\hat{D}^{\complement}$, the hull of any hull $\hat{D}$ is $\hat{D}$ itself.

We say that two vertex-subsets D and D' are *equivalent* if they have the same hull. This clearly defines an equivalence relation, which we will call $\sim$. As a set and its hull have the same crossing edges, we observe that two sets D and D' are equivalent if and only if D and D' have the same crossing edges.

Proposition 14. *Let F, F', $\tilde{F}$ and $\tilde{F}'$ be exact with $\tilde{F} \sim F$ and $\tilde{F}' \sim F'$. Then $F \cup F' \sim \tilde{F} \cup \tilde{F}'$.*

Proof. Clearly, the set of $(F \cup F')$-crossing edges is equal to the set of $(\tilde{F} \cup \tilde{F}')$-crossing edges, which gives the desired result. □

Lemma 15. *For any exact set D, we have $I \cap \hat{D} = I \cap D$.*

Proof. Since $\hat{D} \supseteq D$, clearly $I \cap \hat{D} \supseteq I \cap D$. In order to prove the other inclusion, suppose for a contradiction that there is some $v \in I \cap (\hat{D} \setminus D)$. Since $(I \cap D) + v$ is finite, there is some linkage $\mathcal{L}$ from $((I \cap D) + v)$ to b. Let P be the path from that linkage that starts in v. By Theorem 10, the path P avoids D. So P witnesses that $v \notin \hat{D}$. This is a contradiction, thus $I \cap \hat{D} = I \cap D$. $\square$

Corollary 16. *For any exact D, the hull $\hat{D}$ is exact.* $\square$

3.3 Good functions

We define what a good function is and prove that the existence of a good function in every exact graph implies Theorem 6.

First, we fix some notation. The domain of a function or a partial function f is denoted by $\mathrm{dom}(f)$. Let $\mathcal{E}$ be the set of exact vertex-subsets D, and let $\bar{\mathcal{L}}$ be the set of linkages from finite subsets of I to b. For a vertex-subset D, the set $N(D)$ consists of D together with all endvertices of D-crossing edges.

For $v \in I$ and some linkage $\mathcal{L}$ containing a path starting at v, let $Q_v(\mathcal{L})$ denote the path in $\mathcal{L}$ starting from v. For every exact D, the edges of $Q_v(\mathcal{L})$ contained in $G[N(D)]$ are the edges of some initial path of $Q_v(\mathcal{L})$. We call this initial path $P_v(D; \mathcal{L})$. We follow the convention that $P_v(D; \mathcal{L})$ is empty if $v \notin D$.

A function $f : \mathcal{E} \to \bar{\mathcal{L}}$ is *good* if it satisfies the following:

(i) $f(F)$ is a linkage from $I \cap F$ to b.

(ii) If $v \in I$ and $F, F' \in \mathrm{dom}(f)$ with $F' \subseteq F$, then $P_v(F'; f(F')) = P_v(F'; f(F))$.

(iii) If $\bigcup P_v(F; f(F))$ is a ray, then it dominates b. Here the union ranges over all exact F.

Before proving that there is a good function, we first show how to deduce Theorem 4 from that. Let us abbreviate $P_v(D; f(D))$ by $P_v(D; f)$. If it is clear by the context which function f we mean, we even just write $P_v(D)$.

Lemma 17. *Let $f : \mathcal{E} \to \bar{\mathcal{L}}$ be a partial function satisfying (i) and (ii). Further assume that for any two exact F and F' with $F' \subseteq F$ and $F \in \mathrm{dom}(f)$, also $F' \in \mathrm{dom}(f)$. Let $v \in I$, and let $D, D' \in \mathrm{dom}(f)$ be exact with $v \in D \cap D'$. Then $P_v(D) \subseteq P_v(D')$, or $P_v(D') \subseteq P_v(D)$.*

Proof. $D \cap D'$ is exact by Theorem 11 and in the domain of f. Since f satisfies (ii), we get that $P_v(D \cap D')$ is a subpath of both $P_v(D)$ and $P_v(D')$. Let e be the last edge of $P_v(D \cap D')$, and x be its endpoint in $D^{\complement} \cup D'^{\complement}$.

Now we distinguish three cases. If $x \in D^{\complement} \cap D'^{\complement}$, the edge e is crossing for both D and D', and thus is the last edge of both $P_v(D)$ and $P_v(D')$. So $P_v(D) = P_v(D')$, so the lemma is true in this case.

If $x \in D^{\complement} \cap D'$, then e is the last edge of $P_v(D)$. So $P_v(D) = P_v(D \cap D') \subseteq P_v(D')$, so the lemma is true in this case.

The case $x \in D'^{\complement} \cap D$ follows from symmetry. This completes the proof. $\square$

For the remainder of this subsection, let us fix a good function f. The last Lemma motivates the following definition. For $v \in I$, let P_v be the union of all the paths $P_v(D)$ over all exact D containing v. By the last Lemma P_v is either a path or a ray.

Lemma 18. *If P_v and P_w share an edge, then $v = w$.*

Proof. Let e be an edge in both P_v and P_w. Let D_v be exact with $e \in P_v(D_v)$. Similarly, let D_w be exact with $e \in P_w(D_w)$.

By (i) of Theorem 11, we get that $D_v \cup D_w$ is exact. Since f is good, we have that $P_v(D_v \cup D_w)$ includes $P_v(D_v)$, and that $P_w(D_v \cup D_w)$ includes $P_w(D_w)$. Since $P_v(D_v \cup D_w)$ and $P_w(D_v \cup D_w)$ share the edge e, we must have that $v = w$, which completes the proof. □

Lemma 19. *If P_v is a path, then it ends at b.*

Proof. Suppose for a contradiction that P_v does not end at b. Then P_v does not contain b.

Then by Theorem 12, there is an exact D with $P_v \subseteq D$. Then $P_v(D)$ contains some D-crossing edge whose endvertex does not lie on P_v, which gives a contradiction to the construction of P_v. □

The following lemma tells us that to prove Theorem 4, it remains to show that every exact graph has a good function.

