Vertex degree sums for perfect matchings in 3-uniform hypergraphs

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Abstract

We determine the minimum degree sum of two adjacent vertices that ensures a perfect matching in a 3-uniform hypergraph without an isolated vertex. Suppose that H is a 3-uniform hypergraph whose order n is sufficiently large and divisible by 3. If H contains no isolated vertex and $\deg(u)+\deg(v) > \frac{2}{3}n^2 - \frac{8}{3}n + 2$ for any two vertices u and v that are contained in some edge of H, then H contains a perfect matching. This bound is tight and the (unique) extremal hyergraph is a different *space barrier* from the one for the corresponding Dirac problem.

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1 Introduction

A k-uniform hypergraph (in short, k-graph) H is a pair (V, E), where $V = V(H)$ is a finite set of vertices and $E = E(H)$ is a family of k-element subsets of V. A *matching of size* s in H is a family of s pairwise disjoint edges of H. If the matching covers all the vertices of H, then we call it a *perfect matching*. Given a set $S \subseteq V(H)$, the *degree*

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$\deg_H(S)$ of S is the number of the edges of H containing S. We omit the subscript when the underlying hypergraph is obvious from the context, and simply write $\deg(v)$ when $S = \{v\}$. The *minimum ℓ-degree* of H, denoted by $\delta_\ell(H)$, is the minimum $\deg(S)$ over all ℓ-subsets S of $V(H)$.

Given integers $\ell < k \leqslant n$ such that k divides n, we define the minimum *ℓ-degree threshold* $m_\ell(k,n)$ as the smallest integer m such that every k-graph H on n vertices with $\delta_\ell(H) \geqslant m$ contains a perfect matching. In recent years the problem of determining $m_\ell(k,n)$ has received much attention, see, *e.g.*, [2, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21]. For example, Rödl, Ruciński, and Szemerédi [17] determined $m_{k-1}(k,n)$ for all $k \geqslant 3$ and sufficiently large n. For more Dirac-type results on hypergraphs, we refer readers to surveys [14, 25].

In this paper we focus on 3-graphs. Hàn, Person and Schacht [4] showed that

$$m_1(3,n) = \left(\frac{5}{9} + o(1)\right)\binom{n}{2}. \tag{1}$$

Kühn, Osthus and Treglown [10] and independently Khan [6] later proved that $m_1(3,n) = \binom{n-1}{2} - \binom{2n/3}{2} + 1$ for sufficiently large n.

Motivated by the relation between Dirac's condition and Ore's condition for Hamilton cycles, Tang and Yan [18] studied the degree sum of two $(k-1)$-sets that guarantees a tight Hamilton cycle in k-graphs. Zhang and Lu [22] studied the degree sum of two $(k-1)$-sets that guarantees a perfect matching in k-graphs.

Our objective is to find an Ore's condition that guarantees a perfect matching in 3-graphs. As Ore's theorem concerns the degree sum of two non-adjacent vertices in graphs, we consider the degree sum of two vertices in 3-graphs. In a hypergraph, two distinct vertices are *adjacent* if there exists an edge containing both of them. The following are three possible ways of defining the minimum degree sum of a 3-graph H. Let $\sigma_2(H) = \min\{\deg(u) + \deg(v) : u, v \in V(H) \text{ are adjacent}\}$, $\sigma_2'(H) = \min\{\deg(u) + \deg(v) : u, v \in V(H)\}$, and $\sigma_2''(H) = \min\{\deg(u) + \deg(v) : u, v \in V(H) \text{ are not adjacent}\}$.

The parameter σ_2' is closely related to the Dirac threshold $m_1(3,n)$ – we can prove that *when n is divisible by 3 and sufficiently large, every 3-graph H on n vertices with $\sigma_2'(H) \geqslant 2(\binom{n-1}{2} - \binom{2n/3}{2}) + 1$ contains a perfect matching.* Indeed, such H contains at most one vertex u with $\deg(u) \leqslant \binom{n-1}{2} - \binom{2n/3}{2}$. If $\deg(u) \leqslant (5/9 - \varepsilon)\binom{n}{2}$ for some $\varepsilon > 0$, then we choose an edge containing u and find a perfect matching in the remaining 3-graph by (1) immediately. Otherwise, $\delta_1(H) \geqslant (5/9 - \varepsilon)\binom{n}{2}$. We can prove that H contains a perfect matching by following the same approach as in [10].[1]

On the other hand, no condition on σ_2'' alone guarantees a perfect matching. In fact, let H be the 3-graph whose edge set consists of all triples that contain a fixed vertex. This H contains no two disjoint edges even though it satisfies all conditions on σ_2'' (because any two vertices of H are adjacent).

Therefore we focus on σ_2. More precisely, we determine the largest $\sigma_2(H)$ among all 3-graphs H of order n without isolated vertex such that H contains no perfect matching.

[1] In fact, due to the absorbing method, we only need to verify the extremal case.

(Trivially H contains no perfect matching if it contains an isolated vertex.) Let us define a 3-graph H_n^*, which is one of the so-called *space barriers* for perfect matchings (see Section 5 for their definitions and a connection to a well-known conjecture of Erdős [3]). The vertex set of H_n^* is partitioned into two vertex classes S and T of size $n/3+1$ and $2n/3-1$,

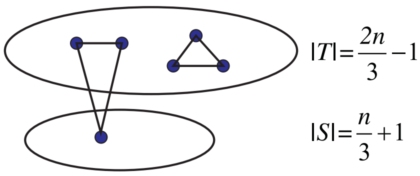

Figure 1: H_n^*: every edge intersects T in two or three vertices.

respectively, and whose edge set consists of all triples containing at least two vertices of T (see Figure 1). For any two vertices $u \in T$ and $v \in S$,

$$\deg(u) = \binom{2n/3-2}{2} + \left(\frac{n}{3}+1\right)\left(\frac{2n}{3}-2\right) > \binom{2n/3-1}{2} = \deg(v).$$

Hence $\sigma_2(H_n^*) = \binom{2n/3-2}{2} + (n/3+1)(2n/3-2) + \binom{2n/3-1}{2} = 2n^2/3 - 8n/3 + 2$. Obviously, H_n^* contains no perfect matching. The following is our main result.

Theorem 1. *There exists $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geqslant n_0$ that are divisible by 3. Let H be a 3-graph of order $n \geqslant n_0$ without isolated vertex. If $\sigma_2(H) > \sigma_2(H_n^*) = \frac{2}{3}n^2 - \frac{8}{3}n + 2$, then H contains a perfect matching.*

Theorem 1 actually follows from the following stability result. For two hypergraphs H_1 and H_2, we write $H_1 \subseteq H_2$ if H_1 is a subgraph of H_2.

Theorem 2. *There exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds for all integers $n \geqslant n_0$ that are divisible by 3. Suppose that H is a 3-graph of order $n \geqslant n_0$ without isolated vertex and $\sigma_2(H) > 2n^2/3 - \varepsilon n^2$, then $H \subseteq H_n^*$ or H contains a perfect matching.*

Indeed, if $\sigma_2(H) > 2n^2/3 - 8n/3 + 2$, then $H \nsubseteq H_n^*$ and by Theorem 2, H contains a perfect matching. Furthermore, Theorem 2 implies that H_n^* is the unique extremal 3-graph for Theorem 1 because all proper subgraphs H of H_n^* satisfy $\sigma_2(H) < \sigma_2(H_n^*)$.

This paper is organized as follows. In Section 2, we provide preliminary results and an outline of our proof. We prove an important lemma in Section 3 and we complete the proof of Theorem 2 in Section 4. Section 5 contains concluding remarks and open problems.

Notation: Given vertices $v_1, \dots, v_t$, we often write $v_1 \cdots v_t$ for $\{v_1, \dots, v_t\}$. The neighborhood $N(u,v)$ is the set of the vertices w such that $uvw \in E(H)$. Let V_1, V_2, V_3 be three vertex subsets of $V(H)$, we say that an edge $e \in E(H)$ is of type $V_1V_2V_3$ if $e = \{v_1, v_2, v_3\}$ such that $v_1 \in V_1$, $v_2 \in V_2$ and $v_3 \in V_3$.

Given a vertex $v \in V(H)$ and a set $A \subseteq V(H)$, we define the *link* $L_v(A)$ to be the set of all pairs uw such that $u, w \in A$ and $uvw \in E(H)$. When A and B are two disjoint sets of $V(H)$, we define $L_v(A, B)$ as the set of all pairs uw such that $u \in A$, $w \in B$ and $uvw \in E(H)$.

We write $0 < a_1 \ll a_2 \ll a_3$ if we can choose the constants a_1, a_2, a_3 from right to left. More precisely there are increasing functions f and g such that given a_3, whenever we choose some $a_2 \leqslant f(a_3)$ and $a_1 \leqslant g(a_2)$, all calculations needed in our proof are valid.

2 Preliminaries and proof outline

We will need small constants

$$0 < \varepsilon \ll \eta \ll \gamma \ll \gamma' \ll \rho \ll \tau \ll 1.$$

Suppose H is a 3-graph such that $\sigma_2(H) > 2n^2/3 - \varepsilon n^2$. Let $W = \{v \in V(H) : \deg(v) \leqslant n^2/3 - \varepsilon n^2/2\}$, $U = V \setminus W$. If $W = \emptyset$, then (1) implies that H contains a perfect matching. We thus assume that $|W| \geqslant 1$. Any two vertices of W are not adjacent – otherwise $\sigma_2(H) \leqslant 2n^2/3 - \varepsilon n^2$, a contradiction. If $|W| \geqslant n/3 + 1$, then $H \subseteq H_n^*$ and we are done. We thus assume $|W| \leqslant n/3$ for the rest of the proof.

Our proof will use the following claim.

Claim 3. *If $|W| \geqslant n/4$, then every vertex of U is adjacent to some vertex of W.*

Proof. To the contrary, assume that some vertex $u_0 \in U$ is not adjacent to any vertex in W. Then we have $\deg(u_0) \leqslant \binom{|U|-1}{2} = \binom{n-|W|-1}{2}$. Since $|W| \geqslant n/4$ and n is sufficiently large,

$$\deg(u_0) \leqslant \binom{n - n/4 - 1}{2} = \frac{9}{32}n^2 - \frac{9}{8}n + 1 < \frac{n^2}{3} - \frac{\varepsilon}{2}n^2,$$

which contradicts the definition of U. □

By Claim 3, when $|W| \geqslant \frac{n}{4}$, we have $\deg(u) \geqslant (2n^2/3 - \varepsilon n^2) - \binom{n-|W|}{2}$ for every $u \in U$. This is stronger than the bound given by the definition of U because

$$\left(\frac{2}{3}n^2 - \varepsilon n^2\right) - \binom{n - |W|}{2} \geqslant \left(\frac{2}{3}n^2 - \varepsilon n^2\right) - \binom{n - \frac{n}{4}}{2} = \left(\frac{37}{96} - \varepsilon\right)n^2 + \frac{3}{8}n > \frac{n^2}{3} - \frac{\varepsilon}{2}n^2.$$

Our proof consists of two steps.

Step 1. We prove that H contains a matching that covers all the vertices of W.

Lemma 4. *There exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph of order $n \geqslant n_0$ without isolated vertex and $\sigma_2(H) > 2n^2/3 - \varepsilon n^2$. Let $W = \{v \in V(H) : \deg(v) \leqslant n^2/3 - \varepsilon n^2/2\}$. If $|W| \leqslant n/3$, then H contains a matching that covers every vertex of W.*

We will prove Lemma 4 in Section 3. The following is an outline of the proof. Consider a largest matching M in H such that every edge of M contains one vertex from W and assume $|M| < |W|$. If $|W| \leqslant (1/3-\gamma)n$, then we choose two adjacent vertices, one from W and the other from $V\setminus W$ to derive a contradiction with $\sigma_2(H)$. If $|W| > (1/3-\gamma)n$, we use three unmatched vertices, one from W and two from $V \setminus W$ to derive a contradiction.

Step 2. We show that H contains a perfect matching.

Because of Lemma 4, we begin by considering a largest matching M such that M covers every vertex of W and suppose that $|M| < n/3$. We distinguish the cases when $|M| \leqslant n/3 - \eta n$ and when $|M| > n/3 - \eta n$. In both cases we derive a contradiction by comparing upper and lower bounds for the degree sum of three fixed vertices from $V \setminus V(M)$. When $|M| > n/3 - \eta n$, we need the Dirac threshold (1).