Lemma 20. *Let G be an exact digraph that has a good function. Let $b \in V(G)$. Let $I \subseteq V(G) - b$ such that every finite subset of I has a linkage into b. Then there is a domination linkage from I to b.*

Proof. Each P_v dominates b: If P_v is a path, this is shown in Theorem 19. If P_v is a ray, this follows from the fact that f is good. By Theorem 18 all the P_v are disjoint, which completes the proof. □

3.4 Intermezzo: The countable case

The purpose of this subsection is to prove that there is a good function under the assumption that G is countable. This case is easier than the general case and some of the ideas can already be seen in this special case. However, in the general case we do not rely on the countable case. At the end of this subsection, we explain why this proof does not extend to the general case. Nonetheless we think that this helps to get a better understanding of the general case.

Lemma 21. *Let G be an exact graph with $V = \{v_0 = b, v_1, v_2, \ldots\}$ countable. Then there is a sequence of exact hulls D_n and linkages $\mathcal{L}_n$ from $I \cap D_n$ to b satisfying the following.*

1. $D_n \subseteq D_{n+1}$;
2. $\{v_1, \ldots, v_n\} \subseteq D_n$;

3. $P_v(D_n; \mathcal{L}_n) = P_v(D_n; \mathcal{L}_{n+1})$ *for any* $v \in I$*;*

4. D_{n+1} *is a forwarder of* D_n *with respect to* $\mathcal{L}_n$*.*

Proof. Assume that for all $i \leqslant n$, we already constructed exact hulls D_i and linkages $\mathcal{L}_i$ satisfying 1-4. Next, we define D_{n+1}. By Theorem 12, there is an exact F_n containing v_{n+1}. By Theorem 11, $D_n \cup F_n$ is exact. Let D'_{n+1} be a forwarder of $D_n \cup F_n$ with respect to the linkage $\mathcal{L}_n$, which exists by Theorem 13. Let D_{n+1} be the hull of D'_{n+1}, which is exact by Theorem 16.

It remains to construct $\mathcal{L}_{n+1}$ so as to make (3) true. Let $\mathcal{L}$ be some linkage from $I \cap D_{n+1}$ to b. By Theorem 9, for each D_n-crossing edge e there is precisely one $P_e \in \mathcal{L}_n$ that contains e, and precisely one $Q_e \in \mathcal{L}$ that contains e. Let $R_e = P_e e Q_e b$. Since $P_e e \subseteq D_n + e$ and $e Q_e b \subseteq D_n^{\complement} + e$, the R_e are disjoint. For $\mathcal{L}_{n+1}$ we pick the set of the R_e together with all $Q \in \mathcal{L}$ that do not contain any D_n-crossing edge. Clearly $\mathcal{L}_{n+1}$ is a linkage from $I \cap D_{n+1}$ to b. And (3) is true by construction, which completes the proof. □

Lemma 22. *Every countable exact graph* G *has a good function* f*.*

Proof. Let D_n and $\mathcal{L}_n$ as in Theorem 21. We let $f(D_n) = \mathcal{L}_n$. Next, we define f at all other exact D. Since there are only finitely many D-crossing edges, there is a number m such that all these crossing edges are in $N(D_m)$. Then $D \subseteq D_m$ as for each $v \notin D_m$ there is a v-b-path included avoiding D_m. Now we let $f(D)$ consist of those paths in $f(D_m)$ that start in D. We remark that this definition does not depend on the choice of m.

Having defined f, it remains to check that it is good: clearly it satisfies (i) and (ii), so it just remains to verify (iii). So assume that for some $v \in I$, the union $R = \bigcup_{F \in \mathcal{E}} P_v(F; f(F))$ is a ray. Then $R = \bigcup_{n \in \mathbb{N}} P_v(D_n; \mathcal{L}_n)$. Let e_v^n be the unique D_n-crossing edge on $Q_v(\mathcal{L}_n)$. Since D_{n+1} is a forwarder of D_n, the path $R_v^n = e_v^n Q_v(\mathcal{L}_n)$ is contained in $D_n + b$ and avoids D_{n+1}. Thus the paths R_v^n are disjoint and witness that R dominates b. So f is good, which completes the proof. □

Remark 23. *Our proof above heavily relies on the fact that we can find a nested set of exact vertex-subsets* D_n *indexed with the natural numbers that exhaust the graph (compare* (2) *in Theorem 21). However if we can find such a nested set, then* I *must be countable since each* D_n *contains only finitely many vertices of* I*. Thus this proof does not extend to the general case.*

3.5 Infinite sequences of exact vertex-subsets

We encourage the reader to read Subsection 3.6 before this subsection.

The purpose of this subsection is to prove Theorem 25 which is applied in Subsection 3.6. In the later Subsection 3.6 we construct a good function in every exact graph. This good function will be constructed recursively as a limit of 'partial good functions' defined on exact subsets. Theorem 25 guarantees a maximal exact subset in a certain collection of exact subsets. Very very roughly, this maximal element will ensure that we can pick these partial good functions in a compatible way in a certain step in the proof.

Lemma 24. *There does not exist a sequence* $(D_n|n \in \mathbb{N})$ *with* $D_n \subsetneq D_{n+1}$ *of exact hulls that all have bounded order.*

Proof. Suppose for a contradiction that there is a such sequence $(D_n|n \in \mathbb{N})$. By taking a subsequence if necessary, we may assume that all D_n have the same order. Since any two D_n are exact and have the same order, we must have $I \cap D_1 = I \cap D_n$ for every n. Let $\mathcal{L}$ be some linkage from $(D_1 \cap I)$ to b. Any $P \in \mathcal{L}$ contains a unique D_n-crossing edge for every n by Theorem 9. Since $D_n \subseteq D_{n+1}$, there is a large number n_P such that for all $n \geqslant n_P$ it is the same crossing edge. Let m be the maximum of the numbers n_P over all $P \in \mathcal{L}$. Then for all $n \geqslant m$, the D_n have the same crossing edges and thus are equivalent. This is a contradiction, completing the proof. $\square$

Lemma 25. *Let* D *be exact and let* $\mathcal{X}$ *be a nonempty set of exact sets* $D' \subseteq D$ *that is closed under* $\sim$ *and taking unions. Then there is some* $D'' \in \mathcal{X}$ *including all* $D' \in \mathcal{X}$.

Proof. Suppose for a contradiction that there is no such $D'' \in \mathcal{X}$. We will construct an infinite sequence $(D_n|n \in \mathbb{N})$ as in Theorem 24.