In Step 2 we will apply three simple extremal results. The first lemma is Observation 1.8 of Aharoni and Howard [1]. A k-graph H is *k-partite* if $V(H)$ can be partitioned into $V_1, \cdots, V_k$, such that each edge of H meets every V_i in precisely one vertex. If all parts are of the same size n, we say H is *n-balanced*.

Lemma 5. *[1] Let F be the edge set of an n-balanced k-partite k-graph. If F does not contain s disjoint edges, then $|F| \leqslant (s-1)n^{k-1}$.*

The bound in the following lemma is tight because we may let G_1 be the empty graph and $G_2 = G_3 = K_n$.

Lemma 6. *Let G_1, G_2, G_3 be three graphs on the same set V of $n \geqslant 4$ vertices such that every edge of G_1 intersects every edge of G_i for both $i = 2, 3$. Then $\sum_{i=1}^{3}\sum_{v\in A}\deg_{G_i}(v) \leqslant 6(n-1)$ for any set $A \subset V$ of size 3.*

Proof. Assume $A = \{u_1, u_2, u_3\}$ and $b := n-3 \geqslant 1$. Our goal is to show that

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\deg_{G_i}(u_j) \leqslant 6b+12.$$

Let ℓ_i denote the number of the vertices in A of degree at least 3 in G_i. We distinguish the following two cases:

Case 1: $\ell_1 \geqslant 1$.

If $\ell_1 \geqslant 2$, say, $\deg_{G_1}(u_j) \geqslant 3$ for $j = 1, 2$, then $E(G_i) \subseteq \{u_1u_2\}$ for $i = 2, 3$ – otherwise we can find two disjoint edges, one from G_1 and the other from G_2 or G_3. Therefore, $\sum_{j=1}^{3}\deg_{G_i}(u_j) \leqslant 2$ for $i = 2, 3$. Moreover, $\sum_{j=1}^{3}\deg_{G_1}(u_j) \leqslant 3b+6$. We have $\sum_{i=1}^{3}\sum_{j=1}^{3}\deg_{G_i}(u_j) \leqslant 3b+10 < 6b+12$.

If $\ell_1 = 1$, say, $\deg_{G_1}(u_1) \geqslant 3$, then G_i is a star centered at u_1 for $i = 2, 3$ – otherwise one edge of G_1 must be disjoint from one edge of G_2 or G_3. In this case we have $\sum_{j=1}^{3}\deg_{G_1}(u_j) \leqslant b+2+4$ and $\sum_{j=1}^{3}\deg_{G_i}(u_j) \leqslant b+4$ for $i = 2, 3$. Therefore, $\sum_{i=1}^{3}\sum_{j=1}^{3}\deg_{G_i}(u_j) \leqslant 3b+14 < 6b+12$ as $b \geqslant 1$.

Case 2: $\ell_1 = 0$.

Let us consider the value of $\max\{\ell_2, \ell_3\}$. First, if $\max\{\ell_2, \ell_3\} = 3$, then $E(G_1) = \emptyset$. Consequently, $\sum_{i=1}^{3}\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 2(3b + 6) = 6b + 12$.

Second, assume $\max\{\ell_2, \ell_3\} = 2$. Without loss of generality, we assume $\ell_2 = 2$ and $\deg_{G_2}(u_j) \geqslant 3$ for $j = 1, 2$. Then $E(G_1) \subseteq \{u_1u_2\}$. In this case $\sum_{j=1}^{3} \deg_{G_1}(u_j) \leqslant 2$ and $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 2b+4+2$ for $i = 2, 3$. Hence $\sum_{i=1}^{3}\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 4b+14 \leqslant 6b+12$ as $b \geqslant 1$.

Third, assume $\max\{\ell_2, \ell_3\} = 1$. Without loss of generality, assume $\ell_2 = 1$ and $\deg_{G_2}(u_1) \geqslant 3$. Then G_1 is a star centered at u_1. We have $\sum_{j=1}^{3} \deg_{G_1}(u_j) \leqslant 4$ and $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant b + 2 + 4$ for $i = 2, 3$. So $\sum_{i=1}^{3}\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 2b + 16 \leqslant 6b + 12$ as $b \geqslant 1$.

At last, assume $\max\{\ell_2, \ell_3\} = 0$. Then $\deg_{G_i}(u_j) \leqslant 2$ for all $i, j \in \{1, 2, 3\}$. Hence $\sum_{i=1}^{3}\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 18 \leqslant 6b + 12$ as $b \geqslant 1$. $\square$

The bound in the following lemma is tight because we may let $G_1 = G_2 = G_3$ be a star of order n centered at a vertex of A.

Lemma 7. *Let G_1, G_2, G_3 be three graphs on the same set V of $n \geqslant 5$ vertices such that for any $i \neq j$, every edge of G_i intersects every edge from G_j. Then $\sum_{i=1}^{3}\sum_{v \in A} \deg_{G_i}(v) \leqslant 3(n + 1)$ for any set $A \subset V$ of size 3.*

Proof. Assume $A = \{u_1, u_2, u_3\}$ and $b := n - 3 \geqslant 2$. Our goal is to show that

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 3b + 12.$$

Let ℓ_i denote the number of the vertices in A of degree at least 3 in G_i. We distinguish the following two cases:

Case 1: $\ell_i \geqslant 1$ for some $i \in [3]$.

Without loss of generality, $\ell_1 \geqslant 1$ and $\deg_{G_1}(u_1) \geqslant 3$. If $\deg_{G_1}(u_2) \geqslant 3$ or $\deg_{G_1}(u_3) \geqslant 3$, say, $\deg_{G_1}(u_2) \geqslant 3$, then $E(G_i) \subseteq \{u_1u_2\}$ for $i = 2, 3$ – otherwise we can find two disjoint edges e_1 and e_2 from two distinct graphs of G_1, G_2, G_3. In this case $\sum_{j=1}^{3} \deg_{G_1}(u_j) \leqslant 3b+6$ and $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 2$ for $i = 2, 3$, which implies that $\sum_{i=1}^{3}\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 3b + 10$.