Let $D_1 \in \mathcal{X}$ be arbitrary. Since $\mathcal{X}$ is $\sim$-closed, we may assume that D_1 is its hull. Now assume that D_n is already constructed. By assumption, there is $D'_n \in \mathcal{X}$ with $D'_n \not\subseteq D_n$. Let $D''_n = D_n \cup D'_n$. Let D_{n+1} be the hull of D''_n. Then $D_{n+1} \in \mathcal{X}$, and $D_n \subsetneq D_{n+1}$. This completes the construction of the infinite sequence $(D_n|n \in \mathbb{N})$, which contradicts Theorem 24 and hence completes the proof. $\square$

3.6 Existence of good functions

The purpose of this subsection is to prove that every exact graph has a good function, which implies Theorem 4 by Theorem 20. We will define when a partial function is good. In order to construct a good function f defined on the whole of $\mathcal{E}$ we will construct an ordinal indexed family of good partial functions f_α such that if $\alpha > \beta$, then the domain of f_α includes that of f_β and agrees with f_β on the domain of f_β. Eventually some f_α will be defined on the whole of $\mathcal{E}$ and will be the desired good function.

A partial function $f : \mathcal{E} \to \bar{\mathcal{L}}$ is *good* if it satisfies the following:

(i) $f(F)$ is a linkage from $I \cap F$ to b.

(ii) If $w \in I$ and $F, F' \in \mathrm{dom}(f)$ with $F' \subseteq F$, then $P_w(F'; f(F')) = P_w(F'; f(F))$.

(iii) If $\bigcup P_w(F; f(F))$ is a ray, then it dominates b. Here the union ranges over all $F \in \mathrm{dom}(f)$.

(iv) Let F and F' be exact with $F' \subseteq F$. If F is in the domain of f, then so is F'.

(v) If F and F' are in the domain of f, then so is $F \cup F'$.

(vi) $\mathrm{dom}(f)$ is closed under $\sim$.

Note that if $F, F' \in \mathrm{dom}(f)$, then so is $F \cap F'$ by (iv). Note that each good partial function defined on the whole of $\mathcal{E}$ is a good function.

Lemma 26. *Let f be a partial function with domain X that satisfies (i)-(v). Then there is a good partial function $\hat{f}$ whose domain is the $\sim$-closure $\hat{X}$ of X such that $\hat{f}\restriction_X = f$.*

Proof. For each $F \in \hat{X}$, there is some $\tilde{F} \in X$ such that $F \sim \tilde{F}$. We let $\hat{f}(F) = f(\tilde{F})$. By Theorem 15, $\hat{f}$ satisfies (i). Clearly $\hat{f}$ satisfies (iii) and (vi). Since f satisfies (v), $\hat{f}$ satisfies (v) by Theorem 14.

To see that $\hat{f}$ satisfies (ii), let $w \in I$ and $F, F' \in \mathrm{dom}(\hat{f})$ with $F' \subseteq F$. Then $P_w(F'; \hat{f}(F')) = P_w(F'; \hat{f}(F))$ as $P_w(\tilde{F}'; f(\tilde{F}')) = P_w(\tilde{F}'; f(\tilde{F}))$.

To see that $\hat{f}$ satisfies (iv), let F and F' be exact with $F' \subseteq F$ and $F \in \hat{X}$. Then $F' \cap \tilde{F}$ is exact, and since f satisfies (iv), it must be in X. Since $F' \cap \tilde{F}$ and F' have the same crossing edges, they are equivalent. So $F' \in \hat{X}$. So $\hat{f}$ satisfies (iv). This completes the proof. □

For $S \subseteq \mathcal{E}$, let $S(\text{iv}) \subseteq \mathcal{E}$ denote the smallest set including S that satisfies (iv). Similarly, let $S(\text{v}) \subseteq \mathcal{E}$ denote the smallest set including S that satisfies (v).

Lemma 27. *$[S(iv)](v) = [S(v)](iv)$ for any set S.*
In particular, $[S(iv)](v)$ is the smallest set included in $\mathcal{E}$ containing S and satisfying (iv) and (v).

Proof. First let $D \in [S(\text{iv})](\text{v})$. Then there are $F_1, F_2 \in S(\text{iv})$ such that $D = F_1 \cup F_2$. Then there are $F_1', F_2' \in S$ such that $F_1 \subseteq F_1'$ and $F_2 \subseteq F_2'$. Then $F_1' \cup F_2' \in S(\text{v})$ by (i) of Theorem 11. Since $F_1 \cup F_2 \subseteq F_1' \cup F_2'$, we deduce that $D \in [S(\text{v})](\text{iv})$. So $[S(\text{iv})](\text{v}) \subseteq [S(\text{v})](\text{iv})$.

Now let $D \in [S(\text{v})](\text{iv})$. Then there is $D' \in S(\text{v})$ with $D \subseteq D'$. Then there are $F_1, F_2 \in S$ such that $D' = F_1 \cup F_2$. Then $F_i \cap D \in S(\text{iv})$ for $i = 1, 2$ by (ii) of Theorem 11. Since $D \subseteq F_1 \cup F_2$, we deduce that $D \in [S(\text{iv})](\text{v})$. This completes the proof. □

Let $X \subseteq \mathcal{E}$, and D be exact. Then $X[D]$ denotes the smallest set including $X + D$ that satisfies (iv) and (v).

3.6.1 Extending good partial functions

The aim of this subsubsection is to prove the following lemma that helps us building a good function in that it allows us to extend a good partial function a little bit.

Lemma 28. *Let f be a good partial function, and let D be exact. Then there is a good partial function g whose domain consists of the $\sim$-closure of $\mathrm{dom}(f)[D]$, and that agrees with f at each point in $\mathrm{dom}(f)$.*

If $D \in \mathrm{dom}(f)$, then we just take $g = f$. So we may assume that $D \notin \mathrm{dom}(f)$. Before we define g, we define auxiliary functions g_1, g_2 and g_3 with domains X_1, X_2 and X_3, respectively, such that $\mathrm{dom}(f) \subseteq X_1 \subseteq X_2 \subseteq X_3 \subseteq \mathrm{dom}(g)$, and g will be defined such that $g\restriction_{X_1} = g_1$, $g\restriction_{X_2} = g_2$, and $g\restriction_{X_3} = g_3$. We let $X_1 = \mathrm{dom}(f) + D$.

For all $D' \in \mathrm{dom}(f)$, we let $g_1(D') = f(D')$. Next we define $g_1(D)$. Since $I \cap D$ is finite, there is some linkage from $I \cap D$ into b. Let $P_1, P_2, \ldots P_n$ be such a linkage.

By Theorem 25, there is some $D'' \in \mathrm{dom}(f)$ with $D'' \subseteq D$ such that $D' \subseteq D''$ for all $D' \in \mathrm{dom}(f)$ with $D' \subseteq D$. Since D'' is exact, each D''-crossing edge lies on one of the P_i by Theorem 9.

If no D''-crossing edge lies on P_i, then we put P_i into $g_1(D)$. If some D''-crossing edge, say e_i, lies on P_i, we take the path Q_i from the linkage $f(D'')$ that contains e_i, and put the path $Q_i e_i P_i$ into $g_1(D)$. This completes the definition of $g_1(D)$, and so of g_1.

Fact 29. *g_1 satisfies (ii).*

Proof. Let $F, F' \in X_1$ with $F' \subseteq F$. If F is not D, then $P_w(F'; f(F')) = P_w(F'; f(F))$ since f satisfies (ii) and (iv).

So we may assume that $F = D$. Then $F' \subseteq D'' \subseteq D$. So

$$P_w(F'; f(F')) = P_w(F'; f(D'')) = P_w(F'; f(D)),$$

which completes the proof. $\square$

We now define X_2 and g_2. We let $X_2 = X_1(\text{iv})$. For each $F \in X_2$ there is some $F' \in X_1$ such that $F \subseteq F'$. We let $g_2(F)$ to consists of those paths from $g_1(F')$ that start in F. By construction, g_2 satisfies (i) and (iv). By Theorem 11(iii), $P_w(F, g_2(F)) = P_w(F, g_1(F'))$ for all $w \in I$. Thus g_2 satisfies (ii) as g_1 does.

Having defined g_2, we now define g_3. We let X_3 be $X_2(\text{v})$, which is equal to $\mathrm{dom}(f)[D]$. We let $P_w = P_w(F; g_2) \cup P_w(F'; g_2)$. By Theorem 17, it must be that $P_w = P_w(F; g_2)$ or $P_w = P_w(F'; g_2)$. Since g_2 satisfies (ii), no vertex of $P_w(F; g_2)$ that is not on $P_w(F'; g_2)$ can be in $F \cap F'$. By (iii) of Theorem 11, it must be that every vertex of $P_w(F; g_2)$ that is not on $P_w(F'; g_2)$ is in $F \setminus F'$. Hence the P_w are disjoint.

By (iii) of Theorem 11, each P_w contains some $(F \cup F')$-crossing edge e_w. Let $\mathcal{L}$ be some linkage from $I \cap (F \cup F')$ to b. Let Q_w be the path in $\mathcal{L}$ that contains e_w. We define $g_3(F \cup F')$ to consist of the paths $P_w e_w Q_w$. Clearly, $g_3(F \cup F')$ is a linkage from $I \cap (F \cup F')$ to b.

By Theorem 27, g_3 satisfies not only (v) but also (iv).

Fact 30. *g_3 satisfies (ii).*

Proof. Let $w \in I$ and $F, F' \in \mathrm{dom}(f)$ with $F' \subseteq F$. Our aim is to prove that

$$P_w(F'; g_3(F')) = P_w(F'; g_3(F)).$$

In the definition of g_3 at F, we have picked F_1 and F_2 in X_2 such that $F = F_1 \cup F_2$ in order to define $g_3(F)$. Similarly, we have picked F'_1 and F'_2 in X_2 such that with $F' = F'_1 \cup F'_2$ to define $g_3(F')$. It suffices to show that $P_w(X_{ij}; g_3(F')) = P_w(X_{ij}; g_3(F))$ where $X_{ij} = F'_j \cap F_i$ and $(i, j) \in \{1, 2\} \times \{1, 2\}$.

By the definition of g_3, we get the following two equations.

$$P_w(F'; g_3(F')) = P_w(F'_1; g_2) \cup P_w(F'_2; g_2) \tag{5}$$

$$P_w(F'; g_3(F)) = P_w(F' \cap F_1; g_2(F_1)) \cup P_w(F' \cap F_2; g_2(F_2)) \tag{6}$$

Since g_2 satisfies (ii), these two equations give the desired result when restricted to X_{ij}. $\square$

Fact 31. *g_3 satisfies (iii).*

Proof. For each $w \in I$, we compare the sets $\bigcup P_w(F; f(F))$ where first the union ranges over all $F \in \mathrm{dom}(f)$ and second it ranges over all $F \in \mathrm{dom}(g_3)$. The second set is a superset of the first and all its additional elements are in $P_w(D, g_3)$, which is finite. In particular, if the second set is a ray, then so is the first set by Theorem 17. In this case, the first set dominates b since f satisfies (iii), so the second set also dominates b. This completes the proof. $\square$

Having defined g_3, we let $g = \hat{g}_3$ as in Theorem 26. Since g_3 satisfies (i) -(v), g is good by Theorem 26. This completes the proof of Theorem 28.

3.6.2 Construction of a good function

In this subsubsection, we construct a good function in every exact graph, which is the last step in the proof of Theorem 4. Each ordinal α has a unique representation $\alpha = \beta + n$ where β is the largest limit ordinal smaller than α, and n is a natural number. We say that α is *odd* if n is odd. Otherwise it is *even*.

Lemma 32. *Let G be an exact graph. Then there is a good function f defined on the whole of $\mathcal{E}$.*

Proof. In order to construct f we will construct an ordinal indexed family of good partial functions f_α such that if $\alpha > \beta$, then the domain of f_α includes that of f_β and agrees with f_β on the domain of f_β. Eventually some f_α will be defined on the whole of $\mathcal{E}$ and will be the desired good function.

Assume that f_β is already defined for all $\beta < \alpha$. First we consider the case that $\alpha = \beta + 1$ is a successor ordinal. If f_β is defined on the whole of $\mathcal{E}$, we stop. Otherwise, we will find some exact F_α. Then we let f_α be the partial function g given to us from Theorem 28 applied to f_β and F_α.