Assume $\deg_{G_1}(u_j) \leqslant 2$ for $j = 2, 3$. We know that G_i, $i = 2, 3$ is a star centered at u_1 – otherwise one edge of G_1 must be disjoint from one edge of G_i, $i \in \{2, 3\}$. If $\deg_{G_2}(u_1) \geqslant 3$ or $\deg_{G_3}(u_1) \geqslant 3$, then G_1 is also a star centered at u_1. In this case $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant b + 4$ for $i \in [3]$, so $\sum_{i=1}^{3}\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 3b + 12$. Otherwise $\deg_{G_i}(u_1) \leqslant 2$ for $i = 2, 3$, hence $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 4$ for $i = 2, 3$. Since $\sum_{j=1}^{3} \deg_{G_1}(u_j) \leqslant b + 6$, we have $\sum_{i=1}^{3}\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant b + 14 \leqslant 3b + 12$.

Case 2: $\ell_i = 0$ for $i \in [3]$.

In this case $\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 6$ for $i = 1, 2, 3$. Hence $\sum_{i=1}^{3}\sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 18 \leqslant 3b + 12$ as $b \geqslant 2$. $\square$

3 Proof of Lemma 4

Choose a largest matching of H, denoted by M, such that every edge of M is of type UUW. To the contrary, assume that $|M| \leqslant |W|-1$. Let $U_1 = V(M) \cap U$, $U_2 = U \setminus U_1$, $W_1 = V(M) \cap W$, and $W_2 = W \setminus W_1$. Then $|U_1| = 2|M|$, and $|U_2| = n - |W| - 2|M|$. We distinguish the following two cases.

Case 1: $0 < |W| \leqslant (\frac{1}{3} - \gamma)n$.

We further distinguish the following two sub-cases:

Case 1.1: A vertex $v_0 \in W_2$ is adjacent to a vertex $u_0 \in U_2$.

Let $M' = \{e \in M : \exists\, u' \in e, |N(v_0, u') \cap U_2| \geqslant 3\}$. Assume $\{u_1, u_2, v_1\} \in M'$ such that $u_1, u_2 \in U_1$, $v_1 \in W_1$, and $|N(v_0, u_1) \cap U_2| \geqslant 3$. We claim that

$$N(u_0, v_1) \cap (U_2 \cup \{u_2\}) = \emptyset. \tag{2}$$

Indeed, if $\{u_0, v_1, u_3\} \in E(H)$ for some $u_3 \in U_2$, then we can find $u_4 \in U_2 \setminus \{u_0, u_3\}$ such that $\{v_0, u_1, u_4\} \in E(H)$. Replacing $\{u_1, u_2, v_1\}$ by $\{u_0, v_1, u_3\}$ and $\{v_0, u_1, u_4\}$ gives a larger matching than M, a contradiction. The case when $\{u_0, v_1, u_2\} \in E(H)$ is similar.

By the definition of M', there are at most $2(|U_1| - 2|M'|)$ edges containing v_0 with one vertex in $U_1 \setminus V(M')$ and one vertex in U_2. This implies that

$$\deg(v_0) \leqslant \binom{|U_1|}{2} + 2|M'||U_2| + 2(|U_1| - 2|M'|) = \binom{|U_1|}{2} + 2|U_1| + |M'|(2|U_2| - 4).$$

By (2), there are at most $|U_1||W_1| - |M'|$ edges consisting of u_0, one vertex in U_1, and one vertex in W_1, and at most $(|U_2|-1)(|W_1| - |M'|)$ edges consisting of u_0, one additional vertex in U_2, and one vertex in W_1. Therefore,

$$\begin{aligned}\deg(u_0) &\leqslant \binom{|U|-1}{2} + |U_1||W_2| + |U_1||W_1| - |M'| + (|U_2|-1)(|W_1| - |M'|)\\ &= \binom{|U|-1}{2} + |U_1||W| + (|U_2|-1)|W_1| - |U_2||M'|,\end{aligned}$$

and consequently,

$$\deg(v_0) + \deg(u_0) \leqslant \binom{|U_1|}{2} + 2|U_1| + \binom{|U|-1}{2} + |U_1||W| + (|U_2|-1)|W_1| + |M'|(|U_2|-4).$$

Since $|W| \leqslant (\frac{1}{3} - \gamma)n$, we have $|U_2| > 3\gamma n > 4$. As $|M'| \leqslant |M| = |W_1| = \frac{|U_1|}{2}$, it follows that

$$\begin{aligned}\deg(v_0) + \deg(u_0) &\leqslant \binom{|U_1|}{2} + 2|U_1| + \binom{|U|-1}{2} + |U_1||W|\\ &\quad + (|U_2|-1)\frac{|U_1|}{2} + \frac{|U_1|}{2}(|U_2|-4)\\ &= \left(\binom{|U|}{2} - \binom{|U_2|}{2}\right) + \binom{|U|-1}{2} + \left(|W| - \frac{1}{2}\right)|U_1|\\ &= (|U|-1)^2 - \binom{|U_2|}{2} + (2|W|-1)|M|.\end{aligned}$$

Since $|M| \leqslant |W|-1$ and $|U_2| \geqslant n-3|W|+2$, we derive that

$$\begin{aligned}\deg(v_0)+\deg(u_0) &\leqslant (n-|W|-1)^2-\binom{n-3|W|+2}{2}+(2|W|-1)(|W|-1)\\ &= \frac{2}{3}n^2-\frac{7}{3}n+\frac{73}{24}-\frac{3}{2}\left(\frac{n}{3}+\frac{7}{6}-|W|\right)^2.\end{aligned}$$

Since $|W| \leqslant (\frac{1}{3}-\gamma)n$, $0<\varepsilon \ll \gamma$ and n is sufficiently large, we have

$$\deg(v_0)+\deg(u_0) \leqslant \frac{2}{3}n^2-\frac{7}{3}n+\frac{73}{24}-\frac{3}{2}\left(\gamma n+\frac{7}{6}\right)^2 < \frac{2}{3}n^2-\varepsilon n^2.$$

This contradicts our assumption on $\sigma_2(H)$ because v_0 and u_0 are adjacent.

Case 1.2: No vertex in W_2 is adjacent to any vertex in U_2.