How we find F_α depends on whether α is an odd or an even successor ordinal. If α is odd, then we pick some $D \in \mathcal{E} \setminus \mathrm{dom}(f_\beta)$, and let $F_\alpha = D$.

If α is even, say $\alpha = \delta + 2n$, where δ is the largest limit ordinal less than α, then for F_α we pick the forwarder of $F_{\delta+n}$ with respect to the linkage $f(F_{\delta+n})$, which exists by Theorem 13.

Having considered the case where α is a successor ordinal, we now consider the case where α is a limit ordinal. For the domain of f_α we take the union of the domains of all f_β with $\beta < \alpha$, and we let $f_\alpha(D) = f_\beta(D)$ for some β where this is defined. It is clear that f_α satisfies (i),(ii),(iv),(v),(vi), so it remains to show that f satisfies (iii). So let $w \in I$ such that $R = \bigcup P_w(D; f_\alpha)$ is a ray. Here the union ranges over all D in the domain of f_α.

Let $\mathcal{O}$ be the set of ordinals $\beta < \alpha$ such that there is some $D \in \mathrm{dom}(f_\beta)$ with $w \in D$. $\mathcal{O}$ is nonempty, so it must contain a smallest ordinal ϵ. Note that ϵ is a successor ordinal. Let ϵ^- be such that $\epsilon = \epsilon^- + 1$.

We will prove that $w \in F_\epsilon$. Suppose not for a contradiction, then $w \notin D$ for all $D \in \mathrm{dom}(f_{\epsilon^-}) + F_\epsilon$. So $w \notin D$ for all $D \in [\mathrm{dom}(f_{\epsilon^-}) + F_\epsilon](\mathrm{iv})$, and hence also $w \notin D$ for all $D \in \mathrm{dom}(f_{\epsilon^-})[F_\epsilon]$ by Theorem 27. By Theorem 16 and since $w \in I$, also $w \notin D$ for all D in the $\sim$-closure of $\mathrm{dom}(f_{\epsilon^-})[F_\epsilon]$. This contradicts the choice of ϵ. Hence $w \in F_\epsilon$. Let x be the unique F_ϵ-crossing edge contained in $P_w(F_\epsilon; f_\alpha)$.

We have a representation $\epsilon = \delta + k$ where δ is the largest limit ordinal less than ϵ. By construction, the $F_{\epsilon(l)}$ with $\epsilon(l) = \delta + 2^l \cdot k$ are nested with each other. To prove that R dominates b, it will suffice just to investigate the $F_{\epsilon(l)}$.

The paths $P_w(l) = Q_w(f_\alpha(F_{\epsilon(l)}))$ are contained in $F_{\epsilon(l+1)} + b$. By Theorem 9, there is a unique $F_{\epsilon(l)}$-crossing edge a_l on $P_w(l)$. The paths $a_l P_w(l)$ meet $F_{\epsilon(l)}$ only in their starting vertex. Thus the paths $a_l P_w(l)$ are disjoint. Since a_l is on $P_w(F_{\epsilon(l)}; f_\alpha)$, it is on R. Hence the paths $a_l P_w(l)$ witness that R dominates b. So f_α is good.

There must be some successor step $\alpha = \beta + 1$ at which we stop. Then f_β is a good function defined on the whole of $\mathcal{E}$. This completes the proof. $\square$

Proof of Theorem 4. Recall that the easy implication is already proved in Theorem 5. For the other implication, combine Theorem 32 with Theorem 20 to get a proof of Theorem 6. Then remember that Theorem 6 implies Theorem 4. $\square$

4 Graph-theoretic applications of Theorem 4

In this section, we show how Theorem 4 implies Theorem 3 and how Theorem 3 implies the Aharoni-Berger theorem for 'well-separated' sets A and B, and the topological Menger theorem for locally finite graphs.

4.1 Variants of Theorem 4

In this subsection, we explain how Theorem 4 implies Theorem 3. Theorem 4 is equivalent to the following.

Theorem 33. *There is a domination linkage from A to B if and only if every finite subset of A can be linked to B.*

Menger's theorem comes in four different versions: the directed edge version, the undirected edge version, the directed vertex version and the undirected vertex version. Depending on the version, we have different notions of path, separator and disjointness. Taking these different notions instead, we know in each of these 4 versions what it means that a ray dominates B, a vertex dominates B, a path dominates B, and what a domination linkage is, and what a linkage is.

The purpose of this subsection is to explain how Theorem 33 implies its undirected-edge-version, directed-vertex-version and undirected-vertex-version. These versions are like Theorem 33 but with the appropriate notions of domination linkage and linkage. The proof is done in the same way how one shows that the directed-edge-version of Menger's theorem for finite graphs implies all the other versions.

Starting with the sketch, one first shows that the directed-edge-version implies the directed-vertex-version for every graph G. For this one considers the auxiliary digraph H of G with $V(H) = V(G) \times \{in, out\}$. The edges of H are of two types: For each $v \in V(G)$, we add an edge pointing from (v, in) to (v, out). For each edge of H pointing from v to w, we add an edge pointing from (v, out) to (w, in). Then the directed-vertex-version for G is equivalent to the directed-edge-version for H.

Next one shows that the directed-vertex-version implies the undirected-vertex-version for every graph G. For this, one considers the directed graph H obtained from G by replacing each edge by two edges in parallel pointing in different directions. Then the undirected-vertex-version for G is equivalent to the directed-vertex-version for H.

Finally, one shows that the undirected-vertex-version implies the undirected-edge-version for every graph G. For this, one considers the line graph H of G. Then the undirected-edge-version for G is equivalent to the undirected-vertex-version for H.

It is clear that the directed vertex version of Theorem 33 is just Theorem 3(i). We call domination linkages in the undirected vertex version *vertex-domination linkages*. Similarly, we define *vertex-linkages*. The undirected vertex version of Theorem 33 is the following.

Corollary 34. *Let G be a graph and $A, B \subseteq V(G)$. There is a vertex-domination linkage from A to B if and only if every finite subset of A has a vertex-linkage into B.* $\square$

Theorem 34 is a reformulation of Theorem 3(ii).

4.2 Well-separatedness

In this subsection, we prove Theorem 35 below, which is used to deduce the Aharoni-Berger theorem for 'well-separated' sets A and B, and the topological Menger theorem for locally finite graphs.

A pair (A, B) of vertex-subsets is *well-separated* if every vertex or end can be separated from one of A or B by removing finitely many vertices.

Corollary 35. *(undirected vertex version) Let (A, B) be a well-separated pair of vertex-subsets. Then there is a vertex-linkage from the whole of A to B if and only if every finite subset of A has a vertex-linkage to B.*

Our next aim is to deduce Theorem 35 from Theorem 34. First we need some lemmas. For this, we fix a graph G and a well-separated pair (A, B) of vertex-subsets. Let $(P_a | a \in A)$ be a vertex-domination linkage from A to B. Let ω be an end that cannot be separated from B by removing finitely many vertices. Let A_ω be the set of those $a \in A$ such that P_a is a ray and belongs to ω.

Lemma 36. *There is a vertex-linkage $(Q_a | a \in A_\omega)$ from A_ω to B such that Q_a and P_x are vertex-disjoint for all $a \in A_\omega$ and all $x \in A \setminus A_\omega$.*

Proof. Given a finite vertex-subset S, we denote by $C(S, \omega)$ the component of the graph $G \setminus S$ that contains the end ω.

As the pair (A, B) is well-separated, there is a finite vertex-subset S that separates the end ω from the set A. Let Z be the set of those vertices a in A such that their path P_a meets the component $C(S, \omega) \cup S$. As each path P_a with $a \in Z$ has to meet the finite set S, the set Z must be finite.

As the set A_ω of vertices with rays to the end ω is a subset of Z, the set A_ω also must be finite. Furthermore there is a finite vertex-subset T such that the component $C(T, \omega)$ meets precisely those paths P_a whose vertex a is in the set A_ω. For each vertex a in the set A_ω, let t_a be the first vertex on the ray P_a such that the subray $t_a P_a$ is contained in the component $C(T, \omega)$, which exists as the ray P_a is eventually contained in the component $C(T, \omega)$. The set of rays $(t_a P_a | a \in A_\omega)$ forms a vertex-domination linkage from $(t_a | a \in A_\omega)$ to the set B in the graph G', where G' is obtained from the graph $G[C(T, \omega)]$ by deleting all edges on the paths $P_a t_a$ with $a \in A_\omega$. By the easy implication of Theorem 34 applied to G', we get a vertex-linkage $(K_a | a \in A_\omega)$ from $(t_a | a \in A_\omega)$ to B. Each walk $P_a t_a K_a$ includes a path Q_a from a to B. From this construction, it is clear that the paths Q_a form a vertex-linkage from A_ω to B and that Q_a and P_x are vertex-disjoint for all $a \in A_\omega$ and all $x \in A \setminus A_\omega$. □

Lemma 37. *There is a vertex-domination linkage $(R_a | a \in A)$ from A to B such that each R_a is a path.*

Proof. We will construct $(R_a | a \in A)$ by transfinite recursion. First we well-order the set Ω of ends: $\Omega = \{\omega_\alpha | \alpha \in \kappa\}$ for $\kappa = |\Omega|$. At each step β we have a current set of vertex-disjoint A-B-paths $\mathcal{Q}_\beta$. The set A_β of start vertices of paths in $\mathcal{Q}_\beta$ consists of those $a \in A$ such that P_a is a ray and belongs to some end ω_α with $\alpha < \beta$. We will also ensure that $\mathcal{R}_\beta = \mathcal{Q}_\beta \cup \{P_a | a \notin A_\beta\}$ is a vertex-domination linkage from A to B.

If β is a limit ordinal, we just set $\mathcal{Q}_\beta = \bigcup_{\alpha<\beta} \mathcal{Q}_\alpha$. It is immediate that $\mathcal{Q}_\beta$ has the desired property assuming that the $\mathcal{Q}_\alpha$ with $\alpha < \beta$ have the property. If $\beta = \alpha + 1$ is a successor ordinal, we apply Theorem 36 to the vertex-domination linkage $\mathcal{R}_\alpha$. Then we let $\mathcal{Q}_{\alpha+1} = \mathcal{Q}_\alpha \cup \{Q_a | a \in A_{\omega_{\alpha+1}}\}$. It is clear from that lemma that $\mathcal{Q}_{\alpha+1}$ has the desired property.

This completes the recursive construction. It is clear that $\mathcal{R}_\kappa = \mathcal{Q}_\kappa \cup \{P_a | a \notin A_\kappa\}$ is the desired vertex-domination linkage. □

Proof that Theorem 34 implies Theorem 35. Let (A, B) be well-separated such that from every finite subset of A there is a vertex-linkage to B. By Theorem 34, there is a vertex-domination linkage $(P_a | a \in A)$ from the whole of A to B. By Theorem 37, we may assume that each P_a is a path. However, $(P_a | a \in A)$ may still contain a path P_u that does not end in B. Then P_u has to contain a vertex ω that cannot be separated from B by removing finitely many vertices. An argument as in the proof of Theorem 36, shows that there is a path Q_u from u to some vertex in B such that $(P_a | a \in A - u)$ together with Q_u is a vertex-domination linkage from A to B. Similar as in the proof of Theorem 37, we can now apply transfinite induction to replace each P_u one by one by such a path Q_u. The final vertex-domination linkage is then a vertex-linkage, which completes the proof. □

4.3 Existing Menger-type theorems

In this subsection, we show how Theorem 35 implies the Aharoni-Berger theorem for 'well-separated' sets A and B, and the topological Menger theorem for locally finite graphs.

The Aharoni-Berger theorem [3] says that for every graph G with vertex-subsets A and B, there is a set of vertex-disjoint A-B-paths together with an A-B-separator consisting of precisely one vertex from each of these paths.

At first glance, it might seem that the Aharoni-Berger theorem does not tell under which conditions there is a linkage from A to B - but actually it does. To explain this, we need a definition. A *wave* is a set of vertex-disjoint paths from a subset of A to some A-B-separator C. It is not difficult to show that the Aharoni-Berger theorem is equivalent to the following: The whole of A can be linked to B if and only if for every wave there is a linkage from A to its separator set C. Thus Theorem 35 implies the Aharoni-Berger theorem for well-separated sets A and B. We remark that neither Theorem 35 nor Theorem 4 follows from the Aharoni-Berger theorem.