Fix $v_0 \in W_2$. Since v_0 is not adjacent to any vertex in U_2, we have $\deg(v_0) \leqslant \binom{|U_1|}{2} = \binom{2|M|}{2}$. Since v_0 is not an isolated vertex, there exists a vertex $u_1 \in U_1$ that is adjacent to v_0. By the assumption, there is no edge of H containing u_1, a vertex from U_2, and a vertex from W_2. Thus $\deg(u_1) \leqslant \binom{|U|-1}{2}+(|U|-1)|W|-|U_2||W_2|$. Since $|M| \leqslant |W|-1$ and $|U|=n-|W|$, it follows that

$$\begin{aligned}\deg(v_0)+\deg(u_1) &\leqslant \binom{2(|W|-1)}{2}+\binom{|U|-1}{2}+(|U|-1)|W|-(n-3|W|+2)\\ &= \frac{3}{2}\left(|W|-\frac{1}{2}\right)^2+\frac{1}{2}n^2-\frac{5}{2}n+\frac{13}{8}.\end{aligned}$$

Furthermore, since $|W| \leqslant (\frac{1}{3}-\gamma)n$ and $0<\varepsilon \ll \gamma$, we derive that

$$\begin{aligned}\deg(v_0)+\deg(u_1) &\leqslant \frac{3}{2}\left(\frac{n}{3}-\gamma n-\frac{1}{2}\right)^2+\frac{1}{2}n^2-\frac{5}{2}n+\frac{13}{8}\\ &= \left(\frac{2}{3}-\gamma+\frac{3}{2}\gamma^2\right)n^2-\left(3-\frac{3}{2}\gamma\right)n+2\\ &< \frac{2}{3}n^2-\varepsilon n^2,\end{aligned}$$

contradicting our assumption on $\sigma_2(H)$.

Case 2: $|W| > (\frac{1}{3}-\gamma)n$.

Claim 8. *$|M| \geqslant n/3-\gamma' n$.*

Proof. To the contrary, assume that $|M| < n/3-\gamma' n$. Fix $v_0 \in W_2$. Then $\deg(v_0) \leqslant \binom{|U|}{2}-\binom{|U_2|}{2}$ because there is no edge of type $U_2U_2W_2$. Suppose $u \in U$ is adjacent to v_0. Trivially $\deg(u) \leqslant \binom{|U|-1}{2}+(|U|-1)|W|$. Thus

$$\begin{aligned}\deg(v_0)+\deg(u) &\leqslant \binom{|U|-1}{2}+(|U|-1)|W|+\binom{|U|}{2}-\binom{|U_2|}{2}\\ &= (n-1)(|U|-1)-\binom{|U_2|}{2}.\end{aligned}$$

Our assumptions imply that $|U| \leqslant 2n/3 + \gamma n$ and $|U_2| \geqslant 2\gamma' n$. As a result,

$$\deg(v_0) + \deg(u) \leqslant (n-1)\left(\frac{2}{3}n + \gamma n - 1\right) - \binom{2\gamma' n}{2} < \frac{2}{3}n^2 - \varepsilon n^2,$$

because $\varepsilon \ll \gamma \ll \gamma'$ and n is sufficiently large. This contradicts our assumption on $\sigma_2(H)$. $\square$

Fix $u_1 \neq u_2 \in U_2$ and $v_0 \in W_2$. Trivially $\deg(w) \leqslant \binom{|U|}{2}$ for any vertex $w \in W$ and $\deg(u) \leqslant \binom{|U|-1}{2} + |W|(|U|-1)$ for any vertex $u \in U$. Furthermore, for any two distinct edges $e_1, e_2 \in M$, we observe that at least one triple of type UUW with one vertex from each of e_1 and e_2 and one vertex from $\{u_1, u_2, v_0\}$ is *not* an edge – otherwise there is a matching M_3 of size three on $e_1 \cup e_2 \cup \{u_1, u_2, v_0\}$ and $M_3 \cup M \setminus \{e_1, e_2\}$ is thus a matching larger than M. By Claim 8, $|M| \geqslant n/3 - \gamma' n$. Thus,

$$\deg(u_1) + \deg(u_2) + \deg(v_0) \leqslant 2\left(\binom{|U|-1}{2} + |W|(U|-1)\right) + \binom{|U|}{2} - \binom{n/3 - \gamma' n}{2}.$$

On the other hand, since $|W| > (\frac{1}{3} - \gamma)n \geqslant n/4$, Claim 3 implies that u_i is adjacent to some vertex in W for $i = 1, 2$. We know that v_0 is adjacent to some vertex in U. Therefore, $\deg(u_i) > (2n^2/3 - \varepsilon n^2) - \binom{|U|}{2}$ for $i = 1, 2$, and $\deg(v_0) > (2n^2/3 - \varepsilon n^2) - \left(\binom{|U|-1}{2} + |W|(|U|-1)\right)$. It follows that

$$\deg(u_1) + \deg(u_2) + \deg(v_0) > 3\left(\frac{2n^2}{3} - \varepsilon n^2\right) - 2\binom{|U|}{2} - \binom{|U|-1}{2} - |W|(|U|-1).$$

The upper and lower bounds for $\deg(u_1) + \deg(u_2) + \deg(v_0)$ together imply that

$$3\left(\binom{|U|-1}{2} + |W|(|U|-1) + \binom{|U|}{2}\right) - \binom{n/3 - \gamma' n}{2} > 3\left(\frac{2n^2}{3} - \varepsilon n^2\right),$$

$$\text{or} \quad (|U|-1)(n-1) - \frac{1}{3}\binom{n/3 - \gamma' n}{2} > \frac{2n^2}{3} - \varepsilon n^2,$$

which is impossible because $|U| \leqslant 2n/3 + \gamma n$, $0 < \varepsilon \ll \gamma \ll \gamma' \ll 1$ and n is sufficiently large. This completes the proof of Lemma 4.

4 Proof of Theorem 2

Choose a matching M such that (i) M covers all the vertices of W; (ii) subject to (i), $|M|$ is the largest. Lemma 4 implies that such a matching exists. Let $M_1 = \{e \in M : e \cap W \neq \emptyset\}$, $M_2 = M \setminus M_1$, and $U_3 = V(H) \setminus V(M)$. We have $|M_1| = |W|$, $|M_2| = |M| - |W|$, $|U_3| = n - 3|M|$.

Suppose to the contrary, that $|M| \leqslant n/3 - 1$. Fix three vertices u_1, u_2, u_3 of U_3. We distinguish the following two cases.

Case 1: $|M| \leqslant n/3 - \eta n$.

Trivially, for every $i \in \{1,2,3\}$, there are at most $3|M|$ edges in H containing u_i and two vertices from the same edge of M. For any distinct e_1, e_2 from M, we claim that

$$\sum_{i=1}^{3} |L_{u_i}(e_1, e_2)| \leqslant 18. \tag{3}$$

Indeed, let H_1 be the 3-partite subgraph of H induced on three parts $\{u_1, u_2, u_3\}$, e_1, and e_2. We observe that H_1 does not contain a perfect matching – otherwise, letting M_1 be a perfect matching of H_1, $(M \setminus \{e_1, e_2\}) \cup M_1$ is a larger matching than M, a contradiction. Apply Lemma 5 with $n = k = s = 3$, we obtain that $|E(H_1)| \leqslant 18$. Therefore $\sum_{i=1}^{3} |L_{u_i}(e_1, e_2)| \leqslant 18$.

For any $e \in M_1$, we claim that

$$\sum_{i=1}^{3} |L_{u_i}(e, U_3)| \leqslant 6(|U_3| - 1).$$

Indeed, assume $e = \{v_1, v_2, v_3\} \in M_1$ with $v_1 \in W$. Apply Lemma 6 with $A = \{u_1, u_2, u_3\}$, $V = U_3$, and $G_i = (U_3, L_{v_i}(U_3))$ for $i = 1,2,3$. Since $|M| \leqslant n/3 - 4$, we have $|B| = |U_3| - 3 \geqslant 2$. By the maximality of M, no edge of G_1 is disjoint from an edge of G_2 or G_3. By Lemma 6, $\sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 6(|U_3| - 1)$. Hence $\sum_{i=1}^{3} |L_{u_i}(e, U_3)| = \sum_{i=1}^{3} \sum_{j=1}^{3} \deg_{G_i}(u_j) \leqslant 6(|U_3| - 1)$.

Similarly, for any $e \in M_2$, we can apply Lemma 7 to obtain that

$$\sum_{i=1}^{3} |L_{u_i}(e, U_3)| \leqslant 3(|U_3| + 1).$$

Putting these bounds together gives

$$\begin{aligned}\sum_{i=1}^{3} \deg(u_i) &\leqslant 18\binom{|M|}{2} + 9|M| + \sum_{i=1}^{3} |L_{u_i}(V(M_1), U_3)| + \sum_{i=1}^{3} |L_{u_i}(V(M_2), U_3)| \\ &\leqslant 18\binom{|M|}{2} + 9|M| + 6|M_1|(|U_3| - 1) + 3|M_2|(|U_3| + 1).\end{aligned}$$

Since $|M_1| = |W|$, $|M_2| = |M| - |W|$, $|U_3| = n - 3|M|$, we derive that

$$\begin{aligned}\sum_{i=1}^{3} \deg(u_i) &\leqslant 18\binom{|M|}{2} + 9|M| + 6|W|(n - 3|M| - 1) + 3(|M| - |W|)(n - 3|M| + 1) \\ &= (3n - 9|W| + 3)|M| + 3|W|n - 9|W|.\end{aligned}$$

Furthermore, $3n - 9|W| + 3 > 0$ and $|M| \leqslant n/3 - \eta n$ implies that

$$\sum_{i=1}^{3}\deg(u_i) \leqslant (3n - 9|W| + 3)\left(\frac{n}{3} - \eta n\right) + 3|W|n - 9|W|$$
$$= (9\eta n - 9)\,|W| + (1 - 3\eta)\,n^2 + (1 - 3\eta)\,n. \tag{4}$$

If $|W| \leqslant n/4$, from (4), we have

$$\sum_{i=1}^{3}\deg(u_i) \leqslant (9\eta n - 9)\,\frac{n}{4} + (1 - 3\eta)\,n^2 + (1 - 3\eta)\,n = \left(1 - \frac{3}{4}\eta\right)n^2 - \left(3\eta + \frac{5}{4}\right)n,$$

which contradicts the condition $\sum_{i=1}^{3}\deg(u_i) \geqslant 3\left(\frac{n^2}{3} - \frac{\varepsilon n^2}{2}\right)$ because $u_i \in U_3$ for $i \in [3]$ and $\varepsilon \ll \eta$.

If $|W| > n/4$, Claim 3 implies that u_i is adjacent to one vertex of W, $i = 1, 2, 3$. Furthermore, $\deg(w) \leqslant \binom{|U|}{2}$ for $w \in W$. So

$$\sum_{i=1}^{3}\deg(u_i) > 3\left(\frac{2n^2}{3} - \varepsilon n^2 - \binom{|U|}{2}\right) = 3\left(\frac{2n^2}{3} - \varepsilon n^2 - \binom{n - |W|}{2}\right).$$

The upper and lower bounds for $\sum_{i=1}^{3}\deg(u_i)$ together imply that

$$(9\eta n - 9)\,|W| + (1 - 3\eta)\,n^2 + (1 - 3\eta)\,n + 3\binom{n - |W|}{2} > 3\left(\frac{2n^2}{3} - \varepsilon n^2\right),$$

which is a contradiction because $|W| > n/4$, $0 < \varepsilon \ll \eta \ll 1$ and n is sufficiently large.

Case 2: $|M| > n/3 - \eta n$.

If $|M| = n/3 - 1$, then $|U_3| = 3$ and we can not apply Lemmas 6 and 7. Fortunately, when $|M| > n/3 - \eta n$, Lemma 5 suffices for our proof.

Let $W' = \{v \in W : \deg(v) \leqslant (5/18 + \tau)n^2\}$. Let M' be the sub-matching of M covering every vertex of W'. If $|W'| \leqslant \rho n$, we claim that $\deg_{H'}(u) \geqslant \left(\frac{5}{9} + \gamma\right)\binom{n}{2}$ for every vertex $u \in V(H')$, where $H' := H[V \setminus V(M')]$. Indeed, from the definition of W', $\deg_H(u) > (5/18 + \tau)n^2$ for every vertex $u \in V(H')$. Hence,

$$\deg_{H'}(u) \geqslant \deg_H(u) - 3n|W'| > \left(\frac{5}{18} + \tau\right)n^2 - 3n|W'|.$$

Since $|W'| \leqslant \rho n$, $0 < \gamma \ll \rho \ll \tau \ll 1$ and n is sufficiently large, we have

$$\deg_{H'}(u) > \left(\frac{5}{18} + \tau\right)n^2 - 3\rho n^2 > \left(\frac{5}{9} + \gamma\right)\binom{n}{2}.$$

In addition, n is divisible by 3, so $|V(H')|$ is divisible by 3. (1) implies that H' contains a perfect matching M''. Now $M' \cup M''$ is a perfect matching of H.