Using this implication, we get the first proof of the topological Menger-Theorem of Diestel [8] for locally finite graphs that does not rely on the Aharoni-Berger theorem. Indeed, the argument of Diestel only relies on the Aharoni-Berger theorem for vertex-subsets A and B that have disjoint closure in $|G|$, which is equivalent to being well-separated if G is locally finite.

5 Infinite gammoids

In this section, we use Theorem 3 to prove Theorem 1 and Theorem 2. Throughout, notation and terminology for matroids are that of [10, 6]. M always denotes a matroid and $E(M)$ and $\mathcal{I}(M)$ denote its ground set and its sets of independent sets, respectively.

Recall that the set $\mathcal{I}(M)$ is required to satisfy the following *independence axioms* [6]:

(I1) $\emptyset \in \mathcal{I}(M)$.

(I2) $\mathcal{I}(M)$ is closed under taking subsets.

(I3) Whenever $I, I' \in \mathcal{I}(M)$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}(M)$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}(M)$, the set $\{I' \in \mathcal{I}(M) \mid I \subseteq I' \subseteq X\}$ has a maximal element.

An *$\mathcal{I}$-circuit* is a set minimal with the property that it is not in $\mathcal{I}$. The following is true in any matroid.

(+) For any two finite $\mathcal{I}$-circuits o_1 and o_2 and any $x \in o_1 \cap o_2$, there is some $\mathcal{I}$-circuit included in $(o_1 \cup o_2) - x$.

Given $\mathcal{I} \subseteq \mathcal{P}(E)$, its *finitarization* $\mathcal{I}^{fin}$ consists of those sets J all of whose finite subsets are in $\mathcal{I}$. Usually, it is made a requirement that $\mathcal{I}$ is the set of independent sets of a matroid [4]. Then $\mathcal{I}^{fin}$ is the set of independent sets of a finitary matroid, called M^{fin} [4]. We will need the following slight strengthening of this fact.

Lemma 38. *If $\mathcal{I}$ satisfies (I1), (I2) and (+), then $\mathcal{I}^{fin}$ is the set of independent sets of a finitary matroid.*

Proof. Clearly $\mathcal{I}^{fin}$ satisfies (I1) and (I2), and it satisfies (IM) by Zorn's Lemma. Thus it remains to check (I3). So let $I, I' \in \mathcal{I}^{fin}$ with I' maximal and I not maximal. So there is some $y \notin I$ with $I + y \in \mathcal{I}$. We may assume that $y \notin I'$ since otherwise we are done. Thus there is some finite $\mathcal{I}$-circuit o with $y \in o \subseteq I' + y$. Suppose for a contradiction that for each $x \in o \setminus (I + y)$, there is some finite $\mathcal{I}$-circuit o_x with $x \in o_x \subseteq I + x$. Applying (+) successively to o and the o_x, we obtain a finite $\mathcal{I}$-circuit o' included in $I + y$, which contradicts the assumption that $I + y \in \mathcal{I}^{fin}$. Thus there is some $x \in o \setminus (I + y)$ such that $I + x \in \mathcal{I}^{fin}$, which completes the proof. $\square$

We will also need the following slight variation of $(I3)$.

(*) For all $I, J \in \mathcal{I}$ and all $y \in I \setminus J$ with $J + y \notin \mathcal{I}$ there exists $x \in J \setminus I$ such that $(J + y) - x \in \mathcal{I}$.

A matroid N is *nearly finitary* if for every base B of N there is a base B' of N^{fin} such that $B \subseteq B'$ and $|B' \setminus B|$ is finite. It is not difficult to show that N is nearly finitary if and only if for every base B' of N^{fin} there is a base B of N such that $B \subseteq B'$ and $|B' \setminus B|$ is finite. The proof of Lemma 4.15 in [4] actually proves the following stronger statement.

Lemma 39. *Let $M = (E, \mathcal{J})$ be a matroid with ground set E. Let $\mathcal{I} \subseteq \mathcal{J}$ satisfying $(I1)$, $(I2)$ $(I3)$, $(*)$ such that for any $J \in \mathcal{J}$ there is some $I \in \mathcal{I}$ such that $|J \setminus I|$ is finite. Then $N = (\mathcal{I}, E)$ is a matroid.*

In the special case where M is finitary, N is nearly finitary.

Next, we will summarise the results from [1] that are relevant to this paper.

Lemma 40 (Afzali, Law, Müller [1, Lemma 2.2])**.** *For any digraph G and $B \subseteq V(G)$, the system $\mathcal{L}(G, B)$ satisfies (I3).*

Lemma 41 (Afzali, Law, Müller [1, Lemma 2.7])**.** *For any digraph G and $B \subseteq V(G)$, the system $\mathcal{L}(G, B)$ satisfies (*).*

Let $B_{AC} = \{b_0, b_1, \ldots\}$. Let $V_{AC} = B_{AC} \cup V^1 \cup V^2$, where $V^i = \{v^i_0, v^i_1, \ldots\}$. The digraph G_{AC} has vertex-subset V_{AC} and three types of edges: For $j \in \mathbb{N}$ it has an edge from v^1_j to b_j. For each $j \in \mathbb{N}$, it has two edges, both start at v^2_j, and end at v^1_j and v^1_{j+1}. The pair (G_{AC}, B_{AC}) is called an *alternating comb (AC)*. A subdivision of AC is drawn in Figure 2. Formally, a *subdivision of AC* is a pair (H_{AC}, B_{AC}) where H_{AC} is obtained from G_{AC} by replacing each directed edge xy by a directed path from x to y that is internally disjoint from all other such paths. Here edges from V_2 to V_1 are not allowed to be replaced by a trivial path[5] but the edges $v^1_j b_j$ are allowed to be replaced by a trivial path. A pair (G, B) *has a subdivision of AC* if there is a subgraph H_{AC} of G and $B_{AC} \subseteq B \cap V(H_{AC})$ such that (H_{AC}, B_{AC}) is isomorphic to a subdivision of AC.

[5] A *trivial path* consists of a single vertex only.