Therefore, we assume that $|W'| \geqslant \rho n$ in the rest of the proof. If one vertex of u_1, u_2, u_3, say, u_1, is adjacent to one vertex in W', the definition of W' implies that $\deg(u_1) > 2n^2/3 - \varepsilon n^2 - \left(\frac{5}{18} + \tau\right) n^2$. Recall that $\deg(u_i) > n^2/3 - \varepsilon n^2/2$ for $i = 2, 3$. Thus

$$\sum_{i=1}^{3} \deg(u_i) > \left(\frac{4}{3}n^2 - 2\varepsilon n^2\right) - \left(\frac{5}{18} + \tau\right) n^2 = \left(\frac{19}{18} - 2\varepsilon - \tau\right) n^2. \tag{5}$$

On the other hand,

$$\sum_{i=1}^{3} \deg(u_i) \leqslant 18\binom{|M|}{2} + 9|M| + 9|M|(n - 3|M| - 1) = 9|M|(n - 2|M| - 1),$$

where, by (3), $18\binom{|M|}{2}$ bounds the number of edges intersecting two members of M, $9|M|$ bounds the number of edges with two vertices in the same member of M, and $9|M|(n-3|M|-1)$ bounds the number of edges with one vertex in $V(M)$ and an additional vertex in U_3 (besides u_i). Since the function $f(x) := 9x(n-2x-1)$ decreases when $x \geqslant \frac{n-1}{4}$, we have $f(x) \leqslant f(\frac{n}{3} - \eta n)$ for all $x \geqslant \frac{n}{3} - \eta n$. It follows that

$$\sum_{i=1}^{3} \deg(u_i) \leqslant 9\left(\frac{n}{3} - \eta n\right)\left(n - 2\left(\frac{n}{3} - \eta n\right) - 1\right) = (1 + 3\eta - 18\eta^2)n^2 - (3 - 9\eta)n.$$

Note that $(1 + 3\eta - 18\eta^2)n^2 - (3 - 9\eta)n < \left(\frac{19}{18} - 2\varepsilon - \tau\right) n^2$ because $0 < \varepsilon \ll \eta \ll \tau \ll 1$ and n is sufficiently large. We thus obtain a contradiction with (5).

We thus assume that none of u_1, u_2, u_3 is adjacent to any vertex in W'. It follows that

$$\begin{aligned}
\sum_{i=1}^{3} \deg(u_i) &\leqslant 18\binom{|M| - |M'|}{2} + 9(|M| - |M'|) + 9(|M| - |M'|)(n - 3|M| - 1) \\
&\quad + 3\binom{2|M'|}{2} + 3(2|M'|)(n - 3|M'| - 1) \\
&= -3\left(|M'| + \frac{1}{2}n - \frac{3}{2}|M|\right)^2 - \frac{45}{4}|M|^2 + \frac{9}{2}n|M| - 9|M| + \frac{3}{4}n^2.
\end{aligned}$$

As before, $18\binom{|M|-|M'|}{2}$ bounds the number of edges intersecting two members of $M \setminus M'$, $9(|M| - |M'|)$ for those with two vertices in the same member of $M \setminus M'$, and $9(|M| - |M'|)(n - 3|M| - 1)$ for those with one vertex in $V(M \setminus M')$ and an additional vertex in U_3 (besides u_i). In addition, $3\binom{2|M'|}{2}$ bounds the number of edges with two vertices in $V(M') \setminus W'$, and $3(2|M'|)(n - 3|M'| - 1)$ for those with one vertex in $V(M') \setminus W'$ and one vertex in $V(H) \setminus V(M')$. Since $-n/2 + 3|M|/2 < 0$ and $|M'| = |W'| \geqslant \rho n$,

$$\begin{aligned}
\sum_{i=1}^{3} \deg(u_i) &\leqslant -3\left(\rho n + \frac{1}{2}n - \frac{3}{2}|M|\right)^2 - \frac{45}{4}|M|^2 + \frac{9}{2}n|M| - 9|M| + \frac{3}{4}n^2 \\
&= -18\left(|M| - \frac{1}{4}n - \frac{1}{4}\rho n + \frac{1}{4}\right)^2 + \left(\frac{9}{8} - \frac{15}{8}\rho^2 - \frac{3}{4}\rho\right) n^2 - \frac{9}{4}\rho n - \frac{9}{4}n + \frac{9}{8}.
\end{aligned}$$

Recall that $0 < \rho \ll 1$, so $\frac{1}{4}n + \frac{1}{4}\rho n - \frac{1}{4} < \frac{n}{3} - \eta n$. Furthermore, $|M| > \frac{n}{3} - \eta n$, hence we have

$$\begin{aligned}\sum_{i=1}^{3} \deg(u_i) &\leqslant -18\left(\frac{n}{3} - \eta n - \frac{1}{4}n - \frac{1}{4}\rho n + \frac{1}{4}\right)^2 + \left(\frac{9}{8} - \frac{15}{8}\rho^2 - \frac{3}{4}\rho\right)n^2 \\ &\quad - \frac{9}{4}\rho n - \frac{9}{4}n + \frac{9}{8} \\ &= \left(1 - 3\rho^2 - 9\eta\rho + 3\eta - 18\eta^2\right)n^2 + (9\eta - 3)n,\end{aligned}$$

which contradicts the condition $\sum_{i=1}^{3} \deg(u_i) \geqslant 3\left(n^2/3 - \varepsilon n^2/2\right)$ because $0 < \varepsilon \ll \eta \ll \rho \ll 1$ and n is sufficiently large. This completes the proof of Theorem 2.