Theorem 42 (Afzali, Law, Müller [1, Theorem 2.6])**.** *Let G be a digraph and $B \subseteq V(G)$ such that (G, B) has no a subdivision of AC. Then $\mathcal{L}(G, B)$ is a matroid.*

For the remainder of this section, let G denote a digraph and $B \subseteq V(G)$. In the following, we will explore for which digraphs G and sets B the system $\mathcal{L}_T(G, B)$ is the set of independent sets of a matroid, and how $\mathcal{L}_T(G, B)$ relates to $\mathcal{L}(G, B)$. If G is finite $\mathcal{L}(G, B)$ is a matroid and thus satisfies (+). The latter easily extends to infinite graphs G.

Lemma 43. *$\mathcal{L}(G, B)$ satisfies (+).*

Sketch of the proof. Given two finite $\mathcal{L}(G, B)$-circuits o_1 and o_2 intersecting in some vertex x, there are separations (A_i, B_i) with $o_i \subseteq A_i$ and $B \subseteq B_i$ of order at most $|o_i| - 1$. Then with a lemma like Theorem 11, one shows that either $(A_1 \cup A_2, B_1 \cap B_2)$ or $(A_1 \cap A_2, B_1 \cup B_2)$ separates some $\mathcal{L}(G, B)$-circuit $o \subseteq (o_1 \cup o_2) - x$ from B. $\square$

Using Theorem 3, we can prove the following slight extension of Theorem 1.

Corollary 44. *$\mathcal{L}_T(G, B) = \mathcal{L}(G, B)^{fin}$ for any digraph G and $B \subseteq V(G)$.*
Moreover, $\mathcal{L}_T(G, B)$ is a finitary matroid.

Proof. By Theorem 3, $\mathcal{L}_T(G, B)$ consists of those sets I all of whose finite subsets can be linked to B by vertex-disjoint directed paths, and thus $\mathcal{L}_T(G, B) = \mathcal{L}(G, B)^{fin}$. As $\mathcal{L}(G, B)$ satisfies (I1), (I2) and (+), $\mathcal{L}_T(G, B)$ is a finitary matroid by Theorem 38. $\square$

Next we prove the following slight strengthening of Theorem 2 from the Introduction. Below we will refer to the definition of dominating as defined in the Introduction.

Corollary 45. *Let G be a digraph with a set B of vertices such that there are neither infinitely many vertex-disjoint rays dominating B nor infinitely many vertices dominating B. Then $\mathcal{L}(G, B)$ is a nearly finitary matroid.*

Proof. $\mathcal{L}(G, B)$ clearly satisfies (I1) and (I2), and it satisfies (I3) and (*) by Theorem 40 and Theorem 41. Let $J \in \mathcal{L}_T(G, B)$. By Theorem 3, we get for each $v \in J$ a ray or path P_v starting at v such that all these P_v are vertex-disjoint. Moreover each such P_v either ends in B or is a ray dominating B or its last vertex dominates B. Let I be the set of those v such that P_v ends in B. By assumption $J \setminus I$ is finite. So by Theorem 44, we can apply Theorem 39 with $\mathcal{J} = \mathcal{L}_T(G, B)$ to deduce that $\mathcal{L}(G, B)$ satisfies (IM), and thus is a nearly-finitary matroid. $\square$

Towards the converse implication of Theorem 45 we observe the following.

Observation 46. *Let G be a digraph with a set B of vertices such that no vertex in B dominates B and such that $\mathcal{L}(G, B)$ is a nearly finitary matroid.*

Then there are neither infinitely many vertex-disjoint rays dominating B disjoint from B nor infinitely many vertices dominating B.

Proof. As $\mathcal{L}(G,B)$ is a nearly finitary, the set B together with just finitely many vertices is a base. Hence there can only be finitely many vertices outside B dominating B and only finitely many rays dominating B disjoint from B. $\square$

A natural question that comes up is to ask how Theorem 42 and Theorem 45 relate to each other. In [1], Afzali, Law and Müller construct a pair (G,B) without AC such that $\mathcal{L}(G,B)$ is not nearly finitary. They also do it in a way to make $\mathcal{L}(G,B)$ 3-connected. Thus Theorem 45 does not imply Theorem 42.

To see that Theorem 42 does not imply Theorem 45, let G be the 3 by $\mathbb{Z}$ grid, formally: $V(G) = \{1,2,3\} \times \mathbb{Z}$, see Figure 2. In G, there is a directed edge from (x,y) to (x',y')

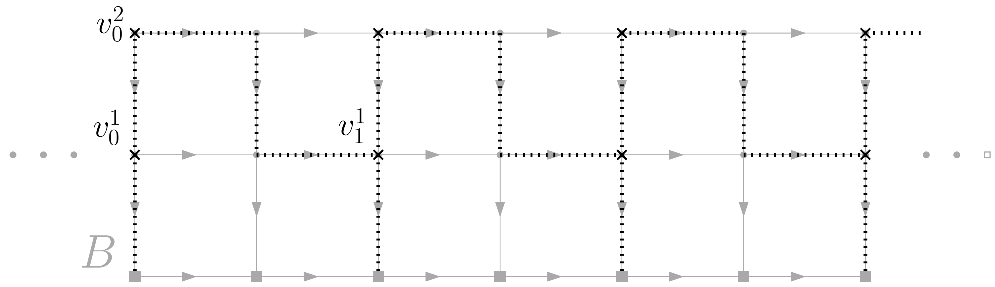

Figure 2: The graph G is depicted in gray. The vertices of B are squares. (G,B) has a subdivision of AC. One is indicated in this figure: The vertices of V^1 and V^2 are black crosses and the subdivided edges are drawn dotted.

if and only if either $x = x'$ and $y' = y+1$ or $y = y'$ and $x' = x+1$. Let: $B = \{3\} \times \mathbb{Z}$. Then it is easy to see that no vertex of G dominates B and there are not infinitely many vertex-disjoint rays dominating B. However (G,B) has a subdivision of AC, which is indicated in Figure 2. Thus, there arises the question if there is a nontrivial common generalization of Theorem 45 and Theorem 42.

During this whole section, we have only considered the directed-vertex-version. Of course, similar results are true if we consider the undirected-vertex-version, the directed-edge-version or the undirected-edge-version instead.

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