5 Concluding remarks

In this paper we consider the minimum degree sum of two adjacent vertices that guarantees a perfect matching in 3-graphs. Given $3 \leqslant k < n$ and $2 \leqslant s \leqslant n/k$, can we generalize this problem to k-graphs not containing a matching of size s? For $1 \leqslant \ell \leqslant k$, let $H^{\ell}_{n,k,s}$ denote the k-graph whose vertex set is partitioned into two sets S and T of size $n - s\ell + 1$ and $s\ell - 1$, respectively, and whose edge set consists of all the k-sets with at least ℓ vertices in T. It is clear that $H^{\ell}_{n,k,s}$ contains no matching of size s. A well-known conjecture of Erdős [3] says that $H^{1}_{n,k,s}$ or $H^{k}_{n,k,s}$ is the densest k-graph on n vertices not containing a matching of size s. It is reasonable to speculate that the largest $\sigma_2(H)$ among all k-graphs H on n vertices not containing a matching of size s is also attained by $H^{\ell}_{n,k,s}$. Note that $H^{k}_{n,k,s}$ is a complete k-graph of order $sk - 1$ together with $n - sk + 1$ isolated vertices and thus $\sigma_2(H^{k}_{n,k,s}) = 2\binom{sk-2}{k-1}$. When $1 \leqslant \ell \leqslant k - 2$, any two vertices of $H^{\ell}_{n,k,s}$ are adjacent and thus $\sigma_2(H^{\ell}_{n,k,s}) = 2\delta_1(H^{\ell}_{n,k,s})$. When $\ell = k - 1$, it is easy to see that $\sigma_2(H^{k-1}_{n,k,s}) = 2\binom{s(k-1)-2}{k-1} + (n - s(k-1) + 2)\binom{s(k-1)-2}{k-2}$.

Assume $s = n/k$. Since $\delta_1(H^{\ell}_{n,k,n/k}) \leqslant \delta_1(H^{1}_{n,k,n/k})$ for $1 \leqslant \ell \leqslant k - 2$ and $H^{k}_{n,k,n/k}$ contains isolated vertices, we only need to compare $\sigma_2(H^{1}_{n,k,n/k})$ and $\sigma_2(H^{k-1}_{n,k,n/k})$. For sufficiently large n, it is easy to see that $\sigma_2(H^{1}_{n,k,n/k}) < \sigma_2(H^{k-1}_{n,k,n/k})$ when $k \leqslant 6$ and $\sigma_2(H^{1}_{n,k,n/k}) > \sigma_2(H^{k-1}_{n,k,n/k})$ when $k \geqslant 7$.

Problem 9. Does the following hold for any sufficiently large n that is divisible by k? Let H be a k-graph of order n without isolated vertex. If $k \leqslant 6$ and $\sigma_2(H) > \sigma_2(H^{k-1}_{n,k,n/k})$ or $k \geqslant 7$ and $\sigma_2(H) > \sigma_2(H^{1}_{n,k,n/k})$, then H contains a perfect matching.

Now assume $k = 3$ and $2 \leqslant s \leqslant n/3$. Note that

$$\sigma_2(H^{3}_{n,3,s}) = 2\binom{3s-2}{2}, \quad \sigma_2(H^{1}_{n,3,s}) = 2\left(\binom{n-1}{2} - \binom{n-s}{2}\right), \text{ and}$$
$$\sigma_2(H^{2}_{n,3,s}) = \binom{2s-2}{2} + (n - 2s + 1)\binom{2s-2}{1} + \binom{2s-1}{2} = (2s-2)(n-1).$$

It is easy to see that $\sigma_2(H^2_{n,3,s}) > \sigma_2(H^1_{n,3,s})$. Zhang and Lu [23] made the following conjecture.

Conjecture 10. [23] There exists $n_0 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph of order $n \geqslant n_0$ without isolated vertex. If $\sigma_2(H) > 2\left(\binom{n-1}{2} - \binom{n-s}{2}\right)$ and $n \geqslant 3s$, then H contains no matching of size s if and only if H is a subgraph of $H^2_{n,3,s}$.

Zhang and Lu [23] showed that the conjecture holds when $n \geqslant 9s^2$. Later the same authors [24] proved the conjecture for $n \geqslant 13s$. If Conjecture 10 is true, then it implies the following theorem of Kühn, Osthus and Treglown [10].

Theorem 11. *[10] There exists $n_0 \in \mathbb{N}$ such that if H is a 3-graph of order $n \geqslant n_0$ with $\delta_1(H) \geqslant \binom{n-1}{2} - \binom{n-s}{2} + 1$ and $n \geqslant 3s$, then H contains a matching of size s.*

Our Theorem 1 suggests a weaker conjecture than Conjecture 10.

Conjecture 12. There exists $n_1 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph of order $n \geqslant n_1$ without isolated vertex. If $\sigma_2(H) > \sigma_2(H^2_{n,3,s})$ and $n \geqslant 3s$, then H contains a matching of size s.

On the other hand, we may allow a 3-graph to contain isolated vertices. Note that $\sigma_2(H^2_{n,3,s}) \geqslant \sigma_2(H^3_{n,3,s})$ if and only if $s \leqslant (2n+4)/9$. We make the following conjecture.

Conjecture 13. There exists $n_2 \in \mathbb{N}$ such that the following holds. Suppose that H is a 3-graph of order $n \geqslant n_2$ and $2 \leqslant s \leqslant n/3$. If $\sigma_2(H) > \sigma_2(H^2_{n,3,s})$ and $s \leqslant (2n+4)/9$ or $\sigma_2(H) > \sigma_2(H^3_{n,3,s})$ and $s > (2n+4)/9$, then H contains a matching of size s.

In fact, we can derive Conjecture 13 from Conjecture 12 as follows. Let $n_2 = \max\{\binom{n_1}{2}, \frac{3}{2}n_1\}$ and H be a 3-graph of order $n \geqslant n_2$ satisfying the assumption of Conjecture 13. If H contains no isolated vertex, then H contains a matching of size s by Conjecture 12. Otherwise, let W be the set of isolated vertices in H. Let $H' = H[V(H) \setminus W']$ and $n' = n - |W|$. Then H' is a 3-graph without isolated vertex and $\sigma_2(H') = \sigma_2(H)$. When $2 \leqslant s \leqslant (2n+4)/9$, we have $\sigma_2(H') > \sigma_2(H^2_{n,3,s}) > \sigma_2(H^2_{n',3,s})$. In addition, since $n \geqslant \binom{n_1}{2}$ and

$$2\binom{n'-1}{2} \geqslant \sigma_2(H') > (2s-2)(n-1) \geqslant 2(n-1),$$

we have $n' \geqslant n_1$. When $s > (2n+4)/9$, we have $\sigma_2(H') > \sigma_2(H^3_{n,3,s}) > \sigma_2(H^2_{n,3,s}) > \sigma_2(H^2_{n',3,s})$. In addition, since $n \geqslant 3n_1/2$ and

$$2\binom{n'-1}{2} \geqslant \sigma_2(H') > 2\binom{3s-2}{2} > 2\binom{2(n-1)/3}{2},$$

we have $n' \geqslant n_1$. In both cases, Conjecture 12 implies that H' contains a matching of size s.

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